U.P. Rajarshi Tandon Open
University, Prayagraj

MSc STAT – 403 (N)A/ MA STAT – 403(N)A
Survival Analysis and Reliability Theory

Block: 1 Survival Analysis

Block: 2 Reliability Analysis

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MScSTAT–403(N)A/MASTAT–403(N)A SUEVIVAL ANALYSIS & RELIABILITY THEORY

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Blocks & Units Introduction

The ***Block - 1 – Survival Analysis,*** is the first block of said self-learning material (SLM), which is divided into five units.

The ***Unit – 1: Basic Concepts,*** is the first unit of present SLM, which serves as the introductory unit, outlining fundamental concepts such as time, order and random censoring, types of censoring and truncation, life tables, failure rate, mean residual life, ageing classes, bathtub failure rate, and the estimation of survival functions through the actuarial estimator, Kaplan-Meier estimator, and log-rank tests.

The ***Unit – 2: Parametric Survival Models,*** focus primarily on the assumptions and characteristics of parametric survival models, life distributions such as exponential, gamma, Weibull, lognormal, pareto, Rayleigh, piece-wise exponential, linear failure rate, parametric inference, likelihood ratio tests, maximum likelihood estimation tests, and estimation under the assumption of IFR/DFR.

The ***Unit – 3: Non-Parametric Survival Models,*** concentrates on the assumptions and characteristics of non-parametric survival models, total time on test, Deshpande test, two-sample problems such as the Gehan test, Log-Rank test, Mantel-Haenszel test, and Ware tests.

The ***Unit – 4: Proportional Hazard Models,*** explores assumptions and characteristics of proportional hazard models, semi-parametric regression for failure rate, Cox's proportional hazard model, rank tests for regression coefficients, competing risks models with parametric and non-parametric inference, and multiple decrement life tables.

The ***Unit – 5: Recurrent Event Survival Analysis,*** introduces recurrent event survival analysis, competing risks survival analysis, competing risk events, and frailty models.

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Unit – 6: Basic Concepts; This unit lays the groundwork for understanding reliability theory by defining key concepts and terms. It covers the basic ideas of reliability, failure rates, and the probability of failure. Topics include the definitions of reliability, maintainability, and availability, and their importance in system design. The unit also introduces the mathematical tools and statistical methods used to analyse reliability data, such as probability distributions and failure rate functions.

Unit – 7: Ageing; In this unit, the focus is on the phenomenon of ageing and its effects on system reliability. It explores the different stages of the life cycle of components and systems, including the infant mortality period, normal life, and wear-out phase. The unit discusses how ageing affects the failure rate and the reliability of components over time. It also covers models that describe ageing processes, such as the bathtub curve, and methods to assess and mitigate the impact of ageing on system performance.

Unit – 8: Reliability Estimation; This unit is dedicated to the techniques and methodologies for estimating the reliability of systems and components. It includes statistical methods for analysing reliability data, such as life data analysis, censored data analysis, and reliability testing. The unit also discusses the use of reliability block diagrams and fault tree analysis to model and estimate system reliability. Practical aspects of conducting reliability tests and interpreting the results to make informed decisions about system design and maintenance are also covered.

Unit – 9: Repairable Systems; This unit examines systems that can be repaired and restored to operational condition after experiencing failures. It covers the concepts of repair and maintenance, including preventive and corrective maintenance strategies. The unit introduces models for analysing the reliability of repairable systems, such as the renewal process, Markov chains, and availability models. It also discusses the impact of repair policies on system reliability and performance, and methods to optimize maintenance schedules to enhance system reliability.

Unit – 10: Growth Models and Accelerated Life Testing; In this unit, growth models and accelerated life testing techniques are explored. Growth models describe how the reliability of a system improves over time as a result of testing and corrective actions. The unit covers various reliability growth models, such as the Duane model and the AMSAA model. Accelerated life testing involves subjecting products to higher stress levels than normal to induce failures quickly and gather reliability data in a shorter period. The unit discusses the principles and methodologies of accelerated life testing, including stress testing, data analysis, and extrapolation of results to normal operating conditions.

At the end of every unit, the summary, self-assessment questions, and further readings are given.

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UNIT - 1 BASIC CONCEPTS

Structure

1.1 Introduction

Survival analysis, a branch of statistics dealing with time-to-event data, is fundamental in various fields, including medicine, engineering, and social sciences. This analysis focuses on understanding the time until an event of interest occurs, such as the failure of a component, the occurrence of a disease, or the death of a patient. Central to survival analysis is the concept of survival time (T), a random variable representing the time from the start of observation to the occurrence of the event. The Hazard Function h(t) provides the instantaneous rate of event occurrence at time t, given survival up to that time. Additionally, the Cumulative Hazard

Function R(t) represents the cumulative risk up to time t, defined as R(t). Censoring is a common challenge in survival analysis, occurring when the exact survival time is unknown. Order censoring happens when only the failure times of the first k items are available, with the remaining censored at the last failure time. Random censoring occurs when some information about survival time is available. Still, the exact time is not known for various reasons such as loss of follow-up or study termination. Censored data are the minimum of survival and censoring times, with an indicator of censoring status. To estimate population survival, life tables are employed, dividing time into intervals and recording the number of individuals alive, dead, lost, or withdrawn at the start and end of each interval. The actuarial method and the Kaplan-Meier method are common approaches for estimating survival functions. The failure (or hazard) rate represents the instantaneous rate of event occurrence, given no previous event. Different classes of distributions exist based on the shape of the failure rate function, such as Increasing Failure Rate (IFR), Decreasing Failure Rate (DFR), and Bathtub Failure Rate.

In conclusion, survival analysis provides valuable insights into time-to-event data, offering methods for estimating survival functions, handling censoring, and understanding failure rates. These tools are essential for various applications in healthcare, enabling a deeper understanding of event occurrence over time.

1.2 Objectives

After going through this unit, you should be able to:

- Explain the basic concepts and terminology of survival analysis, such as survival time, censoring, hazard rate, and survival function.
- Apply parametric survival models to estimate and compare the survival probabilities of different groups/ treatments, using distributions such as exponential, Weibull, and gamma.
- Use non-parametric survival models to estimate the survival function and test the equality of survival curves using methods such as the Kaplan-Meier estimator and log-rank test.

1.3 Concept of Time

Time

Survival analysis is a statistical methodology developed to analyse the time-to-event data. Time is referred to as the duration in years, months, weeks, or days starting from the initiation of an individual's follow-up until a specific event takes place.

Survival time can be defined broadly as the time to the occurrence of a given event. This event can be the development of a disease, response to a treatment, relapse, or death. Therefore, survival time can be tumour-free time, the time from the start of treatment to response, length of remission, and time to death. Survival data can include survival time, response to a given treatment, and patient characteristics related to response, survival, and the development of a disease.

Consider the random variable $T \geq 0$, which we will think of as the lifetime or the survival time of, say, a patient or a lightbulb. We want to know how long the patient or the lightbulb will last. Let T have density f(t) and distribution function F(t).

Define $S(t)=1-F(t)=P(T>t)$,the survival function of T, and define $h(t) = \frac{f(t)}{1-F(t)}$, the hazard rate or hazard function (historically in epidemiology, it was called the force of mortality). The hazard rate has the interpretation:

$$h(t)dt \cong P\{t<T<t+dt|T>t\} \tag{1}$$

$$= P\{\text{expiring in interval}(t, t+dt)|\text{survived past time } t\}$$

Integrating h(t),

$$\int_0^t h(u)du = \int_0^t \frac{f(u)}{1-F(u)} du = -\log\big(1-F(u)\big) |_0^t$$

$$= -\log\big(1-F(t)\big) = -\log S(t) ,$$

Which leads to the important expression:

$$S(t)= e^{-\int_0^t h(u)du} \tag{2}$$

Notice that $F(+\infty)=1$ (i.e., $S(+\infty)=0$) iff $\int_0^\infty h(u)du= \infty$.

Note that the above concepts can be extended to the case when T does not have a density, that is, when the d.f. F has jumped. Our convention will be to assume continuity but to modify concepts and formulas to include jumps in the d.f when it is important to do so.

Event

By "event," we are referring to occurrences such as death, the onset of disease, relapse from remission, recovery (e.g., returning to work), or any specified experience of interest that might occur to an individual.

While it is possible to analyse more than one event simultaneously, we will assume that only a specific event is of primary interest. In cases where multiple events are under consideration, such as death from various causes, the statistical challenge can be categorized either as a recurrent events problem or a competing risk problem.

Some examples of survival analysis are, leukaemia patients in remission over several weeks to see how long they stay in remission and following a disease-free cohort of individuals over several years to see who develops heart disease.

Censored Data

The techniques for reducing experimental time are known as censoring. In survival analysis the observations are lifetimes which can be indefinitely long. So quite often the experiment is so designed that the time required for collecting the data is reduced to manageable levels. Two types of censoring are built into the design of the experiment to reduce the time taken to complete the study.

There are generally three reasons why censoring might happen:

i. A subject does not experience the event before the study ends.
ii. A person is lost to follow up during the study period.
iii. A person withdraws from the study.

Here, we have adopted the notation that T_I is the survival time, C_I is the censoring time, and the observed random variables are $Y_i=T_i \wedge C_i$ and $\delta_i=I(T_i< C_i)$.

1.4 Order and Random Censoring

Order Censoring

Ordered censoring (or Type II censoring) is common in survival analysis, where the goal is to analyse time-to-event data, such as the time until a component fails. If n-identical components are simultaneously put into operation, the study is discontinued when a predetermined number $k(<n)$ of the items fail. Hence, the failure times of the k failed items are available. These are the k smallest order statistics of the complete random sample. For the remaining items the censoring time x_k, which is the failure of the item failing last, is available.

Example: Twelve ceramic capacitors are subjected to a life test. To reduce the test time, the test is terminated after the eight capacitors fail. The remaining are type II censored or ordered censored data.

Random Censoring

Random censoring occurs when some information about individual survival time is available, but exact survival time is not known. The most frequent type of censoring is known as right random censoring. It occurs when the complete lifetimes are not observed for reasons that are beyond the control of the experimenter. *For example*, it may occur in any one of the following situations

1) Loss to follow up
2) Drop Out.
3) Termination of the study

Random censoring arises in medical applications involving animal studies or clinical trials. In a clinical trial, patients may enter the study at different times, and then each is treated with one of several possible therapies.

Let $C_1, C_2 \ldots . C_n$ be iid each with d.f. G. C_I is the censoring time associated with T_i. We can only observe $(Y_1,\delta_1),\ldots\ldots\ldots\ldots,(Y_n,\delta_n)$; where:

$$Y_i = \min(T_i, C_i) = T_i \wedge C_i \tag{3}$$

$$\delta_i = I(T_i < C_i) = \begin{cases} 1 & \text{if} \quad T_i \quad \text{is not censored,} \\ 0 & \text{if} \quad T_i \quad \text{is not censored.} \end{cases}$$

Notice that $Y_1, \ldots\ldots\ldots\ldots, Y_n$ are iid with some d.f. Also $\delta_1, \ldots\ldots\ldots \delta_n$ contains the censoring information. With random censoring we will make the following crucial assumption that T_I and C_I are independent.

Note: In Random censoring, the number of complete(uncensored) observation is random and time for which the study last may also be random.

Example: Let $T \sim \omega(p,\sigma)$.

$$f(x) = \frac{p}{\sigma} \cdot \left(\frac{x}{\sigma}\right)^{p-1} e^{-(x/\sigma)^h}$$

$$L = \prod_{\substack{i=1 \\ i \in 0}}^{n_u} \frac{p}{\sigma} \left(\frac{x_i}{\sigma}\right)^{p-1} \cdot e^{-(t_i/\sigma)^p} \prod_{\substack{i=1 \\ i \in c}}^{n-n_u} e^{-(k_i/\sigma)^p}$$

Where n_u is the number of uncensored samples.

$$L = \frac{p^{n_u}}{\sigma^{n_u p}} \left(\prod_{i=1}^{n_u} x_i^{p-1} {}_{i \in u}\right)^{e^{-\sum_{i=1}^{n_u} di/\sigma)^p}} \cdot e^{-\sum_{i=1}^{n-u_u} (x_i/\sigma)^p}$$

As $n_u + n - n_u = n$

Taking logarithm, we get:

$$\log L = n_u \log p - n_{up} \log\sigma + (p-1)\sum_{i=1}^{n_u} \log x_i - \sum_{i=1}^{n} (x_i/\sigma)^p$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{-n_{up}}{\sigma} + \sum_{i=1}^{n} \frac{x_i^b}{\sigma^{p+1}} = 0$$

$$\Rightarrow -n_{up} + \frac{p}{\sigma^p}\sum_{i=1}^{n} t_i^p = 0$$

$$\hat{\sigma} = \frac{p \cdot t_i p}{n_{u \cdot p}}$$

$$\hat{\sigma} = \left(\frac{\sum_{i=1}^{n} t_i^p}{n_u}\right)^{1/p}$$

$$\frac{\partial \ln L}{\partial p} = \frac{n_u}{p} - n_u \log\sigma + \sum_{i=1}^{n_u} \log t_1 - \sum_{i=1}^{n} \left(\frac{t_i}{\sigma}\right)^p \log\frac{t_i}{\sigma}$$

$$\hat{p} = \frac{n_u}{n_u \log\sigma - \sum_{i=1}^{n_u} \log h_i + \sum_{i=1}^{n} \left(\frac{t_i}{\sigma}\right)^p \log\frac{h_i}{\sigma}}$$

These equations are transcendental equation and be solved by any numerical methods.

1.4.1 Likelihood in these cases

Delta Method

Suppose the random variable y has mean μ and variance σ^2 and suppose we want the distribution of some function g(y). Expand g(y) about μ as:

$$g(Y) = g(\mu) + (Y - \mu) g'(\mu) + \ldots$$

And ignore higher order terms to get:

$$g(Y) \approx (g(\mu),\sigma^2(g'(\mu))^2) \tag{4}$$

Where ≈ denotes "*is approximately distributed as*".

If furthermore, $Y \overset{a}{\sim} N(\mu,\sigma^2)$, then:

$$g(Y) \overset{a}{\sim} N(g(\mu),\sigma^2(g'(\mu))^2) \tag{5}$$

The delta method also has a multivariate version. Suppose:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ & \sigma_y^2 \end{bmatrix} \right)$$

And suppose we want the distribution of g(X, Y) then:

$$g(X,Y)=g\left(\mu_x,\mu_y\right)+\left(X-\mu_x\right)\frac{\partial}{\partial x}g\left(\mu_x,\mu_y\right)+\left(Y-\mu_y\right)\frac{\partial}{\partial y}g\left(\mu_x,\mu_y\right)+\ldots\ldots\ldots,$$

So, $g(x,Y) \approx \left(g\left(\mu_x,\mu_y\right), \sigma_x^2\left(\frac{\partial}{\partial x}g\right)^2 + 2\sigma_{xy}\frac{\partial}{\partial x}g\frac{\partial}{\partial y}g + \sigma_y^2\left(\frac{\partial}{\partial y}g\right)^2\right)$.

If $\begin{bmatrix} X \\ Y \end{bmatrix} \overset{a}{\sim} N$, then $g(X, Y) \overset{a}{\sim} N$

The delta method is very useful. *For example*, we could use it to get an approximate value for $Var\left(\frac{\overline{X}}{\overline{Y}}\right)$ or $Var(\overline{XY})$.

Exponential Distribution:

Under random censoring, let n_u = number of uncensored observations. Then:

$$L=\lambda^{n_u} \exp\left\{ -h\sum_{i=1}^{n} t_i - h\sum_{c=1}^{n} c_i \right\}$$

$$L = n_u \exp\left\{ -\lambda\sum_{i=1}^{n} y_i \right\}$$

$$\log L = n_u \log\lambda - \lambda\sum_{i=1}^{n} y_i$$

$$\frac{\partial \log L}{\partial \lambda}=\frac{n_u}{h}-\sum_{i=1}^{n} y_i$$

$$\hat{\lambda}=\frac{n_u}{\sum_{i=1}^{n} y_i}$$

$$\frac{\partial^2 \log L}{\partial \lambda^2}=-\frac{n_u}{\lambda^2}$$

$$i(\lambda)=\frac{n_u}{\widehat{\lambda^2}}$$

We remark that $\hat{\lambda}=\frac{n_u}{\sum_{i=1}^{n} y_I}$ is also the maximum likelihood estimator (MLE) under type I and type II censoring as well as random censoring.

1.5 Types of Censoring and Truncation

Type-I Censoring

In type I censoring, without accidental losses, all censored observations equal the study period's length. In type I censoring the number of failures is a random variable.

Example: Power supplies are major units for most electronic products. Suppose a manufacturer conducts a reliability test in which 15 power supplies are operated over the same duration. The manufacturer decides to terminate the test after 80000 hrs. Suppose 10 power supplies fail during the fixed time interval. Then remaining five are type I censored

Let t_c be some fixed number which we call the fixed censoring time. Instead of observing $T_1,T_2,\ldots..T_n$ we can only observe $y_1,y_2\ldots.y_n$. Where,

$$y_i=\begin{cases} T_i & \text{if} \quad T_i<t_c \;, \\ t_c & \text{if} \quad t_c\le T_i . \end{cases}$$

Notice that the distribution function of y has positive mass $p(T>t_c)>0$ at $y=t_c$

Type-II Censoring

In type II censoring (also known as order censoring) the time interval over which the observation is taken is a random variable. Let r<n be fixed, and let $T_{(1)}<T_{(2)}<\ldots\ldots\ldots<T_{(n)}$ be the order statistics of $T_1,T_2\ldots.T_n$. Observation ceases after the r^{th} failure so we can observe $T_{(1)},\ldots<T_{(r)}$. The full ordered observed sample is:

$$y_{(1)}=T_{(1)}$$

$$\vdots$$

$$y_{(r)}=T_{(r)}$$

$$y_{(r+1)}=T_{(r)}$$

$$\vdots$$

$$y_{(n)}=T_{(r)}$$

The censoring time for every censored observation in type I and type II censoring is identical, but not so in random censoring.

Example: Twelve ceramic capacitors are subjected to a life test. In order to reduce the test time, the test is terminated after eight capacitors fail. The remaining are type II censored

Right and Left Censoring

When the study ends or when the person is lost to follow-up or is withdrawn, called as right censoring. If the random variable of interest is too large, we do not get to observe it completely. This is called as left censoring.

Example: In random left censoring, we can only observe $(y_1,\varepsilon_1),\ldots.(y_n,\varepsilon_n)$; where, $Y_i= \max(T_i, C_i)= T_i \vee C_i$, $\varepsilon_i=I(C_i< T_i)$. Here both right and left censoring are present.

Example: A Stanford psychiatrist wanted to know the age at which a certain group of African children learned to perform a particular task. When he arrived in the village, there were some children who already knew how to perform the task so these children contributed left censored on observation. Some children learned the task while he was present and their ages could be recorded. When he left, there remained some children who had not yet learned the task, thereby contributing to right-censored observations.

Interval Censoring

Interval censoring is type of censoring for which life time is known only to fall into interval. Both the right and left censoring are special cases of interval censoring in which we may only gent to see that the random variable of interest falls in an interval. If a T_i is random right censored, we gent to observe that T_i falls in the interval $[C_i,\infty)$, and if T_i is random left censored, we gent to observe that T_i falls in the interval $(-\infty,C_i]$.

Truncation

In contrast to interval censoring there is truncation in which if the random variable of interest falls outside some interval, even its existence is unobserved. *For example*, suppose we want to get the distribution and expect size of a certain organelle in the cell, Because of limitations on the measuring equipment, if an organelle is below a certain size, it cannot be detected.

1.6 Life Tables

A life table is a tabular representation of central features of the distribution of a positive random variable, say T, with an absolutely continuous distribution. It may represent the lifetime of an individual, the failure time of a physical component, the remission time of an illness, or some other duration variable. In general, T is the time of occurrence of some event that ends individual survival in a given status. Let its cumulative distribution function be $F(t)=Pr(T\leq t)$ and let the corresponding survival function be $S(t)=1-F(t)$.

In applications to human mortality, which is where life tables originated, the time variable normally is a person's attained age and is denoted x. The function $\mu(x)$ is then called the force of mortality or death intensity. The life table function $l_x=100000S(x)$ is called the decrement function and is tabulated for integer x in complete life tables; in abridged life tables it is tabulated for sparser values of x, most often for five-year intervals of age. The radix l_0 is selected to minimize the need for decimal in the l_x table; a value different from 100,000 is sometimes chosen. Other life tables functions are the expected number of deaths $d_x=l_x-l_{x+1}$ at age x (i.e., between age x and age x + 1), the single year death probability $q_x=Pr(T\leq x+1|T>x) = d_x/l_x$, and the corresponding survival probability $p_x=1-q_x$. Simple integration gives: $q_x=1-\exp\left[-\int_x^{x+1}\mu(s)ds\right]$.

1.7 Failure Rate

Survival time data measure the time to a certain event, such as failure, death, response, relapse, the development of a given disease, parole, or divorce. These times are subject to random variations, and like any random variables, form a distribution. The distribution of survival times is usually described or characterized by three functions: (1) the survivorship function, (2) the probability density function, and (3) the hazard function. These three functions are mathematically equivalent—if one of them is given, the other two can be derived.

Let T denote the survival time. The distribution of T can be characterized by three equivalent functions. The probability that an individual survives beyond the time t is called as survival time of the individual at time t.

The **Survival Function** (or Survivorship Function), denoted by S(t), is defined as the probability that an individual survives longer than t:

S(t) = p (an individual survives longer than t)

$$S(t) = p(T>t)=1-p[T\leq t]=1-F(t)=\bar{F}(t) \qquad (6)$$

T denotes the response variable and $T\geq 0$. It ranges from 0 to ∞ and some properties:

i) $S(0)=1$

ii) $S(\infty)=0$

The **Probability Density Function** (or Density Function) is given as:

$$f(t)=\frac{d}{dt}F(t)=-\frac{d}{dt}\bar{F}(t) \qquad (7)$$

The **Hazard Function** of failure rate function, denoted by h(t) is the instantaneous rate of which event occurs, given no previous event.

$$\begin{aligned} h(t) &= \lim_{0<h\to 0}\frac{1}{h}P[t<T\leq t+dt|T>t] \\ &= \lim_{0<h\to 0}\frac{\bar{F}(t)-\bar{F}(t+dt)}{h\bar{F}(t)} \\ &= \frac{f(t)}{\bar{F}(t)}, \text{ provided } F(t)<1, \text{ and } f(t) \text{ exists.} \end{aligned}$$

Conversely:

$$S(t)=\bar{F}(t)=\exp\left\{-\int_0^t h(u)\,du\right\} \qquad (8)$$

The probability density function of hazard function:

$$f(x)=h(x)*\exp\left\{-\int_0^t h(u)\,du\right\} \qquad (9)$$

The **Cumulative Hazard Functions** describes the cumulative risk up to age x, denoted by:

$$R(t)=\int_0^t h(u)\,du,\ t\geq 0 \qquad (10)$$

Therefore,

$$S(t)=\bar{F}(t)=\exp[-R(t)]=\exp\left(-\int_0^t h(u)\,du\right),\ t\geq 0 \qquad (11)$$

Example-1: Let $x \sim \exp(\theta)$

The probability density function of the given distribution:

$$f(x)=\frac{1}{\theta}e^{-x/\theta};x,\theta>0$$

$$S(t) = P(x>t)$$

$$=\int_t^\infty f(x)dx.$$

$$=\int_t^\infty \frac{1}{e}e^{-x/\theta}dx=e^{-t/\theta}$$

$$S(t) = e^{-t/\theta}$$

Example-2: Suppose that the survival time of a population has the following density function:

$$f(t) = e^{-t}, t\geq 0$$

The cumulative distribution function: $F(t) = \int_0^t f(x)dx = \int_0^t e^{-x}dx = -e^{-x}|_0^t = 1-e^{-t}$

Survival Function: $S(t) = e^{-t}$

The hazard function: $h(t)=\frac{e^{-t}}{e^{-t}}=1$

1.8 Mean Residual Life

Let a unit be of age t. that is, it has survived without failure up to time t. Since the unit has not yet failed at has certain amount of residual life time.

Let, $S(t)=p(T_t>x)=p(T>t+x|T>t)=\frac{\bar{F}(t+x)}{\bar{F}(t)}$

Then the mean residual life function is defined as:

$$L_{F(t)} = E(T_t)=\int_0^\infty xdF(x)=\int_0^\infty [1-F(x)]dx$$

$$L_{F(t)} = \int_0^\infty S(t)\,dt=\int_0^\infty \frac{\bar{F}(t+u)}{\bar{F}(t)}du, t\geq 0 \qquad (12)$$

This gives, $L_F(0)=E(T)=\mu$

***Example*:** Let the distribution has a constant hazard, $\lambda(t)=\lambda$.

The cumulative hazard function is given by: $h(t)= \int_0^t h(u)du = \int_0^t \lambda du = \lambda t|_0^t = \lambda t$

The survival function is: $S(t)= e^{-h(t)}= e^{-\lambda t}$

The probability density function is given by: $f(t)=h(t)S(t)=\lambda e^{-\lambda t}$

The mean of an exponential random variable is given by: $E(T)= \int_0^\infty S(t)\,dt= \int_0^\infty e^{-\lambda t}\,dt=\frac{1}{\lambda}$ and, $Var(T)=\frac{1}{\lambda^2}$.

1.9 Ageing Classes

The choice of ageing plays and important role in the choice of models for the lifetime distributions. Consider the life length of components and the corresponding, life length of system of components. In General, life length is random and so we are failing to study life distribution, so we first consider the concept of ageing. Ageing is studied in terms of Failure/ Hazard rate function.

In the simplest case when no ageing is present, we obtain a constant failure rate corresponding to exponential distribution. The exponential distribution is in several senses the most fundamental distribution in reliability theory.

The survival probability of a fresh unit corresponding to a mission of duration x is:

$$S(x)=1-F(x)$$

$$\overline{F(x)}\equiv 1-F(x)$$

F(x) is life distribution of the unit.

The corresponding conditional survival of a unit of age t is $\overline{F\left(\frac{x}{t}\right)}= \frac{\bar{F}(t+x)}{\bar{F}(t)}$, if $\overline{F(t)}>0$. The conditional probability of failure during the next interval, of duration x of a unit of age t is $\overline{F\left(\frac{x}{t}\right)}= \frac{\bar{F}(t+x)-F(t)}{\bar{F}(t)}=1- \overline{F\left(\frac{x}{t}\right)}$

Suppose now the unit ages adversely in the series that conditional survival probability is a decreasing function of age $\overline{F\left(\frac{x}{t}\right)}= \frac{\bar{F}(t+x)-F(t)}{\bar{F}(t)}$ is decreasing in $-\infty<t<\infty$

As a consequence, we obtain: $h(t)=\lim_{x\to 0}\frac{1}{x}\left[\frac{1-F(t+x)}{\bar{F}(t)}\right]$ is increasing in $t\geq 0$, when density f(t) exists. Conversely $\left(h(t)\right)$ is increasing implies $\bar{F}(x|t) = \exp[- \int_t^{t+x} h(u)du\,]$; decreasing in $t\geq 0$. So that $\overline{F\left(\frac{x}{t}\right)}$ decreasing in $-\infty<t<\infty$ holds. Then, when the density f(t) exists $\overline{F\left(\frac{x}{t}\right)}= \frac{\bar{F}(t+x)-F(t)}{\bar{F}(t)}$ is equivalent to failure rate h(t) increasing.

1.9.1 Class of Distribution Corresponding to Adverse Ageing

Increasing Failure Rate (IFR) class of distribution

We shall first define the concept of stochastic dominance. If X and Y are the two random variables then X is "stochastically smaller" than $Y(X \leq Y)$ if $F(x) \geq G(x) \forall x$, where F and G are distribution function of X and Y respectively. Obviously:

$$F(x) \geq G(x) \forall \Leftrightarrow \overline{F(x)} \leq \overline{G(x)}, \forall x \quad (13)$$

That is, $p[X>x) \leq p[Y>x], \forall x$

In term of survival functions, $\bar{F}_{t_1}(x) \geq \bar{F}_{t_2}(x) \ \forall \ 0 \leq t_1 \leq t_2$

$\Leftrightarrow \bar{F}_t \downarrow t$

$\Leftrightarrow \lim_{x \to 0} \frac{1}{x}(1-\bar{F}_t(x)) \uparrow t$

$\Leftrightarrow \lim_{x \to 0} \frac{1}{x}[1-\frac{\bar{F}(t+x)}{\bar{F}(t)}] \uparrow t$

$\Leftrightarrow \lim_{x \to 0} \frac{1}{x}[\frac{F(t+x)-F(t)}{\bar{F}(t)}] \uparrow t$

$\Leftrightarrow r(t) \uparrow t$, provides the pdf exists.

Thus, the class of distribution known as the increasing failure rate (IFR) class is also exactly the class of distribution F such that Eq. (4) is satisfied for $\forall x \in R$.

Increasing failure rate Average (IFRA) class of distributions

The failure rate average function is defined as:

$$\bar{R}_F(t) = \frac{1}{t} R(t) = -\frac{1}{t} \log \bar{F}(t) \quad (14)$$

If the function $\bar{R}_F(t)$ is increasing, then the distribution F is said to possess the increasing failure rate average property and is said to belong to the IFRA class.

A distribution F is IFRA distribution if and only if: $\bar{F}(\alpha t) \geq [\bar{F}(t)]^{\alpha}$ for $0<\alpha \leq 1 \& t \geq 0$.

It is obvious that IFR⇒IFRA as the average of an increasing function is increasing.

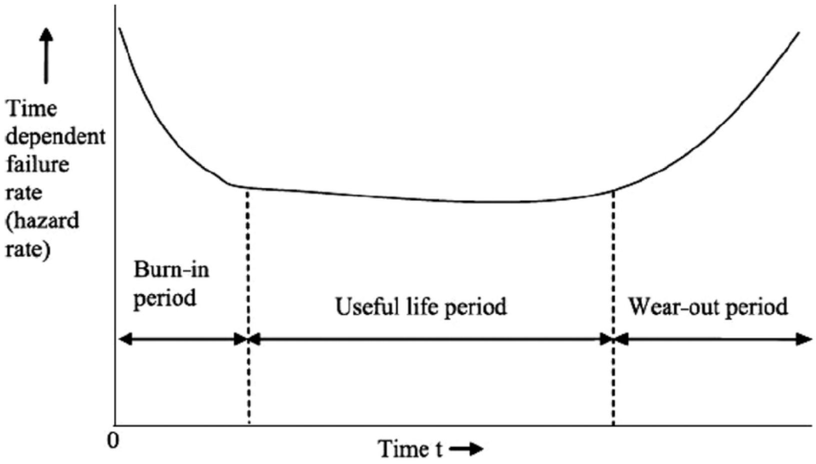
Simply, f or F has an increasing failure rate (we say f or F is IFR) if h(t) is increasing; f or F has an increasing failure rate average (we say f or F is IFRA) if $\frac{1}{t}\int_0^t h(u)du$ is increasing. Analogous definitions can be made for DFR and DFRA.

Constant FR	**IFR**	**DFR**
Exponential	Weibull ($\alpha > 1$)	Weibull ($\alpha < 1$)
	Gamma ($\alpha > 1$)	Gamma ($\alpha < 1$)
	Rayleigh ($\lambda_1 > 0$)	Rayleigh ($\lambda_1 < 0$)
		Pareto ($t > a$)

The concept of IRF and IFRA distributions are useful in engineering applications, particularly in the study of systems of components. In biostatistics they are not usually helpful. *For example*, in epidemiological studies the risk for long term survival usually has a bathtub shape with time divided into three periods.

1.10 Bathtub Failure Rate

Another class of the distributions which arises naturally in human mortality study and in reliability situations is characterized by failure functions having “bathtub shape”. The failure rate decreases initially. This initial phase is known as the “infant mortality” phase. The next phase is known as “useful life” phase, in which the failure rate is more or less constant. Finally, in the third phase, known as “wear out phase”, the failure rate increases. The three phases of failure rates are represented by a bathtub curve.

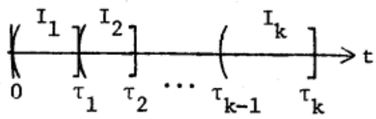


1.11 Estimation of Survival Function

The classical method of estimating S(t) in epidemiology and actuarial science is the actuarial method discussed below. It depends on the life table.

Let time be partitioned into a fixed sequence of intervals I_i…………, I_k,

These intervals are almost always, but not necessarily, of equal lengths, and for human populations the length of each interval is usually one year.



For a life table, let:

n_i= number of alive at the beginning

d_i = # died during I_i^{th} interval,

l_i = # lost to follow-up during I_i,

w_i = # withdraw during I_i interval

p_i = p {surviving through I_i | alive at beginning of I_i}

$q_i = 1 - p_i$

Reduced Sample Method

To estimate S (τ_k), use only those subjects who are at risk during (0, τ_k], the entire interval of interest. Let:

$n = n_1 - \sum_{i=1}^{k} l_i - \sum_{i=1}^{k} w_i$

$d = \sum_{i=1}^{k} d_i$

$\hat{S}(\tau_k) = 1 - \frac{d}{n}$

Table 1. Computation of the 5- Year Survival Rate

Years after diagnosis	Alive at beginning of Interval (n_i)	Died during Interval (d_i)	Lost to Follow-up during Interval (l_i)	Withdraw alive during Interval (w_i)
0-1	126	47	4	15
1-2	60	5	6	11
2-3	38	2	-	15
3-4	21	2	2	7
4-5	10	-	-	6

Suppose we want to estimate $\hat{S}(5)$, then:

$$n = 126 - \sum l_i - \sum w_i$$

$$=126-(4+6+2)-(15+11+15+7+6)$$

$$=126-12-54=60$$

$$d = \sum d_i =(47+5+3+2)=56$$

$$\hat{S}(5 \text{ years}) = 1 - \frac{56}{60} = 0.078$$

Which is the survival probability at the 5th interval.

The drawback with the reduced sample method is that it ignores the information that is contained in l_i and w_i. It is a biased (downward) estimate of survival function S(t).

1.11.1 Actuarial Estimator

We can break up the survival probability S (τ_k) into a product of probabilities:

$$S(\tau_k) = P\{T>\tau_k\},$$

$$= P\{T>\tau_1\}\, P\{T>\tau_2|T>\tau_1\}\ldots\ldots P\{T>\tau_k|T>\tau_{k-1}\},$$

$= p_1 . p_2 \ldots\ldots p_k = \prod_{i=1}^{k} p_i,$

Where, $p_i = P\{T>\tau_1 | T>\tau_{i-1}\}$.

Here we assume that, on the average, those individuals who become lost or withdrawn their self during i^{th} interval (I_i) were at risk for half the interval. The actuarial method gives an example an estimate for each p_i separately and then multiplies the estimates together to estimate $S(\tau_k)$.

For an estimate of p_i , we could use $1 - \frac{d_i}{n_i}$, if there were no losses or withdrawals in I_i. However, with l_i and w_i nonzero, we assume that, on the average, those individuals who became lost or withdrawn during I_i were at risk for half the interval. Therefore, we define the effective sample size $n_i^{'}$ on:

$n_i^{'} = n_i - \frac{1}{2}(l_i + w_i),$

$\widehat{q_i} = \frac{d_i}{n_i^{'}}$,

$\widehat{p_i} = 1 - \widehat{q_i}$.

The actuarial estimate is: $\hat{S}(\tau_k) = \prod_{i=1}^{k} \widehat{p_i}$.

From the above *Table 1.*

Years after diagnosis	Alive at beginning of Interval (n_i)	Died during Interval (d_i)	Lost to Follow-up during Interval (l_i)	Withdraw alive during Interval (w_i)	$n_i^{'}$	$q_i^{'}$	$\widehat{p_i}$
0-1	126	47	4	15	1116.5	0.40	0.6
1-2	60	5	6	11	51.5	0.10	0.9
2-3	38	2	-	15	30.5	0.07	0.93
3-4	21	2	2	7	16.5	0.12	0.88
4-5	10	-	-	6	7.0	0	1

Now, from the above table, $\hat{S}(\tau_k)$, = 0.44

Variance of $\hat{S}(\tau_k)$

To estimate the variance of $\hat{S}(\tau_k)$, consider: $\log \hat{S}(\tau_k) = \sum_{i=1}^{k} \log \hat{p}_i$.

Assuming $n_i' \hat{p}_i \approx$ Binomial (n_i', p_i), the delta method implies:

$$Var(\log \hat{p}_i) \approx Var(\hat{p}_i)\left(\frac{d}{dp_i}(\log p_i)\right)^2 = \frac{p_i q_i}{n_i'} \cdot \frac{1}{p_i^2} = \frac{q_i}{n_i' p_i}$$

and assuming $\log \hat{p}_1, \ldots\ldots, \log \hat{p}_k$ are independent,

$$Var(\log \hat{S}(\tau_k)) = \sum_{i=1}^{k} \frac{q_i}{n_i' p_i} \text{ and } \widehat{Var}(\log \hat{S}(\tau_k)) = \sum_{i=1}^{k} \frac{\hat{q}_i}{n_i' \hat{p}_i} = \sum_{i=1}^{k} \frac{d_i}{n_i'(n_i' - d_i)}$$

Now, using the delta method again:

$$\widehat{Var}(\hat{S}(\tau_k)) = \hat{S}^2(\tau_k) \sum_{i=1}^{k} \frac{d_i}{n_i'(n_i' - d_i)} \qquad (15)$$

which is called ***Greenwood's Formula.***

1.11.2 Kaplan-Meier Estimator

The product limit (PL) estimator is similar to the actuarial estimator except the lengths of the intervals I_i are variables. In fact, let $\tau_{i,}$ the right endpoint of I_i , be the ith ordered censored or uncensored observation.

Recall that we observe the pairs $(Y_1, \delta_1), \ldots\ldots, (Y_n, \delta_n)$. For now, assume no ties. Let $Y_{(1)} < Y_{(2)} < \ldots\ldots Y_{(n)}$ be the order statistics of $Y_1, Y_2, \ldots\ldots, Y_n$ and with an abuse of notation, define $\delta_{(i)}$ to be the value of δ associate with $Y_{(i)}$, i.e., $\delta_{(i)} = \delta_j$ when $Y_{(i)} = Y_j$. note that $\delta_{(1)}, \ldots\ldots, \delta_{(n)}$ are not ordered. Let R(t) denote the risk set at time t, which is the set of subjects still alive at time t, and let:

n_i = #in R $(Y_{(i)})$ = #alive at time $Y_{(i)}$,

d_i = #died at time $Y_{(i)}$,

p_i = P(surviving through I_i | alive at beginning of I_i) = $P\{T > \tau_1 | T > \tau_{i-1}\}$.

$q_i = 1 - p_i$.

From the estimates: $\hat{q}_i = \frac{d_i}{n_i}$,

$$\hat{p}_i = 1 - \hat{q}_i = \begin{cases} 1 - \frac{1}{n_i} & \text{if } \delta_{(i)} = 1 \text{ (uncensored)}, \\ 1 & \text{if } \delta_{(i)} = 0 \text{ (censored)}, \end{cases}$$

The PL estimate when no ties are present is

$$\hat{S}(t) = \prod_{y_{(i)} \le t} \hat{p}_i = \prod_{u: y_{(i)} \le t} 1 - \frac{1}{n_i}$$

$$= \prod_{y_{(i)} \le t} \left(1 - \frac{1}{n_i}\right)^{\delta_{(i)}} = \prod_{y_{(i)} \le t} \left(1 - \frac{1}{n-i+1}\right)^{\delta_{(i)}}$$

$$= \prod_{y_{(i)} \le t} \left(\frac{n-i}{n-i+1}\right)^{\delta_{(i)}}.$$

1.11.3 Log Rank Test

Let $\tau_1 \le \tau_2 \le \ldots\ldots \le \tau_g$ be ordered failure times from the combined sample; d_j = #deaths at τ_j; r_j = #at risk at τ_j; and r_{ij} = #at risk at τ_j from group i, i= 1,2.

The long rank test compares the observed and expected (under Ho). Number of deaths in group 1st, we have:

E = expected number of deaths (under Ho) in group I.

$$= \sum_{j=1}^{g} d_j \frac{r_{1j}}{r_j}$$

$$V = \sum_{j=1}^{g} d_j \frac{r_{1j}\, r_{2j}}{r_j^2}$$

O = observed number of deaths from sample I (or group I).

Then, $z = \frac{O-E}{\sqrt{V}} \sim N(0,1)$.

1.12 Summary

The unit covers the basic concepts of survival analysis, which is essential for understanding time-to-event data. Survival analysis deals with various aspects of time, particularly the time until a specific event occurs. This could be the failure of a component, the death of a patient, or the development of a disease. Key terms in survival analysis include survival time (T), distribution functions (F(t)), density functions (f(t)), survival functions (S(t)), hazard functions (h(t)), and cumulative hazard functions (R(t)). Censoring is a critical concept, where the exact survival time is not known but some information is available. Types of censoring include order censoring, random censoring, right censoring, left censoring, interval censoring, and truncation. Life tables and different methods, such as the actuarial and Kaplan-Meier estimators, are used to estimate survival functions. Additionally, the unit discusses failure rates, ageing classes, and methods for estimating survival functions, as well as the log rank test for comparing survival functions of different groups

1.13 Self-Assessment Exercise

1. Given the hazard function, $h(t) = c$. Derive the survivorship function and the probability density function.
2. Given the survivorship function, $S(t) = \exp(-t^{\gamma})$. Derive the probability density function and the hazard function.
3. Suppose that the survival distribution of a group of patients follows the exponential distribution with G = 0 (year), $\lambda = 0.65$. find:
   a) The mean survival times

   b) The median survival times

   c) The probability of surviving 1.5 years or more

4. In a compound exponential distribution, let the rate be represented by the random variable p. Prove that:
   a) $E(T)=E(1/P)$

   b) $Var(T)=2E(1/P^2)-[E(1/P)]^2$

   Check the results from the case where p has a gamma distribution.
5. For the continuous logistic density with location v and scale parameter τ, having density

$$\frac{\tau^{-1}\exp[(x-v)/\tau]}{\{1+\exp[(x-v)/\tau]\}^2}$$

   Obtain the hazard function and mean survival time.

6. Show that if the hazard function has the form: $\kappa\rho(\rho t)^{\kappa-1}\exp[(\rho t)^{\kappa}]$, the survival function is $\exp\{-[\exp((\rho t)^{\kappa})-1]\}$
7. Summarize the distributions discussed in this chapter, answering the following questions:
   a) What distributions describe constant hazard rates? Give the range of parameter values.
   b) What distributions describe increasing hazard rates? If there are more than one, discuss the differences between them.

c) What distributions describe decreasing hazard rates? If there are more than one, discuss the differences between them.

1.14 References

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1.15 Further Reading

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UNIT - 2 PARAMETRIC SURVIVAL MODELS

Structure

2.1 Introduction

Survival analysis is a specialized statistical field concerned with analyzing time-to-event data, focusing on the duration until an event of interest occurs. This section explores the application of parametric models within this framework, providing a robust approach to understanding the interplay between patient survival time, pre-defined probability distributions, and various explanatory factors. The foundation of survival analysis lies in the survival function, S(t), which quantifies the probability of an individual surviving beyond a specific time point (t). This function complements the more general cumulative density function (CDF), which captures the overall likelihood of the event occurring by time t. To gain a deeper understanding, the probability density function (PDF) is introduced, pinpointing the exact probability of the event occurring at a specific time point. Finally, the hazard function, denoted by h(t), depicts the instantaneous rate at which an event transpires at a given time point (t), taking into account the subject's survival up to that point.

The concept of life distributions within the context of reliability analysis and engineering is also explored. These distributions mathematically model the lifespan of systems or components before failure, playing a crucial role in informed decision-making. The section equips learners with an understanding of various common life distributions, including the exponential, Weibull, gamma, lognormal, Pareto, and Rayleigh distributions. Understanding and selecting the appropriate life distribution is paramount for accurate reliability assessments and informed engineering decisions. Furthermore, the unit delves into the "Linear Failure Rate" model, also referred to as the Bathtub distribution. This model is characterized by a hazard function resembling the shape of a bathtub, reflecting the changing failure rate over time. The method of estimating and testing the unknown parameters of a parametric survival model using the maximum likelihood method is also introduced. Finally, the concepts of increasing failure rate (IFR) and decreasing failure rate (DFR) distributions, with their properties relating to the behaviour of the hazard function over time, are discussed.

By offering a comprehensive exploration of these concepts, this unit empowers learners to effectively utilize parametric models in survival analysis, gaining valuable insights into the factors influencing event times and making informed decisions in various fields.

2.2 Objectives

After going through this unit, you should be able to:

- Apply parametric models to analyze time-to-event data, focusing on the relationship between survival time, probability distributions, and explanatory factors.
- Identify and interpret various life distributions used in reliability analysis, including exponential, Weibull, gamma, lognormal, Pareto, and Rayleigh distributions.
- Understand the concepts of IFR and DFR distributions and their implications for interpreting and analysing survival data

2.3 Assumptions and Characteristics

Parametric Survival Model

A parametric survival model is one in which survival time (the outcome) is assumed to follow a known distribution. Examples of distribution that are commonly used for survival time are: the Weibull, the exponential (a special case of the Weibull), the log-logistic, the lognormal, and the generalized gamma etc.

Assumptions:

1. Outcomes are assumed to follow some family of distributions.
2. Exact distribution is unknown if parameters are unknown.
3. Data used to estimate parameters

2.4 Life Distributions

Life distributions, also known as survival distributions, are a fundamental concept in parametric survival models. These models analyze data where the outcome of interest is the time until an event occurs, like a machine failure or a patient's recovery. Life distributions describe the probability of that event happening at a specific time. Parametric models assume the data follows a specific probability distribution, like the Weibull or exponential distribution. Non-parametric models make fewer assumptions about the data's form. Some frequently used life distributions in parametric models include:

2.4.1 Exponential Distribution

Several survival distributions are available for modelling survival data. The exponential distribution, the simplest survival distribution, has a constant hazard, $h(t)=\lambda$. The cumulative hazard function is given by:

$$H(t)= \int_0^t h(u)du = \int_0^t \lambda du = \lambda t|_0^t = \lambda t \tag{1}$$

The survival function is:

$$S(t)= e^{-R(t)}= e^{-\lambda t} \tag{2}$$

The probability density function is given by:

$$f(t)=h(t)S(t)=\lambda e^{-\lambda t} \tag{3}$$

The mean of an exponential random variable is given by: $E(T) = \int_0^{\infty} S(t)\,dt = \int_0^{\infty} e^{-\lambda t}\,dt = \frac{1}{\lambda}$ and $Var(T) = \frac{1}{\lambda^2}$. The median is the value of t that satisfies $0.5 = e^{-\lambda t}$, so that $t_{med} = \frac{\log(2)}{\lambda}$.

The exponential distribution is easy to work with, but the constant hazard assumption is not often appropriate for describing the lifetimes of humans or animals. The Weibull distribution, which offers more flexibility in modelling survival data, has hazard function:

$$h(t)=\alpha\lambda(\lambda t)^{\alpha-1}=\alpha\lambda^{\alpha}t^{\alpha-1} \tag{4}$$

The cumulative hazard and survival functions are given by, respectively, $H(t)=(\lambda t)^{\alpha}$ and $S(t) = e^{-(\lambda t)^{\alpha}}$. The exponential distribution is a special case with $\alpha=1$. It is monotone increasing for $\alpha>1$ and monotone decreasing for $\alpha<1$.

2.4.2 Gamma Distribution

The gamma distribution (not to be confused with the gamma function) provides yet another choice for survival modelling. The probability density function is given by:

$$f(t) = \frac{\lambda^{\beta} t^{\beta-1} \exp(-\lambda t)}{\Gamma(\beta)} \tag{5}$$

It is monotone increasing for $\beta>1$ and monotone decreasing for $\beta<1$. When $\beta=1$, the gamma distribution reduces to an exponential distribution. Then:

$$E(T) = \frac{\alpha}{\lambda} \text{ and } Var(T) = \frac{\alpha}{\lambda^2} .$$

Unfortunately, the gamma model does not have closed form expression for $S(t)$ and $\lambda(t)$:

$$S(t) = 1 - \int_0^t f(u)du = 1 - \left(\frac{\text{incomplete gamma function}}{\text{complete gamma function}}\right) \tag{6}$$

2.4.3 Weibull Distribution

The Weibull distribution is another generalization of the exponential distribution that is appropriate for modelling the lifetimes having constant, strictly increasing or strictly decreasing hazard functions, it is given by the distribution function $F(t)=1-e^{-\lambda t}, t>0, \lambda>0$

The survival functions:

$$S(t)=e^{(-\lambda t)^{\alpha}} \quad , \qquad \alpha>0 \; , \; \lambda>0 \tag{7}$$

Then,

$$\int_0^t h(u)du = (\lambda t)^{\alpha} \; ,$$

$h(t)=\alpha\lambda(\lambda t)^{\alpha-1}$ and

$f(t)=h(t)\; S(t)=\alpha\lambda(\lambda t)^{\alpha-1}\; e^{-(\lambda t)^{\alpha}}$

For the Weibull model, E(T) and Var(T) have no nice closed form expression, but the forms of λ(t) and S(t) make the Weibull model a useful one in survival analysis.

The mean and median of the Weibull distribution are, respectively, $E(T)=\frac{\Gamma\left(1+\frac{1}{\alpha}\right)}{\lambda}$ and $t_{med}=\frac{[\log(2)]^{\frac{1}{\alpha}}}{\lambda}$.

For integers, the gamma function is given by $\Gamma(n)=(n-1)!$. For the special case $\alpha=1$, of course, the mean and median are identical to those of the exponential distribution. For non-integers, it must be evaluated numerically.

2.4.4 Lognormal Distribution

A random variable T is said to have lognormal distribution when $y=\log_e(T)$ is distributed as normal (Gaussian) with mean μ and variance σ^2.

The d.f. and survival function of lognormal distribution, respectively are:

$$f(t)=\frac{1}{t\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2}\left(\log_e(t)-\mu\right)^2\right), t>0, \sigma>0 \tag{8}$$

$$\overline{F(t)}=\frac{1}{\sigma\sqrt{2\pi}}\int_t^{\infty}\frac{1}{x}\left[-\frac{1}{2\sigma^2}\left(\log_e(x)-\mu\right)^2\right]dx \tag{9}$$

It may be noted that μ and σ^2 which are the location and scale parameters of the normal distribution of Y are scale and shape parameter respectively for the lifetime distribution of T.

The mean and the variance of the distribution are given by $E(T)=\exp[\mu+\sigma^2]$ and $Var(T)=[e^{2\mu+\sigma^2}][e^{\sigma^2}-1]$ respectively. The density curve is positively skew and the skewness increases with σ^2. There are no closed form expressions for survival and hazard function.

The survival function $\overline{F(t)}=p\left[z>\frac{\log_e(t)-\mu}{\sigma}\right]$, where $Z\sim N(0,1)$. Thus:

$$\overline{F(t)}=1-\phi\left(\frac{\log_e(t)-\mu}{\sigma}\right) \tag{10}$$

Where $\phi(.)$ represents distribution function of standard normal variate. So, using the table of the cumulative probability integral for Z, one can evaluate the survival function of T. Similarly, using the table of ordinates of standard normal distribution we can compute f(t) and thus we can get values for hazard function, $h(t)=\frac{f(t)}{\overline{F(t)}}$.

The hazard function is non-monotonic; initially it increases, reaches a maximum and then decreases to zero as time approaches infinity. The log normal distribution may be convenient for use with uncensored data. A log transformation converts the data into the standard linear model setup.

2.4.5 Pareto Distribution

The probability density function of the pareto distribution is:

$$f(t)=\frac{\alpha\, a^{\alpha}}{t^{\alpha+1}} I_{[a,\infty)}(t) \tag{11}$$

Then the survival function of the pareto distribution is:

$$S(t)=\left(\frac{a}{t}\right)^{\alpha} I_{[a,\infty)}(t)\ , \quad \alpha>0\ , a>0 \tag{12}$$

Then,

$$h(t)=\frac{\alpha}{t} I_{[a,\infty)}(t) \tag{13}$$

The moments are easily calculated, but they may be infinite.

2.4.6 Rayleigh Distribution

The probability density function of the Rayleigh Distribution is given by:

$$f(t)=(\lambda_0+\lambda_1 t)e^{-\lambda_0 t-\frac{1}{2}\lambda_1 t^2} \tag{14}$$

Suppose, $\lambda(t)=\lambda_0+\lambda_1 t$

$$\int_0^t \lambda(u)du=\lambda_0\, t+\frac{1}{2}\lambda_1\, t^2$$

Then, $S(t)=e^{-\lambda_0 t-\frac{1}{2}\lambda_1 t^2}$,

The moments have no closed form expressions. The linear risk can be generalized to polynomials:

$$\lambda(t)=\sum_{i=1}^{p}\lambda_i t^i \tag{15}$$

2.4.7 Piece-Wise Exponential Distribution

In a piecewise exponential distribution, the hazard rate is constant on specified confidence intervals. Suppose a three piecewise exponential random variable with cut points $0<T_1<T_2<T_3=\infty$ has hazard function*:*

$$h(t)=\lambda_1 I[0\le t<T_1]+\lambda_2 I[T_1\le t<T_2]+\lambda_3 I[T_2\le t] \tag{16}$$

based on a sample of n i.i.d. observations from a piecewise exponential distribution, the MLE of hazard of the i^{th} piece (for the interval $[T_{i-1},T_i)$) is:

$$\hat{\lambda}_i=\frac{\#\{x_j\in[T_{i-1},T_i)\}}{\sum_{j=1}^{n}[\min(T_i,x_j)-T_{i-1}]^+} \tag{17}$$

where $[t]^+=\max(0, t)$.

2.5 Linear Failure Rate

It is given by:

$$F(x)=1-\exp\left\{-\left(x+\frac{1}{2}\theta x^2\right)\right\},\quad x>0,\quad \theta>0$$

$f(x)=(1+\theta x)e^{-\left(x+\frac{1}{2}\theta x^2\right)}$ and

$$r(x)=\frac{f(x)}{\overline{F}(x)}=(1+\theta x)$$

This too is a generalization of exponential distribution as θ=0 gives the exponential distribution with failure rate 1. This distribution is a suitable model for items which exhibit positive ageing and has particularly simple formula for the failure rate.

2.6 Parametric Inference

Parametric estimation is a fundamental technique in survival analysis that assumes the survival times of subjects follow a specific probability distribution. This approach contrasts with non-parametric methods, which make no distributional assumptions and can yield smoother survival curves. Parametric methods also allow for the inclusion of covariates to examine their effects on survival outcomes. This facilitates the efficient estimation of model parameters, typically achieved through maximum likelihood estimation. In parametric estimation, researchers select a probability distribution that best fits the expected shape of the survival curve, such as the Weibull distribution for a monotonic hazard, the exponential distribution for a constant hazard, or the log-normal distribution for a flexible hazard. Subsequently, they estimate the unknown parameters of the chosen distribution using the available data, enabling them to calculate the probability of survival or hazard at any specific time point.

2.6.1 LR and MLE Tests

Maximum Likelihood Estimation

Let $T_1.T_2....T_n$ be a random sample from a life distribution having probability density $f\left(x;\underline{\theta}\right)$ where $\underline{\theta}=(\theta_1,\theta_2,\ldots,\theta_p)\in\Theta$ is the vector of unknown parameters. Since the lifetimes are independent, the likelihood function $L\left(\underline{t},\underline{\theta}\right)$, is the product of probability density functions evaluated at each sample points. Thus:

$$L(\underline{t},\underline{\theta})=\prod_{i=1}^{n} f(t_i,\underline{\theta}) \tag{18}$$

Where $\frac{t}{n}=(t_1,t_2,\ldots,t_n)$ is the data point. The maximum likelihood estimator $\underline{\hat{\theta}}$ is the value of $\underline{\theta}$ which maximizes $L(\underline{t},\underline{\theta})$ for fixed $\underline{t}$. That is $\underline{\hat{\theta}}$ is the maximum likelihood estimator of $\underline{\theta}$. One may say that $f_{\hat{\underline{\theta}}}(t)$ corresponds to the distribution that is most likely to have produced the data $t_1.t_2\ldots\ldots t_n$ in the family $\{f_{\underline{\theta}}, \underline{\theta} \in \Theta\}$.

Since $L(\underline{t},\underline{\theta})$ is a joint density function, it must integrate over the range of $\underline{t}$ to one. Therefore,

$$\iiint_0^\infty L(\underline{t},\underline{\theta})d\underline{t} = 1 \tag{19}$$

Under regularity conditions which allow interchange of differentiation and integration operations, the partial derivatives of the left side with respect to one on the parameter, θ_i yields

$$\frac{\delta}{\delta\theta_i}\iiint_0^\infty L(\underline{t},\underline{\theta})d\underline{t} = \iiint_0^\infty \frac{\delta}{\delta\theta_i}\log L(\underline{t},\underline{\theta})L(\underline{t},\underline{\theta})d\underline{t}$$

$$=E\left[\frac{\delta}{\delta\theta_i}\log L(\underline{t},\underline{\theta})\right]=E[U_i(\underline{\theta})], i=1,2,\ldots,p \tag{20}$$

Where $U(\underline{\theta})=(U_1(\underline{\theta}),\ldots,U_p(\underline{\theta}))'$ is often called the score vector. The argument $\underline{t}$ is suppressed for compactness. Differentiating the right side of (20) with respect to $\underline{\theta}$, we get:

$$E\left[U_i\left(\underline{\theta}\right)\right]=0, i=1,2\ldots p \tag{21}$$

Further differentiation of (21) with respect to θ_j yields:

$$E\left[U_i\left(\underline{\theta}\right)U_j\left(\underline{\theta}\right)\right]=E\left[\frac{-\delta^2\log L\left(\underline{t},\underline{\theta}\right)}{\delta\theta_i\delta\theta_j}\right] \begin{matrix} i=1,2,\ldots,p \\ j=1,2,\ldots,p \end{matrix} \tag{22}$$

From equations (21) and (22) it follows that:

$$E\left[\frac{-\delta^2\log l(\underline{t},\underline{\theta})}{\delta\theta_i\delta\theta_j}\right]=\mathrm{cov}(U_i(\underline{\theta}),U_j(\underline{\theta})), \begin{matrix} i=1,2,\ldots,p \\ j=1,2,\ldots,p \end{matrix} \tag{23}$$

These elements form the p×p Fisher information matrix, $I(\underline{\theta})$, whose diagonal elements are the variances and the off-diagonal elements are the covariances of the score vector. The solutions of the simultaneous likelihood equations: $U_i(\underline{\theta})=\frac{\delta}{\delta\theta_i}\log L(\underline{t},\underline{\theta})=0$, are $\hat{\theta}_i$, the maximum likelihood estimators of θ_i, $i=1,2\ldots..p$.

The estimators $\hat{\theta}_1,\ldots,\hat{\theta}_p$, under certain regularity conditions are asymptotically normally distributed with mean $\theta_1,\ldots,\theta_p$ and variance covariance matrix given by $V(\hat{\underline{\theta}})=\{I(\underline{\theta})\}^{-1}$.

The observed (sample) information matrix called $i(\underline{\theta})$ is denoted by the elements:

$$\left(-\frac{\delta^2}{\delta\theta_i\delta\theta_j}\log L(\underline{t},\underline{\theta}),i,j=1,\ldots,p\right) \qquad (24)$$

So that $E[i(\underline{\theta})] = I(\underline{\theta})$.

Three broad types of asymptotic procedures, based on the likelihood function, are available for testing of hypothesis $\underline{\theta}=\underline{\theta}_0$.

1. Wilks Likelihood Ratio

Let $L(\hat{\underline{\theta}})=[L(\underline{\theta})]_{\underline{\theta}=\hat{\underline{\theta}}}$.

$$-2\log\frac{L(\underline{\theta}_0)}{L(\hat{\underline{\theta}})}\xrightarrow{a}\chi^2_p \text{ under } H_0,$$

Where $\xrightarrow{a}$ denotes "*asymptotically distributed as*"

2. Wald's method based on MLE's

$$(\hat{\underline{\theta}}-\underline{\theta}_0)'\, I(\underline{\theta}_0)(\hat{\underline{\theta}}-\underline{\theta}_0)\xrightarrow{a}\chi^2_p \text{ under } H_0.$$

3. Rao Scores Method

$$\left[\frac{\delta}{\delta\underline{\theta}}\log L(\underline{\theta}_0)\right]' I^{-1}(\underline{\theta}_0)\left[\frac{\delta}{\delta\underline{\theta}}\log L(\underline{\theta}_0)\right]\xrightarrow{a}\chi^2_p$$

Rao's method does not use the MLE and hence is recommend in practise if only in hypothesis testing. However, in addition to tests, we usually want estimates and confidence intervals, so we need to compute $\hat{\underline{\theta}}$ anyway. Once we have $\hat{\underline{\theta}}$ and $I(\underline{\theta}_0)$, the Wald method is easy.

2.6.2 Point Estimation and Scores

Following are the two commonly used methods for obtaining MLEs when closed form solutions are not possible.

1. **Newton-Raphson Method of Scoring:**

Assume $\hat{\theta}^{(0)}=(\hat{\theta}_1^{(0)},\ldots,\hat{\theta}_p^{(0)})'$ is an initial guess at the solution.

Then $\underline{\hat{\theta}}^{(1)}=\underline{\hat{\theta}}^{(0)}+(i(\underline{\theta}^{(0)}))^{-1}\frac{\delta}{\delta\theta}\log L(\underline{\theta}^{(0)})$,

Where $i(\underline{\theta}^{(0)})=[i(\underline{\theta})]_{\underline{\theta}=\underline{\theta}^{(0)}}$

And $L(\underline{\theta}^{(0)})=L[(\underline{\theta})]_{\underline{\theta}=\underline{\theta}^{(0)}}$

In general,

$$\underline{\hat{\theta}}^{(j+1)}=\underline{\hat{\theta}}^{(j)}+(i(\underline{\theta}^{(j)}))^{-1}\frac{\delta}{\delta\underline{\theta}}\log L(\underline{\theta}^{(j)}),j=1,2,\ldots \qquad (25)$$

2. **Fisher's Method of Scoring**

Replacing sample information matrix $i(\underline{\theta})$ in equation 1 by Fisher's information matrix we get following iterative formula for Fisher's method:

$$\underline{\hat{\theta}}^{(j+1)}=\underline{\hat{\theta}}^{(j)}+(I(\underline{\theta}^{(j)}))^{-1}\frac{\delta}{\delta\underline{\theta}}\log L(\underline{\theta}^{(j)}),j=1,2,\ldots \qquad (26)$$

Fisher's method of scoring produces improved convergence in some instances. However, in many situations, particularly if censoring is present, $I(\underline{\theta})$ is not mathematically tractable. Hence the Newton-Raphson method is used.

2.6.3 Confidence Intervals

To construct confidence intervals and perform test, we need the distribution of $\hat{\lambda}$

a) If no censoring is present

$\hat{\lambda}=\frac{n_u}{\sum_{i=1}^{n} T_i}=\frac{1}{\bar{T}},$

Where $T_1,\ldots\ldots T_n$ are iid each with the exponential distribution

$f_{(T_1)}=\lambda e^{(-\lambda t)}$

Consequently, $\sum_{i=1}^{n} T_i$ h has the gamma density

$$f_s(t)=\frac{\lambda^n}{\Gamma(n)}t^{(n-1)}e^{(-\lambda t)}$$

So $2\lambda\sum_{i=1}^{n} T_i \sim\chi^2_{(2n)}$ or equivalently

$\frac{2n\lambda}{\hat{\lambda}}\sim\chi^2_{(2n)}$

Therefore, $\frac{2n\lambda}{\hat{\lambda}}$ is a pivotal statistic and can be used for test and confidence interval construction.

b) For Type II Censoring

We can rewrite

$$\sum_{i=1}^{n} Y_i = T(1)+T(2)+\ldots+T(r)+(n-r)T(r)$$

$$= nT(1)+(n-1)[T(2)-T(1)]+\ldots+(n-r+1)[T(r)-T(r-1)]$$

Using the results about Poisson processes and exponential waiting times,

$T(1)=\{\text{min of n iid exponential}(\lambda)\text{ r.v.'s}\}\sim n\lambda e^{-n\lambda t}$

$nT(1)\sim\lambda e^{-\lambda t}$

$T(2)-T(1)=\{\text{min of n-1 exponential}(\lambda)\text{ r.v.'s}\}\sim(n-1)\lambda e^{-(n-11)\lambda t}$

$(n-1)[T_2-T_1] \sim \lambda e^{-\lambda t}$,etc.,

and $(n[T_1],(n-1)[T_2-T_1],\ldots,(n-r+1)[T_r-T_{r-1}]$ are independent, so;

$$2\lambda \sum_{i=1}^{n} T_i \sim \chi^2_{(2r)}$$

Thus, $2r\lambda/\hat{\lambda}$ can be used in conjunction with a χ^2 distribution, where the d.f. are twice the number of uncensored ordered statistics, to construct confidence intervals and tests.

c) For Type I Censoring

If random or type I censoring is present, we have no recourse but to use the asymptotic theory.

Thus, $\hat{\lambda}=\frac{n_u}{\sum_{i=1}^{n} y_i}$,

$$\frac{\partial^2}{\partial\lambda^2}\log L=\frac{-n_u}{\lambda^2},$$

So,

$$\frac{\hat{\lambda}-\lambda}{\sqrt{\frac{\lambda^2}{n_u}}} \overset{a}{\sim} N(0,1)$$

Where n_u may be replaced by $E(n_u)$ if the latter is available.

The normality approximation can be improved by transforming the estimate. By the delta method, since, $\hat{\lambda} \sim (\lambda, \frac{\lambda^2}{n_u})$. Then; $\log\hat{\lambda} \sim (\log\lambda, \frac{1}{n_u})$.

Notice that $\frac{1}{n_u}$, the asymptotic variance of $\log\hat{\lambda}$, does not depend on the unknown parameter λ. It is an empirical fact that transforming an estimate to remove the dependence of the variance on the unknown parameter tends to improve the convergence to normality by reducing the skewness.

Example 1: Let $t_1.t_2 \ldots\ldots t_n$ be a random sample from an exponential distribution with parameter λ.

$$f(t;\lambda)=\lambda e^{-\lambda t}, t\geq 0; \lambda>0$$

$$L(\underline{t};\lambda)=\prod_{i=1}^{n} \lambda e^{-\lambda t_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} t_i}$$

The log likelihood function is: $\log L(\underline{t},\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} t_i$

The score is $U(\lambda)=\frac{\delta}{\delta\lambda}\log L\left(\underline{t},\lambda\right)=\frac{n}{\lambda}-\sum_1^n t_i$

$$\left[\frac{\delta}{\delta\lambda}\log L(\underline{t},\lambda)\right]_{\lambda=\hat{\lambda}} = 0 \Rightarrow \hat{\lambda}=\frac{n}{\sum_1^n t_i}$$

$$I(\lambda)=\frac{n}{\lambda^2}.$$

Sample information at $\hat{\lambda}$ is $\frac{n}{\hat{\lambda}^2}$ and $var(\hat{\lambda})$ is $\frac{\hat{\lambda}^2}{n}$.

The maximum likelihood estimator of λ is the ratio of total number of failures to the total lifetime of all the units, i.e., the total time on test. If μ is the mean of the distribution then its maximum likelihood estimator (MLE) is $1/\hat{\lambda}$ which is also the method of moment estimator of μ. We know that $\sum_1^n T_i$ is minimal sufficient statistic. $\overline{T}$ is consistent for μ and $\frac{1}{\overline{T}}$ is a consistent estimator of λ. The asymptotic distribution of $\hat{\lambda}$ is normal with mean λ and variance $\frac{\lambda^2}{n}$. So that,

$$\frac{\sqrt{n}(\hat{\lambda}-\lambda)}{\lambda} \xrightarrow{a} N(0,1).$$

The exact distribution of $\hat{\mu}=\frac{1}{\hat{\lambda}}$ can be derived using following result:

$\sum_i^n T_i$ is the sum of n independent exponential random variables, hence it has gamma distribution and therefore $\frac{2n\overline{T}}{\mu}=\frac{2n\hat{\mu}}{\mu}$ has χ^2_{2n} distribution. Equivalently $\frac{2n\lambda}{\hat{\lambda}}$ has χ^2_{2n} distribution. From the above result we have:

$$E\left[\frac{2n\hat{\mu}}{\mu}\right]=2n \Rightarrow E(\hat{\mu})=\mu$$

Exact confidence interval for λ is obtained by using the pivotal quantity $\frac{2n\lambda}{\hat{\lambda}}$. Let (1-α) be the confidence coefficient and $\chi^2_{\alpha/2,2n}$ and $\chi^2_{1-\alpha/2,2n}$ be such that:

$$P[\chi^2_{2n} \leq \chi^2_{\alpha/2,2n}]=P[\chi^2_{2n} \geq \chi^2_{1-\alpha/2,2n}]=\alpha/2.$$

Then 100(1-α)% equal tailed confidence interval for λ is obtained from:

$$(\chi^2_{\alpha/2,2n} \leq \frac{2n\lambda}{\hat{\lambda}} \leq \chi^2_{1-\alpha/2,2n})=1-\alpha.$$

The required confidence interval is $(\frac{\hat{\lambda}}{2n}\chi^2_{\alpha/2,2n}; \frac{\hat{\lambda}}{2n}\chi^2_{1-\alpha/2,2n})$

Example 2: Let x~exp(θ), then, $f(x)=\frac{1}{\theta}e^{-x/\theta};x,0>0.$

S(t) = P(x>t)

$$=\int_t^\infty f(x)dx$$

$$=\int_t^\infty \frac{1}{e}e^{-x/\theta}dx=e^{-t/\theta}$$

$S(t)=e^{-t/0}$

Maximum Likelihood Estimate: $\hat{\theta}=\bar{x}=\frac{1}{n}\sum_{i=1}^n x_i$

$\hat{S}(t)=e^{-t/\bar{x}}$ (Due to the invariance property)

UMVUE of S(t)

Let us consider the statistic $T(x_1)$

$$T(x_1)=\begin{cases}1 & \text{if } x_1>t\\ 0 & ;\text{ otherneise}\end{cases}$$

Then,

$E[T(x_1)] \quad = 1 \cdot P(x_1 > t).$

$\qquad\qquad = e^{-t/\theta}$

So, $T(x_1)$ is an Unbiased Estimation of survival function.

We know that sample mean $\bar{x}$ is complete and sufficient.

UMVUE of S(t) is $E[\,T(x_1) \mid \bar{x}\,]$ (Lehmann–Scheffé Theorem)

So, to find $f(x_1|\bar{x})$:

Let n numbers of units are put to test. Life time of 1st m units are observed. We call if type II censored (test is terminated after termination of m unit).is called type II censoring failure censored data. So, we get order statistics: $x_{(1)}<x_{(2)}<\cdots<x_{(m)}$. When n units were put on testing, we intend to estimate S(t).

To estimate survival function using EM algorithm, we x follow Weibull distribution under failure censoring:

$$L=\binom{n}{m}\cdot\prod_{i=1}^{m} f(x_i)\cdot[1-f(x_{(m)})]^{n-m}.$$

$$=\binom{n}{m}\cdot\frac{1}{\theta^m}\cdot e^{-\frac{1}{\theta}\Sigma x_{(i)}}\left[e^{-\frac{x(m)}{\theta}}\right]^{n-m}$$

$$=\binom{n}{m}\cdot\frac{1}{\theta^m}\cdot e^{-\frac{1}{\theta}[\Sigma_{i=1}^{m} x(i)+(n-m)x(m)]}$$

Then, $\hat{\theta}=\frac{1}{m}[\sum_{i=1}^{m} x_{(i)}+(n-m)\times(m)]$, which is MLE.

Let, $z_i=(n-i+1)[x_{(i)}-x_{(i-1)}]$ and $x_0=0$.

$z_1=nx_{(1)}$

$z_2=(n-1)[x_{(2)}-x_{(1)}]$

………………

$z_m=(n-m+1)[x_{(m)}-x_{(m-1)}]$

$$\sum_{i=1}^{m} z_i = \sum_{i=1}^{m} x_{(i)} +(n-m)x_{(m)}$$

$$\frac{1}{m}\sum_{i=1}^{m} z_i = \frac{1}{m}[\sum_{i=1}^{m} x_{(i)} +(n-m)x_{(m)}]$$

$$\hat{\theta}=\frac{1}{m}\sum_{i=1}^{m} z_i$$

We know that here z_i's are iid exponential (θ).

Example 3: For the given probability density function of Weibull distribution**:**

$f(x)=\frac{p}{\sigma}x^{r-1}e^{-x^p/\sigma};\sigma,p,x>0$; where σ=scale parameter, p=shape parameter

$$S(t)=\int_t^{\infty} f(x)dx = \int_t^{\infty} \frac{p}{\sigma}x^{p-1}e^{-x/\sigma}dx =e^{-t^p/\sigma}$$

$$h(t)=\frac{f(t)}{S(t)}=\frac{p/\sigma t^{p-1}e^{-t^p/\sigma}}{e^{-t^p/\sigma}}=\frac{p}{\sigma}t^{p-1}.$$

MLE estimation of parameters: Suppose 'n' units are put to test and test is terminated after termination of 1st r unit. Then the likelihood is given by:

$$L(\sigma,p|\underline{x}) = c\cdot \prod_{i=1}^{r} f(x_{(i)}) [1-F(x_r)]^{n-r}$$

Where, $c=\frac{n!}{(n-r)!}$

$$=\prod_{i=1}^{r}\frac{p}{\sigma}\cdot x_i^{p-1}\cdot e^{-\frac{x_{(i)}^p}{\sigma}}\cdot \left[e^{-\left(\frac{x_{(r)}^p}{\sigma}\right)}\right]^{n-r}$$

$$=(\frac{p}{\sigma})^r(\prod_{i=1}^{r} x_i^{p-1})\cdot e^{-\frac{1}{\sigma}[\sum_{i,1}^{r} x_{(i)}^p+(n-r)x_{(r)}^p]}$$

$$\log L = r\log(p) - r\log(\sigma) + (p-1)\sum_{i=1}^{r} \log x_i - \frac{1}{\sigma}[\sum x_{(i)}^p+(n-r)x_{(r)}^p].$$

Differentiating with respect to σ:

$$\frac{\partial \log L}{\partial \sigma} = -\frac{r}{\sigma} + \frac{1}{\sigma^2}\left[\sum x_{(i)}^{b} + (n-r)x_{(r)}^{b}\right]$$

$$\Rightarrow \hat{\sigma} = \frac{1}{r}\left[\sum_{i=1}^{r} x_{(i)}^{b} + (n-r)x_{(r)}^{b}\right]$$

$$\frac{\partial \log L}{\partial p} = \frac{r}{p} + \sum_{i=1}^{r} \log x_i - \frac{1}{\sigma}\left[\sum x_i^{p} \log x_{(i)} + (n-r)x_r^{\cdot p} \log x_r\right]$$

$$\hat{p} \rightarrow r\left[\frac{1}{\sigma}\left[\Sigma x_{(i)}^{p} \log x_{(i)} + (n-r)x_{(r)}^{p}\right] - \Sigma \log x_i\right]$$

These likelihood equations are transcendental equation. And can be solve by using any appropriate iterative procedure.

$$E(x) = \frac{1}{p}\Gamma\frac{1}{p}.\sigma^{\frac{1}{p}}$$

Example 4: **Gamma Distribution:** Let $x_1, x_2 \ldots x_n$ be the complete sample

$$L = \prod_{i=1}^{n} f(x_i) = \frac{1}{\sigma^n(\sqrt{p})^n} \cdot \prod_{i=1}^{n} x_i^{p-1}\, e^{-\Sigma x_i/\sigma}$$

$$\log L = -n\, p \log \sigma - n \log \sqrt{p}_i + (p-1)\sum \log x_i - \sum x_i/\sigma$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{-np}{\sigma} + \frac{\cdot\sum x_i}{\sigma^2} = 0$$

$$\Rightarrow \sigma = \frac{\sum x_i}{n_p} = \frac{\overline{x}}{\overline{p}}$$

$$\frac{\partial \log L}{\partial p} = -n \log(\sigma) - n\frac{\partial \log \sqrt{p}}{\partial p} + \sum_{i=1}^{n} \log x_i$$

Substituting the value of σ in the above equation, we get:

$$\frac{\partial \log L}{\partial p} = -n \log \frac{\overline{x}}{p} - n \frac{\partial \log \Gamma_p}{\partial p} + \sum_{i=1}^{n} \log x_i = 0$$

$$= \log\left(\frac{\overline{x}}{p}\right) - \frac{\partial \log \sqrt{p}}{\partial p} - \frac{1}{n} \sum \log x_i = 0$$

After getting the values of p by Newton Raphson method, we get the value of σ using invariance property of MLE, we get the ML estimation for Survival function S(t).

2.7 Estimation under the assumption of IFR/DFR

Lognormal Distribution

If log x ~ N(μ, σ^2), then x~ lognormal (μ,σ^2).

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-1/2\sigma^2 (\log x - \mu)^2}.$$

$$= \frac{1}{\sigma x} \phi\left(\frac{\log x - \mu}{\sigma}\right).$$

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right)$$

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{1}{\sigma x} \cdot \frac{\phi\left(\frac{\log x - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{\log x - \mu}{\sigma}\right)}$$

In this distribution, maximum portion DFR then IFR, inverted Bathtub, Hazard is non monotonic.

Maximum likelihood Estimation:

If x ~ lognormal (μ,σ^2), then y=log x ~ N (μ, σ^2)

$$\hat{\mu} = \overline{y} = \frac{1}{n} \sum_{i=1}^{n} y_i = \frac{1}{n} \sum_{i=1}^{n} \log x_i$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y})^2$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\log x_i - \frac{1}{n}\sum_{i=1}^{n}\log x_i\right)^2$$

Thus, α = mean = $\exp(\mu+\frac{\sigma^2}{2})$ and β =variance $=(e^{\sigma^2}-1)(e^{2\mu+\sigma^2})$

Normal Distribution

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}; -\infty<x,\mu<\infty \text{ and } \sigma>0$$

If mean is very large than σ, then the distribution can be used as a life time distribution

$$f(x)=\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma}).$$

$$F(x)=\bar{\phi}\left(\frac{x-\mu}{\sigma}\right)$$

The Hazard Rate is given as:

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})}{\bar{\phi}(\frac{x-\mu}{\sigma})}.$$

h(x) is increasing with x so, normal distribution in this case follows IFR distribution.

Pareto Family

Simple (one parameter) form of this distribution is given by:

$$F(x)=1-(1+\theta x)^{-\frac{1}{\theta}}, x>0, \theta>0$$

$$f(x)=(1+\theta x)^{-(1/\theta+1)}, x>0, \theta>0$$

$$r(x)=(1+\theta x)^{-1}, x>0, \theta>0.$$

It is a family of DFR distributions.

The two-parameter version of this distribution known as "Pareto distribution of second kind or Lomax distribution arises as a compound exponential distribution when the parameter of the exponential distribution, is itself distributed as a gamma variate.

$$P[X \leq x \mid \theta]=1-e^{-x/\theta}, x>0, \theta>0$$

And $\mu=1/\theta$ has a gamma distribution. Then:

$F(x)=P[X \leq x]$

$$=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{\infty} t^{(\alpha-1)}e^{-t/\beta}(1-e^{-tx})dt$$

$$=1-(\beta x+1)^{-\alpha}, \alpha,\beta>0; x>0,$$

$f(x)=\beta\alpha(\beta x+1)^{-(\alpha+1)}, \alpha,\beta>0; x>0$

And $r(x)=\frac{\alpha\beta}{(\beta x+1)}, \alpha,\beta>0,\ x>0$

Observe that $r(x)\downarrow x$. Hence this distribution also belongs to the DFR class.

2.8 Summary

This unit covers the application of parametric survival models to analyze time-to-event data. It explores key concepts like the survival function, hazard function, and various life distributions used in reliability analysis. The module equips learners with the skills to estimate model parameters, understand increasing/decreasing failure rates, and make informed decisions based on survival analysis.

2.9 Self-Assessment Exercise

1. Suppose pain relief time follows the gamma distribution with $\lambda=1$, $\gamma=0.5$. Find:
   a) The hazard functions
   b) The Maximum likelihood estimation
2. Suppose that the survival distribution is (1) Gompertz and (2) linear- exponential, and $\lambda=1$, $\gamma=2.0$. Plot the hazard function and find:

a) The mean
b) The probability of surviving longer than 1 unit of time

3. Consider a survival distribution with constant hazard λ=0.07 from t=0 until t=5 and then hazard λ=0.14 for $t > 5$. (This is known as a piecewise constant hazard.) Plot this hazard function and the corresponding survival function for $0 < t < 10$. What is the median survival time?
4. Following are the times (in minutes) to break down of an insulating fluid between electrodes recorded at voltage 36kv. Assume Weibull distribution and estimate the parameters of the distribution .35, .59, .96, .99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77 with and without censoring.
5. Consider the following remission times in weeks for 21 patients with acute leukaemia: 1, 1, 2, 2, 3, 4, 4, 5, 5, 6, 8, 8, 9, 10, 10, 12, 14, 16, 20, 24, and 34. Assume that remission duration follows the exponential distribution. Estimate the parameter λ. Also obtain the Maximum likelihood estimate and confidence interval for mean.
6. Consider the following tumor-free times in days of 10 animals: 2, 3.5, 5,7, 9, 10, 15, 20, 30, and 40. Assume that the tumor-free times follow the log-logistic distribution. Estimate the parameters and α and γ.
7. In a study of deep venous thrombosis, the following blood clot lysis times in hours were recorded from 20 patients: 2, 3, 4, 5.5, 9, 13, 16.5, 17.5, 12.5, 7, 6, 17.5, 11.5, 6, 14, 25, 49, 37.5, 49, and 28. Assume that the blood clot lysis times follow the lognormal distribution.
   a) Obtain MLEs of the parameters μ and σ^2.
   b) Obtain 95% confidence intervals for μ and σ^2.

2.10 References

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UNIT - 3 NON-PARAMETRIC SURVIVAL MODELS

Structure

3.1 Introduction

Survival analysis, or reliability analysis, examines the time until an event occurs, such as failure. This module introduces non-parametric methods in survival analysis, avoiding assumptions about survival time distributions. It covers the hazard function, describing the instantaneous failure risk, and the exponential distribution's role in inference, especially with censored data. However, the exponential distribution's memoryless property limits its use in scenarios where aging affects failure rates. The module introduces alternatives like the Total Time on Test (TTT) transform for assessing the increasing failure rate (IFR) property, indicating higher failure likelihood with age. It also explores U-statistics, useful for non-parametric tests and estimators, including the Deshpande test for checking exponential distribution fit in data, crucial for reliability modelling. The “Two Sample Problem” in survival analysis refers to comparing two independent samples to test whether there is a difference in their survival distributions. The following two sample tests are discussed in this self-learning module - the Gehan Test, Log Rank Test, Mantel-Haenszel Test, and Tarone-Ware Test. The Gehan Test extends the Wilcoxon rank-sum test to handle censored data. It compares the ranks of survival times across two groups, providing a robust option when dealing with censored observations. The Log Rank Test is a widely used method that assesses the null hypothesis of no difference in survival times between groups. By comparing observed and expected event occurrences at each observed event time, it provides a powerful test statistic. The Mantel-Haenszel Test, while not exclusive to survival analysis, offers valuable insights in this context. It allows incorporating additional factors (strata) into the analysis, helping to control for potential confounding variables and assess the association between exposure and survival outcomes across different groups. The Tarone-Ware Test presents itself as an intermediate option between the Gehan and Mantel-Haenszel tests. It assigns weights to observations, accounting for potential variations in the underlying hazard functions, making it particularly useful when the proportional hazards assumption might not hold true. By working through this module, you will gain a comprehensive understanding of non-parametric methods in survival analysis, equipping you to analyse time-to-event data and draw meaningful conclusions from studies investigating the duration until an event of interest occurs.

3.2 Objectives

After going through this unit, you should be able to:

- Explain the exponential distribution and its significance in survival analysis, particularly in dealing with censored data.
- Apply alternative tools such as the Total Time on Test (TTT) transform and U-statistics to assess the increasing failure rate (IFR) property and construct non-parametric tests and estimators, including the Deshpande test for checking exponential distribution fit in data.
- Understand the "Two Sample Problem" in survival analysis and apply the Gehan Test, Log Rank Test, Mantel-Haenszel Test, and Tarone-Ware Test to compare survival distributions between two independent samples, considering their advantages and applications.

3.3 Assumptions and Characteristics of exponentiality against non-parametric classes

The exponential distribution plays an important role in reliability and life-time modelling, just as the normal distribution in classical statistics. For this distribution, explicit and simple forms of survival function, density and hazard are available. It is technically convenient for drawing inferences even in the presence of censoring. Furthermore, it is the only distribution with the memoryless (no ageing) property and therefore is often used to model the lifetimes of electronic and other non-ageing components. However, exponential distribution should be used judiciously since it's no ageing property actually restricts its applicability. For, many mechanical components undergo wear (e.g., bearings) or fatigue (e.g. structural components) whereas certain electronic components undergo reliability growth. These are the reasons why testing for exponentiality is important and why there are many tests of exponentiality. However, out of the several tests for exponentiality we shall only study the three tests:

(i) Hollander and Proschan's test (1972)

(ii) Certain tests based on sample spacings (Hollander and Proschan ,1975), Kllefsjo (1983)

(iii) Deshpande's class of tests (1983).

U-Statistics

Let $X_1,X_2,\cdots,X_n$ be a random sample from the distribution F∈F. A parameter γ is said to be estimable for degree r for the family of distributions F if r is the smallest sample size for which there exists a function $h^*(x_1,x_2,\cdots,x_r)$ such that:

$$E_F[h^*(X_1,X_2,\cdots,X_r)]=\gamma \qquad (1)$$

For every F∈F, the function $h^*(.)$ in Eq. (1) is known as the kernel for the parameter γ

It may be noted that for any kernel $h^*(x_1,x_2,\cdots,x_r)$ we can always create one that is symmetric in its arguments by using:

$$h(X_1,X_2,\cdots,X_r)=\frac{1}{r!}\sum_{A} h^*(X_{\alpha_1},\cdots,X_{\alpha_r}) \qquad (2)$$

where $\alpha_1,\alpha_2,\cdots,\alpha_r$ is a permutation of the number 1,2…. r and A are the set of all permutation $(\alpha_1,\cdots,\alpha_r)$ of the integers 1,2…. r.

A U-statistic for the estimable function γ is constructed with the symmetric kernel h(.) by forming:

$$U(X_1,\cdots,X_n)=\frac{1}{{}^{n}c_r}\sum_{\underline{\beta}\in B} h(X_{\beta_1},\cdots,X_{\beta_r}) \qquad (3)$$

Where $\underline{\beta}=(\beta_1,\cdots,\beta_r)$ is a combination of r integers from (1,2…. n) and B is the set of all such combinations.

Variance of the U-statistic

For a symmetric kernel h(.) consider the random functions $h(X_1,\cdots,X_c,X_{c+1},\cdots,X_r)$ and $h(X_1,\cdots,X_c,X_{r+1},\cdots,X_{2r-c})$ having exactly c variables in common. The covariance between these two random variables is given by:

$$\xi_c = Cov[h(X_1,\cdots,X_c,X_{c+1},\cdots,X_r),h(X_1,\cdots,X_c,X_{r+1},\cdots,X_{2r-c})]$$

$$= E[h(X_1,\cdots,X_c,X_{c+1},\cdots,X_r)h(X_1,\cdots,X_c,X_{r+1},\cdots X_{2r-c})]-\gamma^2$$

Therefore,

$$\xi_c= Cov[h(X_{\beta_1},\cdots,X_{\beta_r}),h(X_{\beta'_1},\cdots,X_{\beta'_r})] \qquad (4)$$

Where $(\beta_1,\cdots,\beta_r)'$ and $(\beta'_1,\cdots,\beta'_r)'$ are subsets of the integers {1,2, ⋯, n} having exactly c integers (out of r) in common. It may be notes that if c=0 then the kernel functions based on β and β' are independent. Hence $\xi_0=0$.

Now the variance of the U-statistic is:

$$Var(U)=E\left[\{\frac{1}{n}\sum_{r\in B} h(X_{\beta_1},\cdots,X_{\beta_r})-\gamma\}^2\right]$$

$$=\frac{1}{({}^nc_r)^2}\sum_{\beta}\sum_{\beta'} Cov[h(X_{\beta_1},\cdots,X_{\beta_r}),h(X_{\beta'_1},\cdots,X_{\beta'_r})$$

All the terms in the above equation for which $\underline{\beta}$ and $\underline{\beta}'$ have exactly c integers in the common have the same covariance, say ξ_c. The number of such terms is $\binom{n}{r}\binom{r}{c}\binom{n-r}{r-c}$. It follows that $Var(U) = \frac{1}{\binom{n}{r}}\sum_{c=1}^{r}\binom{r}{c}\binom{n-r}{r-c}\xi_c$ since $\xi_0=0$.

3.3.1 Total Time on Test

Let $T_1, T_2 \ldots\ldots T_n$ be a random sample of size n from the distribution F which is continuous and have support on the positive half of the real line. Let f be its density $\bar{F}$, its survival function and r_F , its failure rate function. The Total Time on Test Transform (TTT) of F is defined as:

$$H_F^{-1}(t) = \int_0^{\bar{F}^{-1}(t)} \bar{F}(u)du \text{ for } 0 \le t \le 1 \tag{5}$$

$$\bar{F}^{-1}(t) = \inf\{u : F(u) \ge t\} \tag{6}$$

Some Properties of TTT transform

i) $H_F^{-1}(t)$ is integral of a non- negative function and hence $H_F^{-1}(t)$ is an increasing function of t,

ii) If $F^{-1}(t)=\infty$ then $H_F^{-1}(t) = \int_0^{\infty} \bar{F}(u)du = \mu =$ the mean of F. This is the largest value of $H_F^{-1}(t)$

iii) $\frac{d}{dt} H_F^{-1}(t) = \bar{F}\left(F^{-1}(t)\right) \frac{d}{dt} F^{-1}(t)$

$$= \bar{F}(F^{-1}(t)) \cdot \frac{1}{\left[\frac{d}{du}F(u)\right]_{u=F^{-1}(t)}}$$

$$= \bar{F}(F^{-1}(t)) \cdot \frac{1}{f[F^{-1}(t)]}$$

$$= \left[\frac{f(F^{-1}(t))}{\bar{F}(F^{-1}(t))}\right]^{-1} = [r(F^{-1}(t))]^{-1}$$

Where $r(F^{-1}(t))$ is failure rate evaluated at $F^{-1}(t)$

iv) F is IFR if and only if $H_F^{-1}(t)$ is concave function of t.

F is IFR $\Leftrightarrow r(x)\uparrow x$

$\Leftrightarrow r[F^{-1}(t)]\uparrow t \Leftrightarrow \{r[F^{-1}(t)\}^{-1} \downarrow t$

F is IFR $\Leftrightarrow \frac{d}{dt} H_F^{-1}(t) \downarrow t$

$$\Leftrightarrow \frac{d^2}{dt^2}(H_F^{-1}(t)) \leq 0$$

$\Leftrightarrow H_F^{-1}(t)$ is concave function of t.

v) F is NBUE iff $\psi_F(t) \geq t, 0 \leq t \leq 1$

F is NBUE $\Leftrightarrow \int_0^\infty \frac{\overline{F}(t+x)}{\overline{F}(t)} dx \leq \mu, \forall t \geq 0$

$$\Leftrightarrow \int_t^\infty \frac{\overline{F}(y)}{\overline{F}(t)} dy \leq \mu \text{ by putting } x+t=y$$

$$\Leftrightarrow \frac{1}{u} \int_{F^{-1}(1-u)}^\infty \overline{F}(y) dy \leq \mu$$

By writing $\overline{F}(t)=u, 0 \leq u \leq 1$

$$\Leftrightarrow \frac{1}{u}[\mu - \int_0^{\overline{F}^{-1}(1-u)} \overline{F}(y) dy] \leq \mu, 0 \leq u \leq 1$$

$$\Leftrightarrow H_F^{-1}(1-u) \geq \mu(1-u), 0 \leq u \leq 1$$

$$\Leftrightarrow \frac{1}{\mu} H_F^{-1}(1-u) \geq (1-u), 0 \leq u \leq 1$$

$$\Leftrightarrow \psi_F(t) \geq t, 0 \leq t \leq 1$$

vi) let F be exponential (λ) where $\lambda = \frac{1}{\mu}$

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \overline{F}(u) du = \int_0^{F^{-1}(t)} e^{-\lambda u} du$$

$$= \left(-\frac{1}{\lambda} e^{-\lambda u}\right)_0^{F^{-1}(t)} = \left(\frac{1}{\lambda} - \frac{1}{\lambda} e^{-\lambda(F^{-1}(t))}\right)$$

Where $F^{-1}(t) = -\frac{1}{\lambda} \log(1-t)$.

This gives, $H_F^{-1}(t)=t/\lambda=\mu t$.

If F is exponential then $H_F^{-1}(t)=\mu t$, where μ is the mean of the distribution.

Estimation of TTT transform of F

$\hat{F}(t)=F_n(t)=$ empirical distribution function $=\frac{(\#\text{ sample } T_{(i)}\leq t)}{n}$

$H_{F_n}^{-1}(t)=$ Sample TTT transform

$$\hat{H}_F^{-1}(t)=\int_0^{F_n^{-1}(t)}\overline{F}_n(u)du.$$

$$H_{F_n}^{-1}(j/n)=\int_0^{T_{(j)}}\overline{F}_n(u)du$$

$$=\sum_{i=1}^{j}\int_{T_{(i-1)}}^{T_{(i)}}\left[1-\frac{(i-1)}{n}\right]du$$

Note that between $T_{(i-1)}$ and $T_{(i)}$, $\hat{F}_n(t)$ and hence $\hat{\overline{F}}_n(t)$ is constant. Therefore, $H_{F_n}^{-1}(t)$ changes at $1/n, 2/n, \cdots$ and in between is constant.

$$H_{F_n}^{-1}(j/n)=\sum_{i=1}^{j}\frac{(n-i+1)}{n}\left[T_{(i)}-T_{(i-1)}\right]$$

$$H_{F_n}^{-1}(j/n)=\frac{S_j}{n}, j=0,1,2,\cdots,n$$

Where $S_j=\sum_{i=1}^{j}(n-i+1)(T_{(i)}-T_{(i-1)})$

$$=\sum_{i=1}^{j}D_i$$

Where $D_i=(n-i+1)\ (T_{(i)}-T_{(i-1)})$ are the normalized sample spacings

Therefore, $H^{-1}_{F_n}(j/n) = \frac{1}{n}\sum_{i=1}^{j} D_i$ and estimator of scaled TTT transform

$$\psi_{F_n}(j/n) = \frac{1}{\overline{T}_n}\sum_{i=1}^{j}\left(\frac{n-i+1}{n}\right)(T_{(i)}-T_{(i-1)})$$

$$= \frac{1}{\overline{T}_n}\frac{S_j}{n}$$

$$= \frac{S_j}{S_n}, \text{ where } S_n = n\overline{T}_n \text{ and } S_0 = 0.$$

3.3.2 Deshpande Test

A class of tests for exponentiality against increasing failure rate average alternatives.

H_0:F(x)=1-exp(-λx),x≥0,λ>0 (unspecified)

H_1:F is IFRA but not exponential.

Rationale of the test:

$$F \text{ is IFRA} \Leftrightarrow [\overline{F}(x)]^b \le [\overline{F}(bx)], 0\le b\le 1, 0\le x<\infty \quad (7)$$

Equality in Eq. (4) holds iff F is exponential. For F, not exponential, but in IFRA class,

$$[\overline{F}(x)]^b < \overline{F}(bx),\ 0<b<1, 0\le x<\infty \quad (8)$$

Let, $M_F = \int_0^\infty \overline{F}(bx)dF(x)$

Under H_0; $\gamma = M_F = \frac{1}{(b+1)}$ for 0<b<1

Under H_1; $\gamma > \frac{1}{(b+1)}$

For a chosen number b between 0 and 1 (0.5 and 0.9 are possible choices), $(\gamma - \frac{1}{b+1})$.

may be taken as a measure of divergence of F from exponentiality.

Construction of U-statistic for the testing problem

Let $X_1, X_2 \ldots\ldots\ldots X_n$ be a random sample from the distribution F. Let $\tau = \gamma - \frac{1}{(b+1)}$ and

$h(X_1,X_2) = \psi(X_1-bX_2)$

$$= \begin{cases} 1 \text{ if } X_1>bX_2 \\ 0 \text{ otherwise} \end{cases}$$

$$E_{H_0}(\psi(X_1-bX_2)) = P[X_1>bX_2]$$

$$= \int_0^\infty P[X_1>bx]dF(x)$$

$$= \int_0^\infty \bar{F}(bx)dF(x)$$

$$= \int_0^\infty (\bar{F}(x))^b dF(x)$$

$$= \frac{1}{(b+1)}$$

$$=\gamma$$

Thus, γ is an estimable function of degree 2 and $h(X_1,X_2)$ is a kernel of degree 2. However, $h(X_1,X_2)$ Is not symmetric. Hence a symmetric kernel is obtained as follows:

$$h^*(X_1,X_2)=\frac{1}{2}[\psi(X_1-bX_2)+\psi(X_2-bX_1)] \quad (9)$$

Using this symmetric kernel, the corresponding U-statistic is constructed to test the hypothesis of interest:

$$U = J_b=\frac{1}{\binom{n}{2}}\cdot\frac{1}{2}\sum_{i<j}\sum[\psi(X_i-bX_j)+\psi(X_j-bX_i)]$$

$$=\frac{1}{n(n-1)}\sum_{i<j}\sum[\psi(X_i-bX_j)+\psi(X_j-bX_i)].$$

$E(U)=\gamma$ under H_0 asymptotic variance of $\sqrt{n}(U-\gamma)$ is $4\xi_1$. Under H_0, ξ_1 is given by:

$$\xi_1=\frac{1}{4}\{1+\frac{b}{2+b}+\frac{1}{2b+1}+\frac{2(1-b)}{(1+b)}-\frac{2b}{(1+b+b^2)}-\frac{4}{(b+1)^2}\} \quad (10)$$

3.4 Two Sample Problem

For the first sample, $T_1,T_2\ldots\ldots T_m$ be iid each with d.f. F_1 and $C_1,C_2,\ldots,C_m$ be iid each with d.f. G_1. C_i is the censoring time associated with T_i. We can observe $(x_1,\delta_1),\ldots,(x_m,\delta_m)$ where:

$$x_i=T_i\wedge C_i,\delta_i=I(T_i<C_i) \tag{11}$$

And for second sample, let $U_1,U_2,\ldots,U_n$ be iid each with d.f. F_2, and $D_1,D_2,\ldots,D_n$ be the iid each with d.f. G_2. D_j is the censoring time associated with U_j and we observe $(Y_1,\varepsilon_1),\ldots,(Y_n,\varepsilon_n)$ where:

$$Y_j=U_j\wedge D_j,\varepsilon_j=I(U_j<D_j). \tag{12}$$

The usual two sample problems are to test: $H_0:F_1=F_2$

3.4.1 Gehan Test

Gehan's test is an extension of the Wilcoxon test. Let the observations from the two samples be $X_1,\ldots\ldots\ldots X_m;Y_1,\ldots\ldots\ldots\ldots Y_n$. Order the combined sample and define $Z_1<Z_2\ldots..<Z_{m+n},R_{1i}$=rank of X_i , $R_1=\sum_{i=1}^{m}R_{1i}$.

Reject H_0 if R_1 is too small or too large. Use small sample tables or the large sample approximation:

$$\frac{R_1-E_0(R_1)}{\sqrt{var_0(R_1)}}=\frac{R_1-\frac{m(m+n+1)}{2}}{\sqrt{(\frac{mn(m+n+1)}{12}}}\sim N(0,1) \tag{13}$$

Where $E_0(R_1)$ and $Var_0(R_1)$ are the moments calculated under the null hypothesis.

The Mann-Whitney form of the Wilcoxon test will be useful.

$$U(x_1,y_j)=U_{ij}=\begin{cases}+1 & \text{if } x_1>y_j,\\ 0 & \text{if } x_1=y_j,\\ -1 & \text{if } x_1<y_j,\end{cases}=\sum_{i=1}^{m}\sum_{j=1}^{n}U_{1j} \tag{14}$$

It can be shown that $R_1=\frac{m(m+n+1)}{2}+\frac{1}{2}U$.

To see this notice that if we have the total separation $x_{(1)}<\cdots<x_{(m)}<y_{(1)}<\ldots<y_{(n)}$, then $R_1=\frac{m(m+1)}{2}$. For every interchange of contagious x-y pair, R_1 is increased by 1, and the number of such interchanges is $\Sigma_i\Sigma_j\frac{1}{2}(U_{ij}+1)$. Therefore,

$$R_1=\frac{m(m+1)}{2}+\sum_i\sum_j\frac{1}{2}(U_{1j}+1)$$

$$=\frac{m(m+1)}{2}+\frac{mn}{2}+\frac{1}{2}U$$

$$=\frac{m(m+n+1)}{2}+\frac{1}{2}U.$$

The Mann-Whitney test rejects H_0 if U of |U| is too large. Use small sample tables or the large sample approximation:

$$\frac{U\text{-}E_0(U)}{\sqrt{Var_0(U)}}=\frac{U}{\frac{\sqrt{mn(m+n)}}{3}}\sim N(0,1).$$

For censored data, Gehan defines:

$$U_{ij}=\begin{cases}1 \text{ if we know } t_i>u_j, \text{ i.e.,; } (x_i>y_j,\varepsilon_j=1) \text{ or } (x_i=y_j,\delta_i=0,\varepsilon_j=1)\\ 0;\text{otherwise,}\\ -1,\text{if we know } t_i<u_j, \text{ i.e.,}(x_i<y_j,\delta_i=1) \text{ or } (x_i=y_j,\delta_i=1,\varepsilon_j=0),\end{cases} \quad (15)$$

$$U=\sum_{i=1}^{m}\sum_{j=1}^{n}U_{ij} \quad (16)$$

Reject H_0 if U or |U| is large. The statistic U is asymptotically normally distributed by the theory of two-sample U-statistics, but to calculate the critical values we need to know the moments of U.

Mean and variance of U

With no censoring, the mean and variance can be calculated using permutation theory. Under H_0, consider sampling m balls without replacement from an urn containing m+n balls labeled $Z_1,\ldots\ldots, Z_{m+n}$. Think of the labels on the m sampled balls as the values of $X_1,\ldots\ldots,X_m$, and the labels on the n unsampled balls as the values of $Y_1,\ldots\ldots,Y_n$.

Let $E_{(0,P)}(U)=0=E_0(U)$ and $Var_{0,P}(U)=\frac{mn(m+n+1)}{3}=Var_0(U)$.

With censoring, Gehan also uses permutation theory but under the more restrictive null hypothesis: $H_0^*: F_1=F_2$ and $G_1=G_2$. Let the combined sample be denoted by $(Z_1,\zeta_1),\ldots\ldots,(Z_{m+n},\zeta_{m+n})$.

Consider sampling m balls without replacement from an urn containing m+n balls labeled $(Z_1,\zeta_1),\ldots\ldots,(Z_{m+n},\zeta_{m+n})$. Think of the labels on the m sampled balls $(X_1,\delta_1),\ldots\ldots,(X_m,\delta_m)$ and the labels on the n unsampled balls as $(Y_1,\epsilon_1),\ldots\ldots,(Y_n,\epsilon_n)$. Then, $E_{0,P}^*(U)=0$.

Mantel computational form for $Var_{0,P}^*(U)$:

$$U_{kl}=U\left((Z_k,\zeta_k),(Z_l,\zeta_l)\right)=\begin{cases}+1, \text{ if } (Z_k>Z_l,\ \zeta_l=1) \text{ or } (Z_k=Z_l,\ \zeta_k=0,\ \zeta_l=1)\\ 0; \text{ otherwise}\\ -1; \text{ if } (Z_k<Z_l,\zeta_k=1) \text{ or } (Z_k=Z_l,\ \zeta_k=1,\ \zeta_l=0),\end{cases} \quad (17)$$

$$U_k^*=\sum_{\substack{l=1\\ \neq k}}^{m+n} U_{kl}\ ,$$

$$U=\sum_{k=1}^{m+n} U_k^* I(k\in I_1),$$

Where I_1 is the set of integers comprising sample 1. Notice that U is equal to Gehan's statistic because $U_{k_1k_2}=-U_{k_2k_1}$ so if $k_1, k_2 \in I_1$, they cancel each other out in the sum.

To calculate the permutation distribution of U, suppose we are given $U_1^*,\ldots\ldots,U_{m+n}^*$. Under H_0^*, we sample m of these U_k^* without replacement and form U, the sum of these m values. Using results on sampling from finite populations,

$$Var_{0,P}^*(U)=m\left(\frac{1}{m+n-1}\sum_{i=1}^{m+n}(U_i^*)^2\right)\left(1-\frac{m}{m+n}\right)=\frac{mn}{(m+n)(m+n-1)}\sum_{i=1}^{m+n}(U_i^*)^2.$$

Example-1: In the hypothetical clinical trial constructed by Byron Wn. Brown, Jr. let the treatment A patients be the X observations and the treatment B patients be the Y observations.

R_x A:	3	5	7	9^+	18
R_x B:	12	19	20	20^+	33^+

Solution: For Brown's hypothetical clinical trial

Z	Rx	#<Z	#>z	U^*
3	A	0	9	-9
5	A	1	8	-7
7	A	2	7	-5
+9	A	3	0	3
12	B	3	5	-2
18	A	4	4	0
19	B	5	3	+2
20	B	6	2	+4
20+	B	7	0	+7
33+	B	7	0	+7

U= -9-7-5+3+0 = -18, $E^*_{0,P}(U)=0$; $Var^*_{0,P}(U)=\frac{(5)(5)(286)}{(10)(9)}=79.44$.

Under H^*_0,

$$\frac{U}{\sqrt{Var^*_{0,P}(U)}} = -\frac{18}{8.91} = -2.02 \sim N(0,1),$$

So, p = 0.022 is the one-tailed p-value.

3.4.2 Log Rank Test

The Log–Rank Test is a large-sample chi-square test that uses as its test criterion a statistic that provides an overall comparison of the KM curves being compared. This (log–rank) statistic, like many other statistics used in other kinds of chi-square tests, makes use of observed versus expected cell counts over categories of outcomes. The categories for the log–rank statistic is defined by each of the ordered failure times for the entire set of data being analysed.

Let $\zeta_1 \leq \zeta_2 \leq \ldots \leq \zeta_g$ be ordered failure times from the combined sample. Let d_j=deaths at ζ_j; r_j = at risk at ζ_j; r_{ij}=at risk at ζ_j from group i, i=1,2.

The log Rank test compares the observed and expected (under H_0). Number of deaths in group 1st. We have, E = Expected number of deaths (under H_0) group I and O = observed Number of deaths from sample I (or group I).

$$V = \sum_{j=1}^{g} d_j \frac{r_{ij}}{r_j} = \sum_{j=1}^{g} \frac{d_j r_{1j} r_{2j}}{r_j^2} \tag{18}$$

Then,

$$Z=\frac{O-E}{\sqrt{V}} \sim N(0,1) \quad (19)$$

3.4.3 Mantel-Haenszel Test

Suppose we have two population, individual i.e., either population can have one of two characteristics e.g., population 1. Might be cancer population and certain treatment and population 2, cancer population under different treatment. The patient is either group may either die within one year or survive beyond one year. The data may be summarized in a 2×2 table as follows.

	Dead	Alive	
Population 1	a	b	n_1
Population 2	c	d	n_2
	m_1	m_2	n

Where,

a: No. of deaths at complete observation (say 't' time) from sample 1.

n_1: a+b, indicates the no. at risk at t^- time from sample 1.

n_2: define similarly.

m_1: c+d, Total number of deaths at t^- time.

m_2: define similarly.

n: a+b+c+d, total no. of observation

Let,

P_1=P[Dead|Population 1]

P_2=P[Dead|Population 2].

We intend to test, $H_0:P_1=P_2$ and the test statistic,

$$Z=\frac{\widehat{p_1}-\widehat{p_2}}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \sim N(0,1)$$

Where,

$$\widehat{p_1}=\frac{a}{n_1} \quad ,\widehat{p_2}=\frac{c}{n_2} \ , \ \hat{p}=\frac{a+c}{n_1+n_2}=\frac{m_1}{n}$$

$$Z^2=\frac{\left(\frac{a}{n_1}-\frac{c}{n_2}\right)^2}{\frac{m_1}{n}\left(1-\frac{m_1}{n}\right).\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} = \frac{n(ad-bc)^2}{m_1m_2n_1n_2} \sim \chi_1^2 \qquad (20)$$

This is an approximation to the exact conditional distribution under H_0, given n_1,n_2, m_1,m_2 fixed the random variable A, the entry in cell (1,1) of 2×2 table, follow hyper geometric distribution with p.m.f.

$$P(A=a)=\frac{\binom{n_1}{a}\binom{n_2}{m_1-a}}{\binom{n}{m_1}};\ a=0(1)\min(m_1,n_1)$$

$$E(A)=\frac{m_1n_1}{n} \text{ and } Var(A)=\frac{m_1m_2n_1n_2}{n^2(n-1)}$$

Consider,

$$a-E_{H_0}(A)=a-\frac{m_1n_1}{n}=\frac{na-m_1n_1}{n}=\frac{ad-bc}{n}$$

$$ad-bc=n\left[a-E_{H_0}(A)\right]$$

And, $m_1m_2n_1n_2=n^2(n-1)var_{H_0}(A)$

So,

$$\frac{n(ad-bc)^2}{m_1m_2n_1n_2}=\frac{n^3\left[a-E_{H_0}(A)\right]^2}{n^2(n-1)var_{H_0}(A)} = \frac{n\left[a-E_{H_0}(A)\right]^2}{(n-1)var_{H_0}(A)} \sim\chi_1 \text{ [as } n\to\infty]$$

Now suppose we have a sequence of 2 * 2 tables. *For example*, we might have k hospitals; at each hospital, patients receive either treatment 1 or treatment 2 and their responses are observed.

Hospital 1

	D	A	
Treatment 1	a_1		n_{11}
Treatment 2			n_{12}
	m_{11}	m_{12}	n_1

………..

Hospital k

	D	A	
Treatment 1	ak		n_{k1}
Treatment 2			n_{k2}
	m_{k1}	m_{k2}	n_k

Because there may be difference among hospitals, we don't want to combine all k-table into a single 2×2 table. Based on these table (k). We want to test:

$H_0: P_{11}=P_{12}, P_{21}=P_{22}, \ldots\ldots., P_{k1}=P_{k2}$.

Where,

P_{i1}=P[Dead|Population1, Hospital i]

P_{i2}=P[Dead|Population 2, Hospital i], $\forall i=1(1)k$.

Suppose, a_i=no. of patients receiving treatment 1, who died in hospital 'i'.

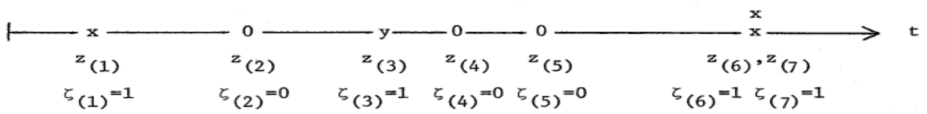
Montel-Haenzel suggested the following statistic:

$$MH=\sum_{i=1}^{k}\frac{\left|a_i-E_{H_0}(A_i)\right|-\frac{1}{2}}{\sqrt{\sum_{i=1}^{k}V_{H_0}(A_i)}}\sim N(0,1) \qquad (21)$$

Here, $\frac{1}{2}$ may be consider as continuity correction

If the tables are independent, then MH~N(0,1) either when k is fixed and $n_1\to\infty$ or when $k\to\infty$ and the tables are also identically distributed.

In survival analysis the MH statistic is applied as follow. As $(Z_1,\zeta_1),\ldots.(Z_{m+n},\zeta_{m+n})$ is the combined ordered sample. Construct a 2×2 table for each uncensored time point.



	D	A
X	1	
Y	0	
		n_1

	D	A
X	0	
Y	1	
		n_2

	D	A
X	2	
Y	0	
		n_3

Example-2: The computations for the MH statistic in Brown's hypothetical clinical trial are given in table. The column labeled z contains the uncensored ordered observations. The next four columns labeled $n, m_1, n_1,$ a construct the 2×2 tables. The next column is $E_0(A)=\frac{n_1 m_1}{n}$. The product of the last two columns, labeled $\frac{m_1(n-m_1)}{n-1}$ and $\left(\frac{n_1}{n_2}\right)(1-\frac{n_1}{n})$, is

$var_0(A)$; it is convenient to break up the calculation of $var_0(A)$ in this way because $\frac{m_1(n-m_1)}{n-1}$ is usually equal to 1 and $\left(\frac{n_1}{n}\right)\left(1-\frac{n_1}{n}\right)$ is the product of the proportions in the two samples.

z	n	m_1	n_1	a	$E_0(A)$	$a-E_0(A)$	$\frac{m_1(n-m_1)}{n}-1$	$\frac{n_1}{n}(1-\frac{n_1}{n})$
3	10	1	5	1	.50	0.50	1	.2500
5	9	1	4	1	.44	0.56	1	.2469
7	8	1	3	1	.38	0.62	1	.2344
12	6	1	1	0	.17	-0.17	1	.1389
18	5	1	1	1	.20	0.80	1	.1600
19	4	1	0	0	0	0	1	0
20	3	1	0	0	0	0	1	0

$$MH=\frac{\text{sum of a-}E_0(A)\text{ column}}{\sqrt{\text{sum of }\left(\frac{m_1(n-m_1)}{n-1}\text{ col.}\times\frac{n_1}{n}\left(1-\frac{n_1}{n}\right)\text{col.}\right)}},$$

$$=\frac{2.31}{1.02}=2.26.$$

p = .012 (one-tailed).

$$MH_c=\frac{2.31-0.50}{1.02},$$

$$=\frac{1.81}{1.02}=1.77.$$

p = .038 (one-tailed)

Asymptotic normality

To show asymptotic normality, we adapt Crowley's representation to our case. Assume no ties. Denote N=m+n,

$$\widehat{H}(t)=\frac{1}{N}\sum_{i=1}^{N}I(Z_i\leq t)\,,\ \widehat{H}_1(t)=\frac{1}{m}\sum_{i=1}^{m}I(X_i\leq t)$$

$$\hat{H}_u(t)=\frac{1}{N}\sum_{i=1}^{N} I(Z_i\leq t,\ \zeta_i=1)\ ;\ \hat{H}_{1u}(t)=\frac{1}{m}\sum_{i=1}^{m} I(X_i\leq t,\ \delta_i=1)\ .$$

Then the numerator of MH is:

$$\sum_{i=1}^{k}\left(a_i-E(A_i)\right)=m\left\{\int_0^{\infty} d\hat{H}_{1u}(s)-\int_0^{\infty}\frac{1-\hat{H}_1(s-)}{1-\hat{H}(s-)}d\hat{H}_u(s)\right\} \tag{22}$$

To see this, recall that $E(A_i)=\frac{m_{i1}n_{i1}}{n_i}$ where a_i, m_{i1}, n_{i1}, n_i are gotten from the 2×2 table corresponding to the ith uncensored observation:

	D	A	
X	a_i		n_{i1}
Y			
	m_{i1}		n_i

Because we have assumed no ties, $m_{i1}=1$. Letting s_i denote the time of the ith uncensored observation,

$n_{i1}=\#\left(X's \text{ remaining at time } s_i-\right)$,

$\quad = m\left(1-\hat{H}_1(s_i-)\right)$

$n_i=\#\left(Z's \text{ remaining at time } s_i-\right)$

$\quad =N\left(1-\hat{H}(s_i-)\right)$.

Now that we have the numerator of MH expressed in terms of empirical (sub) distribution functions, we may apply arguments similar to those used in showing the asymptotic normality of the PL estimator.

3.4.4 Tarone-Ware Test

Tarone-ware suggested the following statistic:

$$\sum_{i=1}^{k} W_i\left|a-E_{H_0}(A)\right| \tag{23}$$

where w_i denotes the weight of ith class or observation.

Variance,

$$\sum_{i=1}^{k} w_i^2 \, var_H(A) = \sum w_i^2 \left[\frac{m_{i1}(n_i - m_{1i})}{n_i - 1} \cdot \frac{n_{1i}}{n_i} \cdot \left(1 - \frac{n_{1i}}{n_i}\right) \right] \quad (23)$$

Cases:

I. If $w_i=1$, it become Montel-Haenzel statistic.

II. If $w_i=n_i$; then it given Gehan's statistic with its variance given as above.

III. If $w_i=\sqrt{n_i}$, this suggested by Tarone-Ware.

It may be noted that Gahen's statistic put more weight to small observations while MH statistic puts equal weight to each observation. Tarone and ware's suggestion is intermediate between the two. They claim that $w_i=\sqrt{n_i}$, have greater efficiency over the range of alternatives.

Notes:

1. The Gehan statistic puts more weight on the beginning observations, while the MH statistic puts equal weight on each observation. Tarone and Ware's suggestion is intermediate between the two, and they claim that the weights $w_i = \sqrt{n_i}$ have high efficiency over a range of alternatives.

2. Although Tarone equation is identical to the Gehan statistic U, $Va\hat{r}_{TW}(U)$, is not the same as $Var^*_{0,P}(U)$. Asymptotically, $\widehat{Var}_{TW}(U)$ is equivalent to the variance of U under H_0 while $Var^*_{0,P}(U)$ is the variance under H_0^*.

Example-3: Using the above Example-2, where we calculate the MH statistic for Brown's clinical trial,

$$\sum_{i=1}^{k} n_i\big(a_i - E(A_i)\big) = (10)(.50)+(9).56)+(8).62)+(6)(-.17)+5(.80)$$

$$= 17.98$$

Which is what we got for Gehan's statistic U except for sign and roundoff. Also,

$$\widehat{Var}_{TW}(U) = \sum n_i^2 \left[\frac{m_{i1}(n_i - m_{i1})}{n_i - 1} \right] \left[\left(\frac{n_{i1}}{n_i}\right) \left(1 - \frac{n_{i1}}{n_i}\right) \right]$$

$$= (10^2)(.25)+(9^2)(.2469)+(8^2)(.2344)+(6^2)(.1389)+(5^2)(.16),$$

$= 69$

$\mathrm{Var}^{*}_{0,P}(U) = 79.44$,

Which give: $\sqrt{\widehat{\mathrm{Var}}_{TW}(U)}=8.31$ and $\sqrt{\mathrm{Var}^{*}_{0,p}(U)}=8.91$

3.5 Summary

This unit covers the application of survival analysis, also known as reliability analysis, which investigates the time until an event like failure occurs. Non-parametric methods are introduced to avoid assumptions about survival time distributions, along with concepts like the hazard function and the role of the exponential distribution, especially with censored data. Alternatives like the TTT transform and U-statistics are explored, including the Deshpande test for assessing exponential distribution fit, crucial for reliability modelling. The module also delves into the "Two Sample Problem" in survival analysis, discussing tests like the Gehan Test, Log Rank Test, Mantel-Haenszel Test, and Tarone-Ware Test. These tests offer various approaches to comparing survival distributions between independent samples, providing robust options for handling censored observations and controlling for confounding variables.

3.6 Self-Assessment Exercise

1. Following table shows failure time of two machines, new and old.

	Failure times (day)
New machine	$250, 476^{+}, 355, 200, 355^{+}$
Old machine	191, 563, 242, 285, 16, 16, 16, 257, 16

   (+ indicates censored times).

   Test whether the new machine is more reliable than the old one by using log rank test.

2. Ten female patients with breast cancer are randomized to receive either CMF (cyclic administration of cyclophosphamide, methatrexate, and fluorouracil) or no treatment after a radical mastectomy. At the end of two years, the following times to relapse (or remission times) in months are recorded:

CMF(Group1)	23	16^{+}	18^{+}	20^{+}	24^{+}
Control (Group 2)	15	18	19	19	20

   Test whether the CMF more efficient than no treatment (Control Group) using the Gehan Test.

3. Five hundred and ninety-five persons participate in a case control study of the association of cholesterol and coronary heart disease (CHD). Among them, 300 persons are known to have CHD and 295 are free of CHD. To find out if elevated cholesterol is significantly associated with CHD, the investigator decides to control the effects of smoking. The study subjects are then divided into two strata: smokers and non-smokers. The following tables give the data for smokers:

Elevated cholesterol?	With CHD	Without CHD	Total
For smokers			
Yes	120	20	140
No	80	60	140
Total	200	80	280
For non-smokers			
Yes	30	60	90
No	70	155	255
Total	100	215	315

Compare the survival distributions of the two treatment groups using the Mantel Haenszel Test.

4. The table below gives the survival times in weeks of 30 brain tumour patients receiving four different treatments. Are the four treatments equally effective?

1	**2**	**3**	**4**
4	1	3	5
5	4	7	15
9	9	14	20
12	12	20	31
20^+	15	27	39
25	23	30	47
30^+	30	32^+	55^+
		50^+	67^+

5. Form the below table, use Mantel's procedure for Gehan's generalized Wilcoxon test to compute a score for each observation and the sum of scores for each of the three treatment groups.

Group – 1	4, 5, 9, 10, 12, 13, 10, 23, 28, 28, 28, 29, 31, 32, 37, 41, 41, 57, 62, 74, 100, 139, 20+, 258+, 269+
Group – 2	8, 10, 10, 12, 14, 20, 48, 70, 75, 99, 103, 162, 169, 195, 220, 161+, 199+, 217+, 245+
Group – 3	8, 10, 11, 23, 25, 25, 28, 28, 31, 31, 40, 48, 89, 124, 143, 12+, 159+, 190+, 196+, 197+, 205+, 219+

3.7 References

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3.8 Further Reading

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UNIT - 4 PROPORTIONAL HAZARD MODELS

Structure

4.1 Introduction

Survival analysis is a fundamental statistical tool used to analyze time-to-event data, where the primary interest lies in understanding the time until an event of interest occurs. In this chapter, we focus on the Proportional Hazard Models, with a particular emphasis on the Cox Proportional Hazards Model. This model is a cornerstone in survival analysis due to its ability to analyze the effect of covariates on survival times without assuming a specific form for the underlying survival distribution. The Cox Proportional Hazards Model is a versatile tool in survival analysis as it does not require knowledge of the underlying survival distribution. It assumes that the hazard rates among individuals change in step functions over time, under the proportionality assumption. This assumption implies that the hazard functions for different individuals are proportional and time-independent, allowing for the inclusion of covariates that can be time-dependent or constant. Statistical inference in the Cox model is based on the partial likelihood function, which considers the relative ordering of event times rather than the exact times themselves. This makes the Cox model particularly useful in identifying significant prognostic factors when the exact form of the survival distribution is unknown. To ensure the validity of the Cox Proportional Hazards Model, it is crucial to assess the proportionality assumption. This can be done using both graphical and analytical tests. The graphical test involves comparing survival functions for two groups using a log-log plot, while the analytical test utilizes a test statistic based on pairs of observed survival times. These tests help researchers determine if the hazard functions of different individuals are parallel, indicating proportionality. The Cox Proportional Hazards Model can be extended to include several covariates, allowing for the analysis of the effect of multiple factors on survival times. This extension does not require specifying the baseline hazard function and provides a framework for interpreting regression coefficients and constructing the likelihood function for observed failure times. In the context of competing risks, where multiple risks compete to cause failure, the Cox model can be extended to account for these complexities. Each risk is associated with its own cause-specific hazard function, and the overall hazard is the sum of all cause-specific hazards, assuming the failure types are mutually exclusive. This section also covers modeling the joint distribution of survival time and cause of failure using sub-survival functions and constructing multiple decrement life tables to reflect competing risks. The chapter concludes with a discussion on multiple decrement life tables, which are used to compute survival probabilities when multiple, mutually exclusive risks affect the population. This section introduces cause-specific hazard functions and decrement probabilities for each cause, providing a comprehensive framework for understanding the impact of individual risks on survival probability.

4.2 Objectives

After going through this unit, you should be able to:

- Explain the principles of Proportional Hazard Models, with a focus on the Cox Proportional Hazards Model and its significance in survival analysis.
- Apply the Cox Proportional Hazards Model to analyze survival data effectively, demonstrating proficiency in assessing the impact of covariates on survival times without requiring assumptions about the underlying survival distribution.
- Evaluate the validity of the proportionality assumption inherent in the Cox Proportional Hazards Model through both graphical and analytical tests, ensuring the appropriate application of the model in survival analysis.

- Extend the application of the Cox Proportional Hazards Model to accommodate multiple covariates and competing risks, enabling the analysis of factors influencing survival times and understanding the complexities of competing failure types in survival analysis.

4.3 Assumptions and Characteristics

Proportional Hazard models

The parametric survival methods for model fitting and for identifying significant prognostic factors are powerful if the underlying survival distribution is known. The estimation and hypothesis testing of parameters in the models can be conducted by applying standard asymptotic likelihood techniques. However, in practice, the exact form of the underlying survival distribution is usually unknown, and we may not be able to find an appropriate model. Therefore, the use of parametric methods in identifying significant prognostic factors is somewhat limited. We discuss a most commonly used model, the Cox (1972) proportional hazards model, and its related statistical inference. This model does not require knowledge of the underlying distribution. The hazard function in this model can take on any form, including that of a step function, but the hazard functions of different individuals are assumed to be proportional and independent of time. The usual likelihood function is replaced by the partial likelihood function. The important fact is that the statistical inference based on the partial likelihood function is similar to that based on the likelihood function.

Assumption

The proportional hazards assumption states that the hazard ratio associated with each covariate is constant over time. In other words, the relative hazard of experiencing the event remains constant across different levels of the covariate(s) throughout the follow-up period. This assumption implies that the hazard curves for different levels of the covariate(s) are parallel over time.

Graphical Test

Consider the case of single variate at two levels denoted by z=0 (control) and z=1 (treatment). Let X and Y denote the lifetime of the subjects for the two values of the covariates Z_0. Let $\overline{F}$ and $\overline{G}$ be the survival functions and h_f and h_G be the hazard function of X and Y respectively. Under the proportional hazard assumption, we have:

$$h_G(t)=\delta h_F(t) \tag{1}$$

Where $\delta>0$ is the constant of proportionality

From equation 1), in term of survival functions is: $e^{-\int_0^t h_G(u)du} = e^{-\delta \int_0^t h_F(u)du}$

$$\overline{G(t)}=[\overline{F(t)}]^{\delta} \tag{2}$$

Under commonly used log linear function $\delta=e^{\beta}$ where β is the regression coefficient. Using equation 2) we have:

$\Rightarrow -\log\overline{G}(t) = \delta(-\log\overline{F}(t))$

$\Rightarrow -\log(-\log\overline{G}(t)) = -\log\delta - \log(-\log\overline{F}(t)) = -\beta - \log(-\log\overline{F}(t))$

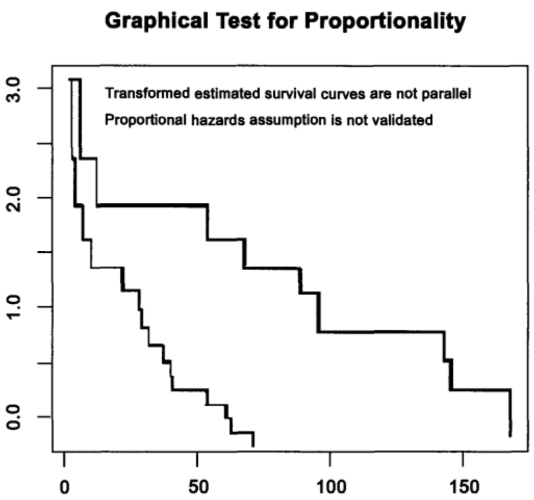
Now consider the estimated transformed survival curves corresponding to two groups indexed by 0 and 1. The difference between their transformed survival curves is β, a constant not depending on t. So, if we plot the transformed survival curves for the two groups on the same plot, they should be parallel if the proportionality assumption holds.

Example: Suppose a data set from randomized clinical trial investigate prednisolone therapy. These are the survival times in months until death from chronic active hepatitis patients (+ denotes censored data)

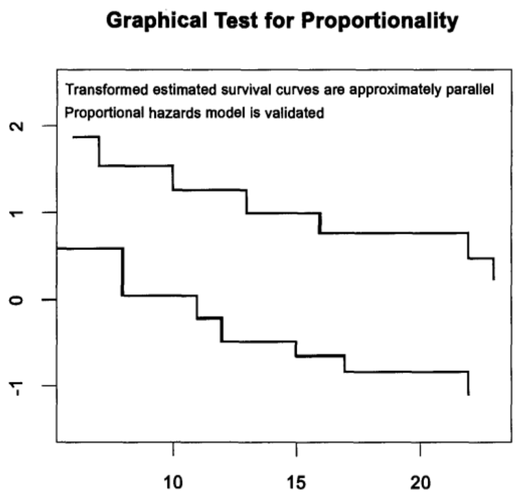
Treatment Group: 2, 6, 12, 54, 56+, 68, 89, 96, 96, 125+, 128+, 131+, 140+, 141+, 143, 145+, 146, 148+, 162+, 168, 173+, 181+

Control Group: 2, 3, 4, 7, 10, 22, 28, 29, 32, 37, 40, 41, 54, 61, 63, 71, 127+, 140+, 146, 158+, 167+, 182+

Data set 1:



Data set 2:



Analytical Test

Let X~F (control groups) and Y~ (treatment group) under proportional assumption (or under PH assumption) $\overline{G}(t)=[\overline{F}(t)]^{\delta}$. $\delta>0\forall t$, or, $h_G(t)=\delta h_F(t)$.

The value of δ=1 indicates that X and Y have the same probability distribution. Given $x_1,x_2\ldots\ldots x_n$ and $y_1,y_2\ldots y_m$, two independents' samples from F and G respectively. We assume that there is no censoring. Define a statistic:

$$U=\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\Phi(x_i,y_j) \tag{3}$$

Where

$$\Phi_{ij}=\Phi(x_i,Y_j)=\begin{cases}1 & ;\text{ if } x_i\leq y_j\\ 0 & ;\text{ otherwise.}\end{cases}$$

Then,

$$Z=\frac{U\text{-}E(U)}{\sqrt{V(U)}}\sim N(0,1)$$

$$E(\Phi_{ij}) = 1\cdot P(x_i\leq y_j)+0\cdot P(x_i>y_j)=P(x_i\leq y_j)$$

$$=\sum_{x=0}^{\infty}\sum_{y=x}^{\infty}G(y)\cdot dF(x)=\int_{x=0}^{\infty}\overline{G}(x)dF(x)=\int_{x=0}^{\infty}(\overline{F}(x))^{\delta}dF(x)$$

$$=-\int_{1}^{0}t^{\delta}dt\,,\ \left(\text{let } \overline{F}(x)=t\Rightarrow\text{-}DF(x)=dt\right)$$

$$=\int_{0}^{1}t^{\delta}dt=\frac{t^{\delta+1}}{\delta+1}\Big]_0^1=\frac{1}{\delta+1}$$

$$E(u)=\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=q}^{m}\frac{1}{\delta+1}=\frac{1}{\delta+1}$$

We intend to test whether, H_0:δ=0 vs. H_1:δ≠0

Under H_0:δ=0, we have E(U)=$\frac{1}{\delta+1}$ and under H_1:δ≠0 we have E(U)>$\frac{1}{\delta_0+1}$. Hence a test which rejects for large value of U is reasonable in this context.

$$\mathrm{Var}(u)=\sum_{i=1}^{n}\sum_{j=1}^{m}\mathrm{Var}(\Phi_{ij})+\sum_{i=1}^{n}\sum_{\leq j<m\neq l}\sum \mathrm{Cov}((\Phi_{ij},\Phi_{il})+\sum_{i=1}^{m}\sum_{i\leq i\neq k<n}\sum \mathrm{Cov}((\Phi_{il},\Phi_{kl})+$$

$$\sum_{l\leq i\neq k\leq n}\sum\sum\sum_{i\leq j\neq l<n}\mathrm{Cov}((\Phi_{ij},\Phi_{kl})$$

$$\mathrm{Var}(\Phi_{ij})=E[\Phi_{ij}^2]-\{E[\Phi_{ij}]\}^2$$

$$=\frac{1}{\delta+1}-(\frac{1}{\delta+1})^2=\frac{1}{\delta+1}(1-\frac{1}{\delta+1})$$

$$=\frac{\delta}{(\delta+1)^2}$$

$$\mathrm{Cov}(\Phi_{ij},\Phi_{ii}) = E[\Phi_{ij},\Phi_{il}]-E(\phi_{ij})\cdot E(\Phi_{il})$$

$$E[\Phi_{ij}\Phi_{il}] = P[x_i<y_j \text{ and } x_i<y_l]$$

$$=\int_{x=0}^{\infty}\int_{y=x}^{\infty}dG(y)^2dF(x)$$

$$\int_0^{\infty}\{\overline{G(x)}\}^2dF(x)=\int_0^{\infty}(\overline{F}(x))^{2\delta}dF(x)$$

$$=-\int_0^1 t^{2\delta}dt=\left.\frac{t^{2\delta+1}}{2\delta+1}\right]_0^1=\frac{1}{2\delta+1}$$

$$\mathrm{cov}(\Phi_{ij},\Phi_{il})=\frac{1}{2\delta+1}-(\frac{1}{\delta+1})^2=\frac{\delta^2}{(\delta+1)^2(2\delta+1)}$$

When j=l

$$E[\Phi_{il,}\Phi_{kl}] = P[x_i<y_l \quad \& x_k<y_l]$$

$$= \int_{y=0}^{\infty} \int_{0}^{y} dF(x)^2 dG(y) = \int_{y=0}^{\infty} (F(y))^2 dG(y).$$

$$= \int_{y,0}^{\infty} (1-\overline{F}(y))^2 dG(y) = \int_{y=0}^{\infty} (1-(G(y))^{1/\delta})^2 dG(y)$$

$$= \int_{0}^{1} (1-y^{1/\delta})^2 dy = \delta\, B(3,\delta) = \frac{\delta\sqrt{3}\sqrt{\delta}}{\sqrt{3+\delta}} = \frac{2\delta\sqrt{\delta}}{(\delta+2)(\delta+1)\delta\sqrt{\delta}} = \frac{2}{(\delta+2)(\delta+1)}$$

$$\text{cov}\left[\Phi_{il},\Phi_{kl}\right] = \frac{2}{(\delta+2)(\delta+1)} - \left(\frac{1}{\delta+1}\right)^2 = \frac{2\delta+2-\delta-2}{(\delta+1)^2(\delta+2)} = \frac{\delta}{(\delta+2)(\delta+1)^2}$$

$$V(u) = \frac{1}{(mn)^2}\left[\frac{\delta}{(\delta+1)^2}\right] + \frac{\delta^2}{(\delta+1)^2(2\delta+1)} * \frac{n(m-1)m}{m^2n^2} + \frac{\delta}{(\delta+2)(\delta+1)^2} \cdot \frac{(n-1)}{mn}$$

$$V(u) = \frac{1}{(\delta+1)^2 mn}\left[\frac{\delta}{mn} + \frac{\delta^2(m-1)}{(2\delta+1)} + \frac{\delta(n-1)}{(\delta+2)}\right] = \frac{\delta}{(\delta+1)^2 mn}\left[\frac{1}{mn} + \frac{\delta(m-1)}{2\delta+1} + \frac{n-1}{\delta+2}\right]$$

4.4 Semi – Parametric Regression for Failure Rate

Cox-Proportional Hazard Model

Here, we study certain models which incorporate the effect of covariates of explanatory variable on the distribution of life times. With this model, we will be able to test whether the covariates affect the lifetimes significant or not covariates may be in several form:

Covariates: these are the characteristic or features of the experimental units which are though to affect the lifetimes of individuals. Following are some examples of covariates:

i) Treatment

In simple comparison of 2 treatment, say a new treatment with a central or standard treatment. We consider a binary covariate;

$$z = \begin{cases} 1; & \text{if an individual recieve a new treatment} \\ 0; & \text{if an individual recieve control or standard treatment} \end{cases}$$

ii) Intrinsic Properties

Explanatory variables or covariates, measuring intrinsic properties of the individual, include (in medical context) such variables as sex, age on entry in the medical trial and variables describing medical history before admission to the study.

iii) **Exogeneous Variables**

This type of covariates exhibits environmental features of the problem, for example, grouping of individuals according to observers or apparatus, month in which the experiment was carried out, etc.

The covariates could be constant over time or dependent on time. For example,

i) Suppose that a treatment is applied at time $t_0>0$. Then one can incorporate a time dependent binary covariate $(Z(t))$ defined as

$$Z(t)=\begin{cases} 0 \text{ if } t<t_0 \\ 1 \text{ if } t\geq t_0 \end{cases}$$

ii) In some industrial applications, a time varying stress may be applied. So, the covariate process will be the entire history of the stress process.

Model formulation

Suppose that for every individual there is defined a vector $\mathbf{z}$ of covariates, such models are developed in two parts.

i) A model of distribution of lifetime where $\mathbf{z} = 0$, which may be called baseline model

ii) A representation (link function) of the changes introduced by the non-zero vector $\mathbf{z}$.

4.4.1 Cox's Proportional Hazards Model with one covariate

The simplest form of cox proportional hazard model for single covariate x is given by:

- $h(t,\mathbf{z})=h_0.\Psi(\mathbf{z})=h_0.\exp(\beta x)$
- $\Psi(\mathbf{z})=$ link function bringing in the covariate;
- $h_0(t)$ = baseline hazard rate (Hazard rate of control grp)
- $\Psi(\mathbf{z})$ satisfies, $\Psi(0)=1$ and $\Psi(\mathbf{z})>0\ \forall\ \mathbf{z}$

4.4.2 Cox's Proportional Hazards Model with several covariates

The simplest form of proportional hazard model is,

$$h(t,\mathbf{z})=h_0.\Psi(\mathbf{z})$$

$\Psi(\mathbf{z})=$ link function bringing in the covariate;

$h_0(t)$ = baseline hazard rate (Hazard rate of control grp).

$\Psi(z)$ satisfies, $\Psi(0)=1$ and $\Psi(z)>0\ \forall\ z$

The following two forms of Ψ are commonly used;

i) $\Psi(z,\beta)=\exp(\beta' z)$ (log linear form);

ii) $\Psi(z,\beta)=1+\beta' z$ (linear form)

We shall consider $h_0(t)$ unknown and the covariates as fixed quantities, thus leading to semi-parametric model.

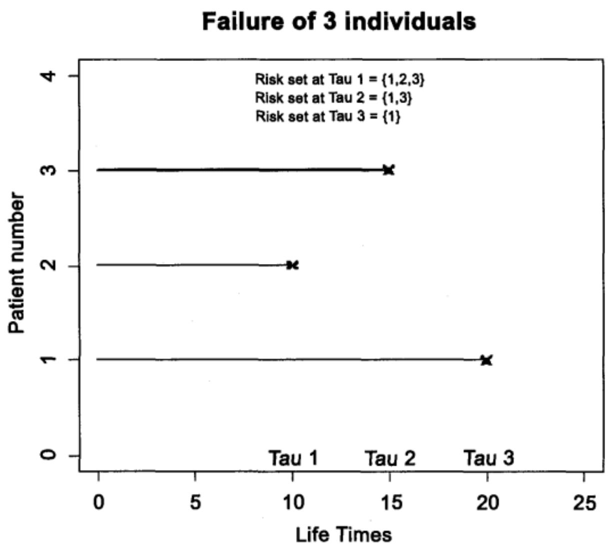
Complete data Modelling

Let $t_1, t_2, \ldots\ldots, t_n$ be the observation and $\zeta_1<\zeta_2<\ldots<\zeta_n$ denote ordered failure times of n-individuals. Let l_i be the label of the subject which fails at ζ_j. Thus, $l_j=i$ iff $t_i=\zeta_j$.Let $R(\zeta_j)$ be the risk set at ζ_j. i.e., $R(\zeta_j)=\{i;t_i\geq\zeta_j\}$.

For example: $R(\zeta_1)=\{1,2,3\}$

$R(\zeta_2)=\{1,3\}$, $R(\zeta_3)=\{1\}$

Individual i	failure time t_i	j	$R(\tau_j)$
1	20	3	1
2	10	1	1,2,3
3	15	2	1,3



Likelihood Function

The basic principle of the derivation of the likelihood is as follows. The $\{\tau_j\}$ and $\{l_j\}$ are jointly equivalent to the original data, namely the unordered failure times t_i. In the absence of knowledge of $h_0(t)$, the τ_j can provide little or no information about β as their distribution depends heavily on $h_0(t)$.

The conditional probability that $l_j=i$ given $H_j=\left\{\zeta_1, \ldots\ldots, \zeta_j ;\ i_1, i_2, \ldots\ldots, i_{j-1}\right\}$

$P[l_j=i \mid H_j]$ upto j^{th} failure time ζ_j can be written as follows.

$P[l_j=i \mid H_j]$ = The conditional prob that individual i fail at ζ_j given that one individual from $R(\zeta_j)$ fails at ζ_j

$$\frac{h_i(\zeta_j)\Delta t}{\sum_{(k\epsilon R(\zeta_j))} h_k(\zeta_j)\Delta t} = \frac{h_i(\zeta_i)}{\sum_{k\epsilon R(\zeta_j)} h_k(\zeta_j)}$$

$$P[l_j=i \mid H_j] = \frac{h_0(t)\Psi(i)}{\sum_{k\epsilon R(\zeta_j)} h_0(t).\Psi(k)}$$

$$= \frac{\Psi(i)}{\sum_{k\epsilon R(\zeta_j)} \Psi(k)}$$

$$L(i_1, i_2, \ldots\ldots, i_n) = \prod_{i=1}^{n} P_j[e_j=i \mid H_j]$$

$$= \prod_{i=1}^{n} \frac{\Psi(i)}{\sum_{k\epsilon R(\zeta_j)} \Psi(k)}$$

Example: In a laboratory, test was conducted with 3 rats. First and second rats were given 100 mg dose of a drug whereas the third rat was given 50 mg dose. The purpose is to determine whether increment in dose of drug has any influence on the survival of rats. Let:

$t_1=60$ days.(100 mg dose)

$t_2=30$ days. (100 mg dose)

$t_3=50$ days. (50 mg dose)

$\zeta_1=30=t_2$

$\zeta_2=50=t_3$

$\zeta_3=60=t_1$

Let the covariate z assumes value 0 for 50 mg dose and 1 for 100 mg dose.

$t_1 \rightarrow z_1=1$

$t_2 \rightarrow z_2=1$

$t_3 \rightarrow z_3=0$

w.r.t. z_j

i	t_i	l_j	R_j
1	60	3	1
2	30	1	(x, 3, 1)
3	50	2	(1,3)

$$L(2,3,1)=\frac{\Psi(2)}{\Psi(2)+\Psi(3)+\Psi(1)} \times \frac{\Psi(3)}{[\Psi(1)+\Psi(3)]} \times \frac{\Psi(1)}{\Psi(1)}$$

Let

$$\Psi(\beta' z)=e^{\beta' z}$$

$$\Psi(1)= e^{\beta}, \quad \Psi(2)=e^{\beta}, \quad \Psi(3)=1$$

$$L(2,3,1) = \frac{e^{\beta}}{1+e^{\beta}+e^{\beta}} \times \frac{1}{1+e^{\beta}} \times 1$$

$$= \frac{e^{\beta}}{(1+e^{\beta})(1+2e^{\beta})}$$

$$\log L = \beta - \log[1+e^{\beta}] - \log(2e^{\beta}+1)$$

$$\frac{\partial \log L}{\partial \beta}=1-\frac{1}{1+e^{\beta}} \,.\, e^{\beta}-\frac{2e^{\beta}}{2e^{\beta}+1}=0$$

$$\beta= -0.347$$

$$e^{\beta} = 0.746$$

$$h(t)= \begin{cases} h_0(t) & ;z=0 \\ h_0(t)*0.706 & ;z=1 \end{cases}$$

This shows that, when z=0, i.e., dose of 50 mg is provided, the hazard rate remains same but with increment of dose (100 mg); i.e., z =1, the hazard rate decreases, and so the chances of survival increase with use of 100 mg dose.

Censored Data

Suppose that there is ‘d’ observed failures. From the sample of size ‘n’. Let the ordered observed failure times be $\zeta_1 \leq \zeta_2 \leq \ldots \leq \zeta_d$. Let $l_j = i$ if the subject I fails at ζ_j And let $R(t_j) = \{ i : t_i \geq \zeta_j \}$ be the corresponding risk set. The likelihood function can be obtained similar to the precious case. Here, H_j will also include censored observation in $(0, \zeta_j)$, as well as information regarding the failure and the combination of these condition probabilities gives the overall partial likelihood:

$$L = \prod_{i \epsilon D} \frac{\Psi(i)}{\sum_{k \epsilon R_i} \Psi(k)} = \prod_{j=1}^{d} \frac{\Psi(k)}{\sum_{k \epsilon R_i} \Psi(k)}$$

Where D is the set of complete observations.

$R_i = R(t_i)$ and $t_i^{'}$s are the unordered failure times. It may be noted here that we have omitted terms corresponding to censored observation from each risk set. Since censoring mechanism, itself does not depend on β, such terms can be ignored for the purpose of likelihood inference about β.

For the general case the log likelihood is:

$$\log(L) = \sum_{i \in D} \left[\log\psi(i) - \log \sum_{k \in R_i} \psi(k) \right] = \sum_{i \in D} L_i \qquad (4)$$

Assume that ψ(i) possesses first and second derivatives with respect to $\underline{\beta}$ for all i.

Then,

$$\frac{\delta L_i}{\delta \beta_r} = \frac{\psi_r(i)}{\psi(i)} - \frac{\sum_{k \in R_i} \psi_r(k)}{\sum_{k \in R_i} \psi(k)}$$

Where, $\psi_r(i) = \frac{\delta}{\delta \beta_r} \psi(i), r = 1, 2, \cdots, q$

Let:

$$\psi_{rs}(i)=\frac{\delta^2}{\delta\beta_r\delta\beta_s}\psi(i) \quad r=1,2,\cdots,q,s=1,2,\cdots,q$$

Then:

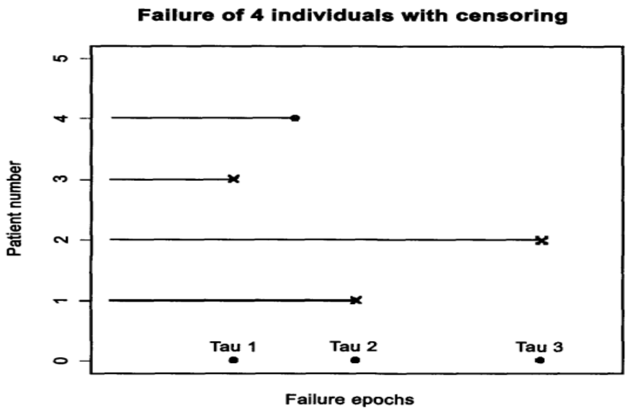
$$\frac{\delta^2 L_i}{\delta\beta_r\delta\beta_s}=\frac{\psi_{rs}(i)}{\psi(i)}-\frac{\psi_r(i)\psi_s(i)}{[\psi(i)]^2}-\frac{\sum_{k\in R_i}\psi_{rs}(k)}{\sum_{k\in R_i}\psi(k)}+\frac{\sum_{k\in R_i}\psi_r(k)\sum_{k\in R_i}\psi_s(k)}{[\sum_{k\in R_i}\psi(k)]^2}$$

For the log linear form:

$$\frac{\delta L_i}{\delta\beta_r}=z_{ir}-\frac{\sum_{k\in R_i} z_{kr}\exp[\underline{\beta}'\underline{Z}_k]}{\sum_{k\in R_i}\exp[\underline{\beta}'\underline{Z}_k]} \tag{5}$$

Where z_{ir} = value of the r-th covariate for ith subject.

Example: The following figure summarize the information regarding the failure of four individuals with censoring.



From this figure: $R_3=\{1,2,3,4\}=R(\tau_1)$

$R_1=\{1,2\}=R(\tau_2)$ (3 has failed and 4 censored before τ_2)

$R_2=\{2\}=R(\tau_3)$.

The partial likelihood is:

$$L=\frac{\Psi(3)}{\Psi(1)+\Psi(2)+\Psi(3)+\Psi(4)}\times\frac{\Psi(1)}{\Psi(1)+\Psi(2)}\times\frac{\Psi(2)}{\Psi(2)}$$

$$\Psi(i)=e^{\beta' z}=\begin{cases}1 & ; \quad i=1.\\ e^{\beta} & ; \quad i=2,3,4\end{cases}$$

$$L = \frac{e^{\beta}}{3e^{\beta}+1} \cdot \frac{1}{1+e^{\beta}} \ .$$

4.5 Rank Test for the Regression Coefficients

The likelihood function for censored case is:

$$L=\prod_{i\in D} \frac{\psi(i)}{\sum_{k\in R_i} \psi(k)}$$

$$\log L=\sum_{i\in D} \left[\log\psi(i)-\log \sum_{k\in R_i} \psi(k) \right]$$

$$=\sum_{i\in D} L_i$$

Where,

$$L_i=\left[\log\psi(i)-\log \sum_{k\in R_i} \psi(k) \right]$$

$$\frac{\partial L_i}{\partial \beta_r}=\frac{1}{\psi(i)} \cdot \frac{\partial}{\partial \beta_r} \psi(i)-\frac{1}{\sum_{k\in R_i} \psi(k)} \sum \frac{\partial}{\partial \beta_r} \psi(k); \ [r=1(1)q.]$$

$$\frac{\partial L_i}{\partial \beta_r}=\frac{\psi_r(i)}{\psi(i)}-\frac{\sum_{k\in R_i} \Psi_r(x)}{\sum_{k\in R} \psi(k)}$$

Further, $\frac{\partial^2 L_i}{\partial \beta_r \partial \beta_s}=\left\{\frac{\psi_{rs}(i)}{\psi(i)}-\frac{\psi_r(i)\psi_s(i)}{[\psi(i)]^2}\right\}-\left\{\frac{\sum_{k\in R_i} \psi_{rs}(k)}{\sum_{k\in R_i} \psi(k)}-\frac{\sum_{k\in R_i} \psi_r(k) \sum_{k\in R_i} \psi_s(k)}{(\sum_{k\in R_i} \psi(k))^2}\right\}$

Where:

$$\frac{\partial}{\partial p_r \partial \beta_s} \psi(i)=\psi_{rs}$$

$$\frac{\partial^2 L_i}{\partial \beta r \gamma \beta_s}=\frac{\left(\sum_{k\in R_i} z_{kr} e^{Rz_i'}\right)\left(\sum_{k=z_i} z_{ks} e^{\beta' z'}\right)}{(\sum_{k\in R} e^{R' z_k})^2}-\sum_{k\in R_i} \frac{e^{\beta_1 \underline{z}_k} z_{kr} z_{ks}}{\sum_{k\in R_i} e^{\beta' z_k}}$$

$$= C_{(irs)}(\beta) \ (\text{Say})$$

Suppose we intend to test H_0:β=0 vs. H_1:β≠0

The value of $U_r(0) = \sum_{i\in D} z_{ir} - \frac{\sum_{k\in R_i} z_{kr}}{n_i}$

Where n_i= # is R_i (Risk set)]

Then, $C_{(irs)}(0)=-\frac{\sum_{k\in R_i} z_{ks} z_{kr}}{n_i} + \frac{\left(\sum_{k\in F_i} z_{kr}\right)\left(\sum_{k\in R_i} z_{ks}\right)}{n_i^2}$

Thus, the approximate variance is:

$$C_{irr}=\frac{-\sum z_{kr}^2}{n_i}+\frac{\left(\sum_{k\in f_i} z_{kr}\right)^2}{n_i^2}$$

It can be assumed that, $E(u_r)=0$

Therefore:

$$x_i^2=\frac{v_r^2(0)}{\sum_{i\in n} c_i rr} \sim x_{(q)}^2.$$

Let $\psi(i) = e^{\beta' z_i}$

Then, $\frac{\partial L_i}{\partial \beta_r} = \frac{e^{\beta' z_i} * z_{ir}}{e^{\beta' z_i}} - \sum_{k\in R_i} \frac{e^{\beta_i' z_k} \cdot z_{kr}}{\sum_{k\in R_i} e^{\beta' z_k}}$

$$=z_{ir}-\frac{\sum_{k\in R_i'} e^{\beta_n' z_k} * z_{kr}}{\sum_{k\in R_i} e^{\beta_n' z_k}}$$

$z_{ir}-A_{ir}(\beta_\alpha)$

The score function is,

$$v_\gamma(\beta)=\sum_{i\in D}\frac{\partial L_i}{\partial \beta_r}=\sum_{i\in D}[z_{ir}-A_{ir}(k)]\ ;[r=1(1)q.]$$

Again,

$$\frac{\partial^2 L_i'}{\partial \beta_r \partial \beta_s}=\frac{e^{\beta' z_i} z_{ir} z_{is}}{e^{\beta_1' z_i}}-\frac{e^{\beta_2' z_i} z_{ir} e^{\beta_2' z_is'}}{[e^{\beta'^2 z_i}]^2}-\frac{\sum_{k\in R_i} e^{\beta_2' z_i'} z_{Kr} z_{kS}}{\sum_{k\in R_i} e^{\beta' z_k}}+\frac{\left(\sum_{k\in R_i} z_{kr} e^{R' z}\right)\left(\sum_{k\in R_i} z_{ks}\, e^{\beta' z_k}\right)}{\left(\sum_{k\in R} e^{\beta' z_k}\right)^2} \qquad (6)$$

4.6 Competing Risks Model

Competing Risk

In many situations there are several possible risks of modes of failure. The unique actual risk which claims the life of the unit is called the cause of failure. The risks are said to complete each other for the life the unit, hence the probabilistic models used for the life times in presence of several risks are called competing risk model. Let T be the survival time, x the covariate vector, and J the type or cause of failure. We defined a type or cause specific hazard function $h_j(t;x)$

$$h_j(t;x)=\lim_{\Delta t\to 0}\frac{P(t\leqslant T<t+\Delta t,J=j|T\geqslant t,x)}{\Delta t},j=1,\dots,m \qquad (7)$$

In words, $h_j(t;x)$ is the instantaneous failure rate of cause j at time t given x and in the presence of other (m-1) cause of failure. The overall hazard of failure is the sum of all the type-specified hazards, that is $h(t;x)=\sum_j h_j(t;x)$; provided that the failure types are mutually excluded.

The survival function:

$$S_j(t;x)=\exp[-\int_0^t h_j(u;x)du],\ j=1,\dots,m \qquad (8)$$

These functions cannot, in general be interpreted as survivorship functions when m>1.

The Model for General Competing Risks

In all the models described above, the observation consists of a positive values continuous random variable T indicating the time at which the event takes place and the outcome of event which is a discrete random variable δ, taking values 1,2…..k; assuming there are k possible outcomes, risks, component etc. therefore we need to model the probabilistic behaviour of the random pair (T,δ). If we have n independent pairs (T_i,δ_i), i=1,2…..n.

The joint distribution of (T,δ). May be specified in term of the k sub survival functions:

$$\bar{F}(t,j)=P(T>t,\delta=j),j=1,2,\cdots,k \tag{8}$$

The sub density functions $f(t,j)=-\frac{\partial}{\partial t}\bar{F}(t,j)$. The Proper density of T is then $f(t)=\sum_{j=1}^{k} f(t,j)$. Similarly, the marginal probability distribution of δ is given by $P(\delta=j)=\bar{F}(0,j)=p_j$ say. Then $\sum_{j=1}^{k} p_j=1$.

The overall hazard rate of T is $h(t)=-\frac{\partial}{\partial t}\log\bar{F}(t)=f(t)/\bar{F}(t)$ and the cause specific hazard rate is defined as $h(t,j)=\frac{f(t,j)}{\bar{F}(t)}$ leading to the relationship $h(t)=\sum_{j=1}^{k} h(t,j)$.

The cause specific hazard rates are also known as crude hazard rates. The relative risk of the jth competing risk is defined as the ratio $h(t,j)/h(t)$. This ratio is constant (independent of t) and if and only if T and δ are independent.

The constant relative risk phenomenon is also known as “proportional hazards” in the context of competing risks

The conditional survival function of the jth cause of failure as:

$$\bar{F}(t \mid j)=P(T>i \mid \delta=j]=\frac{P[T>t,\delta=j]}{P(\delta=j]}=\frac{\bar{F}(t,j)}{p_j} \tag{9}$$

Therefore, in case, T and δ are independent, $\bar{F}(t)=\bar{F}(t \mid j)$, for every j. In general:

$$\bar{F}(t)=\sum_{j=1}^{k} p_j\bar{F}(t|j) \tag{10}$$

Which is a mixture of the conditional survival function $\bar{F}(t|j)$ with weights p_j.

4.6.1 Parametric and Non-Parametric Inference

The Likelihood for Parametric Models with Independent Latent Lifetimes

Suppose that n units were under investigation and each unit faces the risk of failure due to J risk. Let

$$\delta_{ij}=\begin{cases}1; \text{ if the ith unit is falied due to jth cause}\\ 0;\text{otherwise}\end{cases}$$

$$L=\prod_{j=1}^{J}\prod_{i=1}^{n}\{f(t_i)\}^{\delta_{ij}}\{\overline{F}(t_i)\}^{j-\delta_{ij}}$$

$$=\prod_{j=1}^{J} L_j \text{ where } L_j=\prod_{i=1}^{n}\{F(t_i)\}^{\delta ij}\cdot\{\overline{F}(t_i)\}^{1-\delta ij}$$

Let us further assume that the latent lifetimes are independent exponential random variable mean $T\sim Exp(\sigma)$; $\sigma=\sigma_j$ for jth cause.

$$L_j=\prod_{i=1}^{n}\left(\frac{1}{\sigma_j}e^{-t_i/\sigma_j}\right)^{\delta_{ij}}\left(e^{-ti/\sigma_j}\right)^{1-\delta_{ij}}$$

$$=\left(\frac{1}{\sigma_j}\right)^{\sum_{i=1}^{n}\delta_{ij}} e^{-\sum_{i=1}^{n} t_i/\sigma_j}$$

$$=\left(\frac{1}{\sigma_j}\right)^{n_j} e^{-\frac{1}{\sigma_j}\sum_{i=1}^{n} t_i}$$

Let, $\sum_{i=1}^{n}\delta_{ij}=n_j$=number of units failed due to the j^{th} cause and $\sum_{i=1}^{n} t_i=t$.

Maximum Likelihood estimation:

$$\log L_j=-n_j\log\sigma_j-\frac{1}{\sigma_j}\sum_{i=1} t_i$$

$$\frac{\partial}{\partial\sigma_j}\log L_j=\frac{-n_j}{\sigma_j}+\frac{1}{\sigma_j^2}\sum_{i=1}^{n} t_i=0$$

$$\Rightarrow\frac{1}{\sigma_j^2}\sum_{i=1}^{n} t_i=\frac{n_j}{\sigma_j}$$

$$\hat{\sigma}_j=\sum_{i=1}^{n}\frac{t_i}{n_j}$$

$\hat{s}_j(t)=e^{-t/\hat{\sigma}_j}$; survival function due to jth cause and where $t\sim Weibull(p,\sigma)$

***Example*:** Suppose t_j~Weibull(p_j,σ_j) under jth cause

$$L_j = \prod_{i=1}^{n} \left\{ \frac{p_j}{\sigma_j} t_i^{p_{j-1}} e^{-t_i^{k_j}/\sigma_j} \right\}^{\delta_{ij}} \left\{ e^{-t_i^{r_j}/\sigma_j} \right\}^{1-\delta_{ij}}$$

$$= \left(\frac{p_j}{\sigma_j}\right)^{\sum_{i=1}^{n} \delta_{ij}} \prod_{i=1}^{n} (t_i^{p_j-1})^{\sum_{i=1}^{n} \delta_{ij}} * e^{-\frac{1}{\sigma_j}\sum_{i=1}^{n} t_i^{p_j}}$$

$$L_j = \left(\frac{p_j}{\sigma_j}\right)^{n_j} \prod_{i=1}^{n} (t_i^{p_j-1}) * e^{-\frac{1}{\sigma_j}\sum_{i=1}^{n} t_i^{p_j}}$$

Test for the Stochastic dominance of independent Competing risks

Let us consider the situation where only two risks are operating so the data consists of $T_1, T_2 \ldots T_n$ where $T_i = \min(X_{1i}, X_{2i})$,the lifetimes of n units and $\delta_i = I(x_{1i} > x_{2i})$, the lifetimes, the indicate of the unit that the second risk claimed the life. Let F and G be the distribution function of X_1 and X_2 respectively, we intend to test H_0:F(x)=G(x) vs. H_1:F(x)⩽G(x); i.e. second risk is more likely to be the cause of failure than the first.

We look simultaneously at the pair (T_i,δ_i) and (T_i,δ_j) and define a statistic:

$$\Phi_{i<j}(T_i,\delta_i,T_j,\delta_j) = \begin{cases} 1, & \text{if } \delta_i=1, T_i<T_j \\ 1, & \text{if } \delta_j=1, T_j<T_i \\ 0, & \text{otherevise} \end{cases}$$

Then define the statistic:

$$U^* = \frac{1}{\binom{n}{2}} \sum_i \sum_j \phi(T_i,\delta_i,T_j,\delta_j) \qquad (11)$$

Which is equal to n.

$$U^* = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} (n-R_i)\delta_i \qquad (12)$$

R_i is rank of T_i

Let us consider the statistic:

$$U=\frac{1}{\binom{n}{2}}\sum_{i=1}^{n}(n-R_i+1)\,\delta_i \qquad (13)$$

And if $S=\binom{n}{2}U$; then $E_{H_0}(s)=\frac{n(n+1)}{4}$ and $V_{H_0}(s)=\frac{n(n+1)(2n+1)}{24}$

For large n, $\frac{3}{\sqrt{n}}(u^x-\frac{1}{2})\sim N(0,1)$.

Non parametric Estimation

Consider a random sample with observed data $\{(Y_i,\delta_i,\tilde{C}_i),i=1,\ldots,k\}$, Let $0<y_1<\cdots<y_N$ be the ordered distinct observed time points. We define the following quantities:

- d_{ij} is the number of subjects failing from cause j at time y_i,
- $d_i=\sum_{j=1}^{k} d_{ij}$ are the number subjects failing at time y_i from any cause,
- $n_i=\sum_{l=1}^{n} I_l(y_i)$, with $I_l(t)=I(y_l\geq t)$ is number of individuals at risk at y_i , that is, alive and uncensored just prior to this time.

The estimate of the cause-specific hazard for cause j at time y_i is given by $\hat{\lambda}_j(y_i)=\frac{d_{ij}}{n_n}$, and it is 0 at any other time. Hence, the Nelson-Aalen estimator for the cumulative cause-specific hazard function is given by:

$$\hat{\Lambda}_j(t)=\sum_{it_i\leq t}\frac{d_{ij}}{n_i};\ j=1,\ldots,k$$

With variance estimated by:

$$\hat{Var}[\hat{\Lambda}_j(t)]=\sum_{it_i\leq t}\frac{d_{ij}}{n_i^2};\ j=1,\ldots,k.$$

The overall survival function for T can be obtained by using the Kaplan-Meier estimate:

$$\hat{S}(t)=\prod_{i:y_i<t}\left(1-\frac{d_i}{n_i}\right)^{\delta_i} \quad (14)$$

Alternative, S(t) can be obtained through $\hat{S}(t)=\exp[-\sum_{j=1}^{k}\hat{\Lambda}_j(t)]$.

Since the cumulative incidence function for cause j can be obtained from the cause specific hazard trough $F_j(t)=\int_0^t \lambda_j(u)S(u)du$, a natural non-parametric estimate of $F_j(t)$ is:

$$\hat{F}_j(t)=\int_0^t \hat{\lambda}_j(u)\hat{S}(u)du=\sum_{i:y_i\le t}\frac{d_{ij}}{n_i}\hat{S}(y_i^-);\ j=1,\ldots,k. \quad (15)$$

4.7 Multiple Decrement Life Table

When two or more mutually exclusive risks operate on the study population (see Competing Risks), one may correspondingly compute a multiple-decrement table to reflect this. For instance, a period of sickness can end in death or, alternatively, in recovery. Suppose that an integer random variable K represents the cause of decrement and define $F_k(t)=\Pr(T\le t,K=k)$, $f_k(t)=dF_k(t)/dt$ and $\mu_k(t)=f_k(t)/S(t)$, Assuming that all $F_k(\cdot)$ are absolutely continuous. Then $\mu_k(\cdot)$ is the cause-specific hazard for risk cause k and $\mu(t)=\sum_k \mu_k(t)$ is the total risk of decrement at time t. For the multiple decrement table, we define the decrement probability:

$$q_x^{(k)}=\Pr(T\le x+1,K=k|T>x)$$

$$=\int_0^1 \exp\left[[-\int_0^t \mu(x+s)ds]\right].\mu_k(x+t)dt$$

For given risk intensities, $q_x^{(k)}$ can be computed by numerical integration of the above equation. The expected number of decrements at age x as a result of cause k is $d_x^{(k)}=l_x q_x^{(k)}$. When estimates are available for the cause-specific risk intensities, one or two columns can therefore be added to the life table for each cause to include estimates of $d_x^{(k)}$ and $q_x^{(k)}$.

Several further life-table functions can be defined by formal reduction or elimination of one or more of the intensity functions in formulas like those above. In this manner, a single-decrement life table can be computed for each cause k, depicting what the normal life table would look like if cause k were the only one that operated in the study population and if it did so with the risk function estimated from the data. The purpose is to see the effect of the risk cause in question without interference from other causes. Some demographers call this abstraction the risk's pure effect. No assumption is made that in practice the total attrition risk can actually be reduced to the level of the one which is in focus or that this cause operates independently of other causes. For instance, a single-decrement life table of recovery from an illness reflects the pure timing effect of the duration structure of the intensity of recovery even though the elimination of mortality is unattainable.

A single-decrement life table is at an extreme end of a class of tables produced by deleting one (or more) of the cause intensities in formulas like those above. To obtain a cause-deleted life table, where only cause k has been eliminated, one may introduce:

$\mu_{-k}(t)=\mu(t)-\mu_k(t)$,

$$q_x^{(-k)}=\int_0^1 \exp\left[-\int_0^t \mu_{-k}(x+s)ds]\right] \mu_{-k}(x+t)dt$$

$$=1-\exp[\int_x^{x+1} \mu_{-k}(s)ds]$$

And so on, and a normal life table may be computed with $\mu(t)$ replaced by $\mu_{-k}(t)$ everywhere. A corresponding cause-deleted multiple-decrement life table may be based on reduced cause-specific decrement probabilities like:

$$\int_0^1 \exp\left[-\int_0^t \mu_{-k}(x+s)ds\right] \mu_j(x+t)dt\,;\ \text{for } j\neq k$$

Such a table would show what a normal table would look like if it were possible to eliminate cause k without changing the risk of any other cause. Again, no assumption needs to be made about the feasibility of such elimination in real life nor about cause independence. The computations are based on a pure abstraction. The interpretation for real-life applications must be based on substantive considerations and is a different matter.

4.8 Summary

The unit covers the fundamentals of survival analysis, focusing on Proportional Hazard Models, particularly the Cox Proportional Hazards Model. This model is essential for analyzing time-to-event data without assuming a specific survival distribution. It allows for the analysis of covariate effects on survival times and is useful for identifying significant prognostic factors. The unit discusses the proportionality assumption, which implies that hazard functions are proportional and time-independent among individuals. Statistical inference is based on the partial likelihood function, which considers the relative ordering of event times. Graphical and analytical tests are used to assess the proportionality assumption. The Cox model can be extended to include multiple covariates and account for competing risks. It also covers the joint distribution of survival time and cause of failure using sub-survival functions and constructing multiple decrement life tables. Multiple decrement life tables help compute survival probabilities when multiple, mutually exclusive risks affect the population, providing a comprehensive framework for understanding the impact of individual risks on survival probability.

4.9 Self-Assessment Exercise

1. Suppose $h(t|Z) = h_0(t)e^{\beta_1 Z_1+\beta_2 Z_2+\beta_3 Z_3}$ where:

   $Z_1=\begin{cases}0;\text{tr.0}\\1;\text{tr.1}\end{cases}$, $Z_2=\begin{cases}0;\text{ female}\\1;\text{male}\end{cases}$ and $Z_3=Z_1\cdot Z_2$.

   What values of β_1,β_2,β_3 correspond to

   a) treatment hazard ratio same in males as in females

   b) no treatment effect in males, but an effect in females

   c) no treatment effects

2. Following is a survival data set from 30 patients with AML (acute Myelogenous Leukaemia). Following possible prognostic factors are considered:

   $x_1:\begin{cases}1;\text{ if the patients } \geq 50 \text{ year old}\\ 0;\text{otheriwse}\end{cases}$

   $x_2:\begin{cases}1;\text{ if celluraity of marrow clot section is 100\%}\\ 0;\quad\text{otheriwse}\end{cases}$

Survival Time	x_1	x_2	Survival Time	x_1	x_2
18	0	0	8	1	0
9	0	1	2	1	1
28+	0	0	26+	1	0
31	0	1	10	1	1
39+	0	1	4	1	0
19+	0	1	3	1	0

45+	0	1	4	1	0
6	0	1	18	1	1
8	0	1	8	1	1
15	0	1	2	1	1
23	0	0	14	1	1
28+	0	0	3	1	0
7	0	1	13	1	1
12	1	0	13	1	1
9	1	0	35+	1	0

Cary out the analysis using proportional hazard's model.

3. What does the proportional hazards assumption state about the hazard ratio associated with each covariate in the Cox Proportional Hazards Model, and how can this assumption be graphically tested using survival functions?
4. Describe how the Cox model can be extended to account for competing risks in survival analysis.

4.10 References

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4.11 Further Reading

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UNIT - 5 RECURRENT EVENT SURVIVAL ANALYSIS

Structure

5.1 Introduction

Competing risks are a fundamental concept in survival analysis, particularly in scenarios where subjects are at risk of experiencing multiple, mutually exclusive failure events. This chapter provides a comprehensive overview of competing risks in survival data, emphasizing the need to understand and model the probabilities associated with each potential event. The chapter begins by discussing the model specification, which involves defining the failure time T and the cause of failure C, where the joint distribution is specified by cause-specific hazards or cumulative incidence functions. Cause-Specific Hazard Function $\lambda j(t)$ is introduced as a key component, representing the rate of occurrence of the jth failure. This function is utilized to define the Cumulative Incidence Function Fj(t), which represents the probability of a subject failing from cause j in the presence of all competing risks. The likelihood function for a sample is then presented, considering a random sample of individuals with observed data, including failure times Yi, censoring indicators δi, and causes of failure Ci. Various interpretations of probabilities in competing risk scenarios are discussed, emphasizing the distinction between survival functions and cumulative incidence functions. The chapter also delves into the interpretation of the function Sj(t) and its estimation. A key modeling approach discussed in this chapter is the Cox Proportional Hazard Model, which is used for each cause-specific hazard, treating other causes as censored observations. The model assumes a multiplicative effect of covariates on an unknown baseline hazard function and estimates parameters using the partial likelihood approach. Furthermore, the chapter explores the use of Cumulative Incidence Functions to compare the observed incidence of events from a given cause between groups. Estimates of cumulative incidence functions can be obtained using the estimated hazards from Cox's analyses and distinct failure times. The chapter also introduces Fine and Gray's Model, which fits the sub-hazard with a Cox model, considering a new function derived from the sub-distribution function for the cumulative incidence. Lastly, the chapter introduces Frailty models, which account for unobserved heterogeneity or dependencies in survival analysis. These models are particularly useful for related individuals, recurrent events, and multiple treatments. Frailty models introduce a random effect, or 'frailty,' into the hazard function, often following a gamma distribution, to capture variability. The chapter discusses the expression of hazard conditionally on frailty and the integration of unknown frailty to obtain a marginal distribution of survival times, showcasing the utility of these models in various survival analysis scenarios.

5.2 Objectives

After going through this unit, you should be able to:

- Understand the concept of competing risks in survival analysis and explain why it is important in scenarios with multiple potential failure events.
- Define and explain the model specification in competing risks, including the concepts of failure time, cause of failure, cause-specific hazards, and cumulative incidence functions.
- Interpret probabilities in competing risk scenarios, distinguishing between survival functions and cumulative incidence functions.
- Apply the Cox Proportional Hazard Model to model cause-specific hazards, estimate parameters using the partial likelihood approach, and understand its implications in competing risk analysis.
- Gain an understanding of frailty in survival analysis and its function in addressing unobserved heterogeneity or dependencies among subjects.

5.3 Competing Risks Survival Analysis

The survival data in which each subject can experience only one of different types of events over follow-up. The probabilities of these events are referred as competing risks. Competing risks occur when there are at least two possible ways that a person can fail, but only one such failure type can actually occur.

***Example*:**

a) A person can die from lung cancer or from a stroke, but not from both (although he can have both lung cancer and atherosclerosis before he dies);
b) Patients with advanced-stage cancer may die after surgery before their hospital stay is long enough for them to get a hospital infection.
c) In a clinical trial, patients with nonmetastatic limb sarcoma undergoing chemotherapy and
d) surgery might develop a local recurrence, lung metastasis, or another metastasis after follow-up.

For each of the above examples, the possible events of interest differ, but only one such event can occur per subject.

5.4 Competing Risk Events

The objective of the competing risks data is to assess the relationship of relevant predictors to the failure rate or corresponding survival probability of any one of the possible events, allowing for the competing risks of the other ways to fail.

Model Specification

Define for each individual, the pair (T, C), where T us the failure time, and C is the failure cause. T is assumed to be a continuous and positive random variable, while C takes values in the finite set {1,2....,k). Assume that the individual fails from one and only one cause. The joint distribution of (T, C) is completely specified through either the cause-specific hazards $\lambda_j(t)$ or through the cumulative incidence functions $F_j(t)$. The cause-specific hazard function for the j^{th} cause is defined as:

$$\lambda_j(t)=\lim_{\Delta t\to 0}\frac{\Pr(T<t+\Delta t, C=j|T\geq t)}{\Delta t} j=1,\ldots,k \quad (1)$$

and represents the rate of occurrence of the j^{th} failure.

The cumulative incidence function from type j failure is defined by:

$$F_j(t)=\Pr(T\leq t, C=j) j=1,\ldots,k, \quad (2)$$

and corresponds to the sub-distribution function for the probability of a subject failing from cause j in the presence of all the competing risks.

The cause-specific cumulative hazards $\Lambda_j(t)$, the overall hazard $\lambda(t)$. The overall cumulative hazard $\Lambda(t)$ and the overall survival function S(t) are defined, respectively, as:

$$\Lambda_j(t)=\int_0^t \lambda_j(t);\ j=1,\ldots,k,$$

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{\Pr(T<t+\Delta t|T\geq t)}{\Delta t} = \sum_{j=1}^{k} \lambda_j(t),$$

$$\Lambda(t) = \int_0^t \lambda(u)du = \sum_{j=1}^{k} \Lambda_j(t)\text{, and}$$

$$S(t) = \Pr(T>t) = e^{-\Lambda(t)}$$

The survival function can be factorized into the following k functions $S_j(t)=e^{-\Lambda_j(t)}$ as follows:

$$S(t)=e^{-\sum_{j=1}^{k} \Lambda_j(t)} = \prod_{j=1}^{k} e^{-\Lambda_j(t)} = \prod_{j=1}^{k} S_j(t) \quad (3)$$

The sub density function $f_j(t)$ from the j. the marginal distribution F(t) of T, and the marginal distribution of C are respectively given by:

$$f_j(t) = \frac{d}{dt} F_j(t)=\lambda_j(t)S(t),$$

$$F(t) = P(T\leq t)=\sum_{j=1}^{k} F_j(t),\ \text{and}$$

$$\pi_j(t) = \Pr(C=j) = \lim_{t\to\infty} F_j(t) j=1,\ldots,k.$$

The cumulative incidence function for cause j, $F_j(t)$, can be obtained from the cause specific hazard λ_j and the overall survival function S(t) from the relationship:

$$F_j(t)=\int_0^t \lambda_j(u)S(u)du;\ j=1,\ldots,k \quad (4)$$

Likelihood Function

Consider a random sample of n individual, $(T_1,C_1),\ldots,(T_n,C_n)$,where T_i is the time of failure and C_i is the cause of failure for subject i. For each individual, there exists a non-negative right censoring time V_i. Let $\delta_i=I(T_i\leq V_i)$ be the censoring indicator, and define $\overline{C}_i=\delta_i C_i$ is the cause of failure for failing individuals or 0 for censored individuals, The observed data for each individual are given by $Y_i=\min(T_i,V_i),\delta_i,C_i,i=1,\ldots,n$.

In the following these conditions are assumed:

1) V_i is independent of (T_i,C_i).

2) If $Y_i=T_i(T_i$ is not censored), then C_i is observed (we exclude cases when the time of failure is observed, but to information about the cause of failure is available).

3) The support of T and V are disjoint.

The likelihood function for the sample is given by:

$$L=\prod_{i=1}^{n} f_{c_i}(y_i)^{\delta_i}S(y_i)^{1-\delta_i}\prod_{i=1}^{n} Q(y_i)^{\delta_i}q(y_i)^{1-\delta_i} \qquad (5)$$

Since the censoring time V is independent from the failure time T, and their supports are disjoint (assumptions 1 and 3 respectively), the censoring terms in the likelihood do not provide information on the failure process and can be removed. The likelihood function is then proportional to:

$$L\propto L=\prod_{i=1}^{n} f_{c_i}(y_i)^{\delta_i}S(y_i)^{1-\delta_i} \qquad (6)$$

Denoted by $\delta_{ij}=I(C_i=j)$, where $\delta_i=\sum_{j=1}^{k}\delta_{ij}$. If $\delta_i=1$, then it exists some j with $\delta_{ij}=1$. From the factorization of the survival $S(t)=\prod_{j=1}^{k}S_j(t)$ and defining $g_j(t)=-S_j'(t)=\lambda_j(t)S_j(t)$. The likelihood function can be rewritten as product of k separate components for each failure cause:

$$L=\prod_{i=1}^{n}\prod_{j=1}^{k} f_j(y_i)^{\delta_{ij}}S(y_i)^{1-\delta_i}=\prod_{i=1}^{n}\prod_{j=1}^{k}(\lambda_j(y_i)S(y_i))^{\delta_{ij}})S(y_i)^{1-\delta_i}$$

$$=\prod_{i=1}^{n}\left\{\left(\prod_{j=1}^{k}(\lambda_j(y_i)S(y_i))^{\delta_{ij}})\right)\prod_{j=1}^{k}S_j(y_i)^{1-\delta_i}\right\}$$

$$=\prod_{i=1}^{n}\prod_{j=1}^{k}\left(\lambda_j\left(y_i\right)^{\delta_{ij}}\left[\prod_{l=1}^{k}S_l\left(y_i\right)^{\delta_{ij}}\right]S_j\left(y_i\right)^{1-\delta_i}\right)$$

$$=\prod_{i=1}^{n}\prod_{j=1}^{k}g_j(y_i)^{\delta_{ij}}S_j(y_i)^{1-\delta_{ij}}$$

$$L=\prod_{i=1}^{n}\left(\prod_{j=1}^{k}g_j(y_i)^{\delta_{ij}}S_j(y_i)^{1-\delta_{ij}}\right)=\prod_{j=1}^{k}L_j \tag{7}$$

The above expression provides a factorization of the overall likelihood L in terms of cause-specific likelihood L_j.

Interpretating Probabilities in Competing Risk

In classical survival analysis, a lifetime endpoint T is usually described by its survival function S(t)=P(T>t) which satisfies that S(t)=1-F(t), F(t) being its distribution function. The survival function could be derived from the hazard function of T, λ(t), by $S(t)=e^{-\int_6^t\lambda(u)du}$. And in competing risks, given the cause specific hazard for cause j $\lambda_j(t)$, a similar function $S_j(t)$ could be considered for each cause of failure: $S_j(t)=e^{-\int_6^t\lambda_j(u)du}=e^{-\Lambda_j(t)}$. The $S_j(t)$ $S_j(t)$ do not correspond to the complementary of the incidence function $F_j(t)$, that is $S_j(t)\neq F_j(t)$, neither to the joint probability of failing from cause j after t, P(T>t,C=j). This consideration led us to define two more functions that may play the role of cause-specific survivals. We define $S^*(t)$ as the complement of the cumulative incidence function $S_j^*(t)=1-F_j(t)$.

A function S(t) is a survival function if:

i) It is defined in [0, ∞)

ii) it is non-negative and non-increasing

iii) it is right-continuous,

iv) $S(0)=1$ and $\lim_{t\to\infty}S(t)=0$

Interpretation of Function $S_j^*(t) = 1\text{-}F_j(t)$

$S_j^*(t) = 1\text{-}F_j(t)$ represents the probability of not failing from cause j before t. It is not a proper survivor function because $\lim_{t\to\infty}(S_j^*(t)) = 1 - \lim_{n\to\infty} F_j(t) = 1 - P\ (C = j)$, which is strictly positive if there are at least two causes of failure.

Moreover, $S_j^*(t) = 1\text{-}F_j(t) = 1\text{-}\ F(t) + \sum_{l\neq j} F_l\ (t) = S(t) + \sum_{l\neq j} P(T\leq t, C=l)$. That is, the probability of not failing from cause j before t is the sum of the probability of having not failed for any cause by t plus the probability of having failed before t from other causes than j. This probability $S_j^*(t)$ is used to build Fine and Gray's regression model for the cumulative incidence function.

Interpretation of Function $\tilde{S}_j(t) = P(T>t, C=j)$

By analogy with the cumulative incidence functions F_j were defined, $\tilde{S}_j(t) = P(T>t, C=j)$ represents the probability of failing from cause j after t. It is not a proper survivor function because $\tilde{S}_j(0) = P\ (C = j)$, which is strictly below 1 if there are at least two causes of failure.

The relationship with $F_j(t)$ is given by:

$$\tilde{S}_j(t) = P(T>t, C=j) = P_r\{T>j|C=j\}P\ (C=j) = [1\text{-}P\{T\leq j|C=j\}]P(C=j)$$

$$= P(C=j)\text{-}P\ (T\leq t, C=j) = P(C=j) - F_j(t).$$

Hence, it behaves like a complementary probability for $F_j(t)$, complementary on the probability of failing from cause j, P (C = j). Note as well that S(t) could be decomposed in terms of $\tilde{S}_j(t)$ as follows:

$$S(t) = 1\text{-}\ F(t) = 1\text{-}\ \textstyle\sum_{j=1}^{k} F_j(t) = 1\text{-}\ \sum_{j=1}^{k} P(C=j) + \sum_{j=1}^{k} P(T>t, C=j) \ = \sum_{j=1}^{k} \tilde{S}_j(t)\ .$$

The expression of S(t) as a sum of $\tilde{S}_j(t)$ is indeed different from the alternative decomposition $S(t) = \prod_{j=1}^{n} S_j(t)$, and shows that $\tilde{S}_j(t)$ and $S_j(t)$ are different. A consistent estimate for $\tilde{S}_j(t)$ is given by:

$$\tilde{S}_j(t) = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} I(Y_i>t, C_i=j)\ ;\ \ j = 1,\ldots\ldots\ldots,k.$$

Despite these are estimable functions and could specify the competing risks model, they have been scarcely used in the competing risks literature (Peterson, 1976).

Interpretation of Function $S_j(t) = e^{-\Lambda_j(t)}$

We have come across functions $S_j(t)$ repeatedly in the previous sections. Firstly, we encountered them in the factorization of the survival function S(t) in eq. (3). Later, in the factorization of the likelihood function, where $S_j(t)$ corresponds to the survival function obtained from the cumulative hazard function $\Lambda_j(t)$, the cumulative risk when failure times from other causes are treated as censoring times. The functions $S_j(t)$ hold the mathematical properties of a survival function, however they are not survival functions of any observable random variable.

When failures from other causes are treated as censored observations, the assumption of independence between failure time and censoring time is possibly violated. Thus, only when distinct causes of failure are assumed to be independent, $1-S_j(t))$ is fully interpretable as the probability of failing from cause j if the other causes of failure were removed (Gooley et al., 1999).

Often $1 - S_j(t)$ has been used incorrectly to estimate $F_j(t)$ partly because of the availability of software to obtain the Kaplan-Meier estimate for $S_j(t)$:

$$KM_j\,(t) = \prod_{i:yi<t} \left(1-\frac{d_{ji}}{n_i}\right)^{\delta_{ij}} \tag{8}$$

Where d_{ji} , n_i and δ_{ij} are defined in equation (7) and failures from other causes are treated as censored observation. However, 1- KM_j (t) provides a biased estimate of the cumulative probability of failure from type j, $F_j(t)$ (Putter et al., 2007). This is clear intuitively since $S_j(t)$ only depends on the cause-specific hazard $\lambda_j(t)$, whereas $F_j(t)$ depends on all cause-specific causes $\widetilde{\lambda_l}(t)$, $l \in \{1,\ldots,k\}$ through the survival function S(t) . Moreover, $1 - KM_j(t)$ as an estimate of $1 - S_j(t)$ overestimates the probability of failure from cause j, $F_j(t)$. This is reasonable, because if an individual failing from other causes is treated as a censored observation, one assumes that the individual WILL fail from the cause of interest j somewhen in the future, which in some situations may be unfeasible: if an individual dies due to cancer, he/she would not certainly die (again) due to a heart attack. By censoring individuals, we expect a higher incidence of failures. In effect, there always exist $t^* > 0$ such as:

$$F_j(t^*) = \int_0^{t^*} S(u)\,\lambda_j\,(u)du < \int_0^{t^*} Sj(u)\,\lambda_j\,(u) = 1\text{-}\,S_j(t^*) \tag{9}$$

Regression Modelling of Competing Risks Events

In a survival analysis with competing risks, two different regression modelling strategies are possible, modelling the cause-specific hazards or modelling the cumulative incidence functions. When the cause-specific hazards are modelled, each hazard is analysed separately by treating individuals failing from other causes as censored observations, On the other hand, the cumulative incidence functions are used to determine factors associated to the incidence of a given cause.

Modelling the Cause-Specific Hazard $\Lambda_J(T)$

The classical regression analysis of competing risks establishes a **Cox Proportional Hazard (PH) Model** for each cause-specific hazard:

$$\lambda_j(t|Z)=\lambda_{0j}e^{\beta' Z} j=1,\ldots,k \tag{10}$$

Where Z is a p×1 vector of covariates and β_j is a p×1 vector of regression coefficients for each cause. Each cause of failure is analysed separately, treating individuals failing from other causes as censored observations. The effect of the covariates is assumed to act multiplicatively on an unknown baseline hazard function λ_{0j} . As in classical PH analysis, the validity of the models does not depend on the true form of the baseline hazard, provided the multiplicative form of the model is correct. The PH assumption is a strong one that must be carefully checked for each cause.

Estimation of the regression parameters β_j is based on the partial likelihood approach. Let's suppose that a censored random sample $(y_i,\delta_i,\delta_i c_i)$,i=1,…,n yields N distinct observed times of failure $t_1<\cdots<t_N$ and n-N censored times (no ties considered here). Consider the probability that an individual fails by cause j at time t_i , given that one of the individuals at risk (alive and uncensored) at time t_i fails by cause j:

$$\frac{e^{\beta' Z_i}}{\sum_{l=1}^{n} Y_l(t_i)e^{\beta_j' Z_l}} \tag{11}$$

Where $Y_l(t)=I(t_l \geq t)$. The partial likelihood function is defined only in the N times of failure, yielding:

$$L(\beta_1,\ldots,\beta_k)= \prod_{i=1}^{n} \prod_{j=1}^{k} \left(\frac{e^{\beta' Z_i}}{\sum_{l=1}^{n} Y_l(t_i)\, e^{\beta' Z_l}} \right)^{\delta_{ij}} = \prod_{j=1}^{k} L_j(\beta_j) \qquad (12)$$

Where $\delta_{ij}=I(C_i=j)$. The risk set can be diminished by the occurrence of an event from any cause. Maximizing each factor in eq(12) provides an estimator $\hat{\beta}_j$ consistent and d asymptotically normal under suitable conditions, and score, information and likelihood ratio statistics based on $L(\hat{\beta}_j)$ behave as if they were deduced from ordinary likelihood.

Given $\hat{\beta}_j$, the generalized Nelson-Aalen estimates for the cause-specific baseline cumulative hazard functions are:

$$\hat{\Lambda}_{0j}(t)= \sum_{i:t_l \leq t} \left(\frac{\delta_{ij}}{\sum_{l=1}^{n} Y_l(t_i) e^{\hat{\beta}' Z_l}} \right); j=1,\ldots,k \qquad (13)$$

Inference for the $\beta_j^{'}$s and for the Λ_{0j}'s can be conducted then as in the standard Cox model where a single cause of failure is considered. Overall survival and cumulative hazard functions for T given Z are obtained by:

$$\hat{S}(t|Z)=\exp\left\{- \sum_{j=1}^{k} \hat{\Lambda}_{0j}(t) e^{\hat{\beta}' Z}\right\} \text{ and } \hat{\Lambda}_j(t|Z)=\hat{\Lambda}_{0j}(t) e^{\hat{\beta}' Z}; j=1,\ldots,k.$$

The cumulative incidence function $F_j(t|Z)$ can be obtained by plugging-in the estimation in equation $F_j(t)=\int_0^t \lambda_j(u) S(u) du$; j =1,…,k.

$$\hat{F}_j(t|Z)= \int_0^t \hat{S}(u|Z) d\hat{\Lambda}_j(u|Z)$$

$$\sum_{i:t_i \leq t} \delta_{ij} \exp\left\{- \sum_{l=1}^{k} \hat{\Lambda}_{0l}(u) e^{\hat{\beta}_l' Z}\right\} \frac{e^{\hat{\beta}' Z}}{\sum_{r=1}^{n} Y_r(t_i) e^{\beta_j' Z_r}}$$

Modelling the Cumulative Incidence Functions $F_J(T)$

The modelling of the cause-specific hazards applies when the goal is to assess if a factor is associated with the risk of a specific cause of failure. However, when the goal is to compare the observed incidence of events from a given cause between groups, the cumulative incidence functions should be used. Estimates of these functions can be obtained via:

$$\hat{F}_j(t|Z)=\sum_{t_i \le t} \hat{\lambda}_j(t_i|Z)\hat{S}(t|Z) \qquad (14)$$

Where $\hat{\lambda}_j(t|Z)$ are the estimated hazards resulting from Cox's analyses, and t_i the distinct failure times. The overall survivor function is:

$$\hat{S}(t|Z)=\exp\left\{\sum_{j=1}^{k}\sum_{t_i \le t} \hat{\lambda}_j(t_i|Z)\right\} \qquad (15)$$

Fine and Gray's Model

Fine and Gray considers a new function, the sub-hazard $\gamma_j(t)$derived from the sub-distribution function:

$$\gamma_j(t \mid Z)=\lim_{\Delta t \to 0}\frac{\Pr\left(T<t+\Delta t, C=j \mid Z, T \ge t \text{ or } (T<t \text{ and } C \ne j)\right)}{\Delta t}=\frac{f_j(t|Z)}{1-F_j(t|Z)} \; j=1,\ldots,k \qquad (16)$$

This would be the hazard obtained from F_j if it were a proper distribution. The conditional expression includes two different scenarios: i) the event has not occurred at time t, ii) the event has occurred from a different cause before t. Thus, the risk set at time t is formed by two types of individuals, corresponding to the two different scenarios. Contrary to the analyses based on the cause-specific hazards, a patient failing from other causes would not be removed from the risk set at his/her time of failure. The sub-distribution function is expressed in terms of the sub-hazards as $F_j\{t|x\} = 1- \exp(-\int_0^t \gamma_j\{t|x\})$.

Fine and Gray proposed to fit the sub hazard with a Cox model, that is:

$$\gamma_j\{t|x\} = \gamma_{0j}(t)\, e^{\beta_j' x}, \quad j=1,\ldots..k \qquad (17)$$

where the covariates are linear on a complementary log-log transformed cumulative incidence function. When censoring is absent or is always observable, Fine and Gray (1999) showed that the partial likelihood approach is valid for estimation. In the case of right-censoring, they developed a weighted score function based on the non-censored case to deal with dependent censoring. If there are N failures at the times $t_1 < t_2 < \cdot\cdot\cdot < t_N$, the partial likelihood was defined by:

$$L(\beta_j)=\prod_{i=1}^{N}\left(\frac{e^{\beta_j' Z_i}}{\sum_{i\in\tilde{R}_i} w_{il} e^{\beta_j' Z_l}}\right) \quad (18)$$

Now the risk set for cause j at time t_i is $\widetilde{R_i} = \{l: t_l \geq t_i \text{ or } (t_l \leq t_i \text{ and } C \neq j)\}$ where subjects experiencing a competing cause remain in the risk set. The weight w_{il} given to such an individual is $\widetilde{G}(t_i)/\ \widetilde{G}(\min(t_l, t_i))$, where $\widetilde{G}$ is the survivor function for the censoring distribution. An individual satisfying$t_l \geq t_i$ is given a weight of 1.

5.5 Frailty Models

Frailty models represent a specific class of random effects models widely employed in survival analysis. Their core function lies in addressing unobserved heterogeneity or dependence within observational data. Unlike conventional random effects models, frailty models introduce an additional layer of random variation, termed frailty, that supplements the inherent randomness. In the context of survival analysis, frailty models offer a particularly valuable tool for modeling survival times when observations exhibit dependencies or when the data displays overdispersion relative to the baseline random variation. Their application extends to diverse scenarios, encompassing:

- **Related individuals:** This includes studies involving families, twins, or other groupings where shared genetic or environmental factors might influence survival outcomes.
- **Similar organs within an individual:** Frailty models can be used to analyze the survival of paired organs (e.g., kidneys) where the health of one might influence the other.
- **Recurrent events:** These models can account for dependencies between the times of repeated events for a single individual (e.g., hospital admissions).
- **Experimental designs involving multiple treatments:** Frailty models can be employed to account for unobserved heterogeneity among experimental units that might influence treatment response times.

By incorporating frailty, these models provide a more nuanced understanding of survival data, allowing researchers to account for unobserved factors that contribute to variability in event times.

The frailty model introduces a random effect, denoted as Y, which is assumed to follow a specific distribution, often a gamma distribution. This random effect is incorporated into the hazard function, which describes the instantaneous probability of an event occurring at a given time. Mathematically, the frailty model specifies that the hazard, conditional on the frailty, is of the form $Y^*\mu(t)$, where $\mu(t)$ represents the hazard function (or conditional hazard function) at time t. Since Y is unknown, this formulation may initially seem impractical. However, similar to other random effects models, the approach is to integrate out the random components, resulting in a marginal distribution of the survival times, which are the observed quantities.

One key advantage of the frailty model is its ability to create dependence between survival times by allowing the frailty Y to be shared among several individuals. This shared frailty can capture the effects of unobserved covariates, similar to how random effects in other models can reflect unobserved factors influencing the outcome of interest.

Purpose of a Frailty Model

One of the key characteristics of a frailty model is its capacity to model dependence among multiple time variables. This is achieved by positing that the frailty is shared across these variables. Fundamentally, the time variables are considered conditionally independent given the frailty; however, when the frailty is integrated out, the observations exhibit dependence. Consequently, the frailty methodology serves as a mechanism for inducing dependence among time variables.

Example for Multivariate Data

In this case, the frailty model is created by the conditional independence setup. It is, for example, useful to model the genetic effect in a twin. In most cases, the actual genes are not known, but it is known that identical twins have the same genes. The conditional independence assumption means that when the genes are accounted for, the survival times of twins are independent. As the genes are unknown, their effects have to be integrated out, and this implies that the times will be dependent. If some genes are known, they can be included as fixed effects. Inclusion of known covariates is therefore a key aspect of the model, and this turns the random effects model into a mixed effects model. In the twin case, it is the dependence as modelled by the frailty that is the interesting aspect of the model.

In other cases, the most interesting aspect in the study may be the effect of covariates, and the frailty is only included in order to account for the dependence. Still, the frailty model is useful and the frailty setup does the job of describing the dependence. This dependence is sometimes a nuisance, but in other cases, such as a cross-over experiment, the dependence is a design tool that reduces the unexplained random error and therefore allows for more precise evaluation of the effect of key covariates (typically, treatment).

Example for Univariate Data

While the multivariate data are the real drivers of frailty models, they may have some use even in the case of univariate (independent) data. If $\mu(t)$ is a restricted parametric model, including a frailty on top of this can create a more flexible model. This could be interpreted as a model with overdispersion compared to the model given by $\mu(t)$ but alternatively, it could be used pragmatically just as a model with more parameters than the original model.

As an example of creating a more flexible model, one can take the proportional hazards regression model. Depending on the choice of distribution for the frailty, this leads to a model with non-proportional hazards. This can be used not only to derive a test for hazard proportionality but also as a model in its own right, for use when the proportional hazards model is not fulfilled.

Model for Univariate data

The frailty model will be presented in the general case, meaning that at this stage, the calculations apply to all distributions for the frailty, of course, satisfying that $Y \geq 0$. The univariate model for the hazard is simply given by:

$$Y\mu(t)=Y\exp(\beta z)\,\mu_0(t) \tag{19}$$

Where z is a vector of covariates with corresponding regression coefficient β and $\mu_0(t)$ is the conditional hazard function corresponding to z=0. The function $\mu_0(t)$ can be parametric or non-parametric. From this expression, on can derive the conditional survival function, which in the absence of covariates is:

$$S(t \mid Y)=\exp(-YM(t)) \tag{20}$$

Where $M(t)=\int_0^t \mu(u)du$ is the integrated conditional hazard. As Y is unobserved and independent for all times, it has to be considered random and integrated out. Then, for continuous distribution $S(t)=\int_0^\infty \exp(-yM(T))g(y)dy$

Using the Laplace transformation defined as L(s)=E{exp(-sY)}, the survivor function becomes:

$$S(t)=ES(t|Y)=L(M(t)) \tag{21}$$

Using this expression, we can derive the distribution of the frailty among the survivors at time t.

Example: Assume a frailty model given by. The hazard function of the population $\mu(t)=\frac{f(t)}{S(t)}$ is generally $\mu(t)=E(\mu(t,Z)|T>t)$, or more specifically,

$$\mu(t)=\int_0^\infty \mu(t,z)f(z|T>t)dz=\mu_0(t)\int_0^\infty zf(z|T>t)dz,$$

Where $f(z|T>t)$ represents the density of frailty among the survivors of age t. Then:

$$\mu(t,z)=\frac{f(t|z)}{S(t|z)}=z\mu_0(x)$$

$$f(t|z)=z\mu_0(t)S(t|z)$$

$$f(t,z)=z\mu_0(t)S(t|z)f_Z(z)$$

$$f(t)=\mu_0(t)\int_0^\infty zS(t|z)f_Z(z)dz$$

With f_z as p.d.f. of the frailty distribution. Hence:

$$\mu(t)=\frac{\mu_0(t)\int_0^\infty zS(t|z)f_Z(z)dz}{S(t)}.$$

Because survival at age t implies an age of death greater that t, it holds that:

$$f(z,T>t)=f_Z(z)\int_t^\infty \mu_0(s)S(s|z)ds\, z=f_Z(z)S(t|z)$$

$$f(z|T>t)=\frac{f_Z(z)S(t|z)}{S(t)}$$

Shared Frailty Models for Multivariate Data

When the frailty is shared among several individuals, it leads to dependence between the times. To be more precise, conditionally on the frailty, the individuals are assumed to have independent times, modelled as described in Equation $Y^*\mu(t)$, but the frailty is shared, the actual times are dependent. Thus, one can say that the frailty generates dependence between the times. For *example*, the frailty can describe the effect of shared genes among family members. In this setup, the dependence is necessarily positive. So, the hazard function model conditional on the frailty will have the form $Y\mu_j(t)$ for the j^{th} individual, where $\mu_j(t)$ can denoted either of $u_j(t)$ (one hazard function per coordinate), $\mu(t)$(symmetric) or $\exp(\beta' z_j)\mu_0(t)$ (proportional hazards). The likelihood function can be given as:

$$\int_0^\infty y^{D_1+D_2}\mu_1(T_1)^{D_1}\mu_2(T_2)^{D_2}\exp(-y\{M_1(T_1)+M_2(T_2)\})g(y)dy \tag{22}$$

In the Bivariate Case

Laplace transform allows for direct computation of the survivor function. This is based on the bivariate conditional survivor function being of the form:

$$S(t_1,t_2|Y)=\exp(-Y\{M_1(t_1)+M_2(t_2)\}) \tag{23}$$

The integration is essential the same as in the univariate case, giving the bivariate survivor function as:

$$S(t_1,t_2)=L\{M_1(t_1)+M_2(t_2)\} \tag{24}$$

To handle possible censored data, this expression needs to be differentiated towards the coordinates, which correspond to actual events. This gives:

$$(-1)^{D_1+D_2}\mu_1(T_1)^{D_1}\mu_2(T_2)^{D_2}L^{(D_1+D_2)}(M_1(T_1)+M_2(T_2))$$

The expressions above are given using the conditional hazard function $\mu(t)$ and this is the standard way of thinking in a random effects model. In a normal distribution repeated measurements model, this is known as a “subject-specific model.” Alternatively, one might invert the expression in Equation (21) to give M(t) as function of S(t) for each coordinate and insert this in Equation (23). This gives the bivariate survivor function $S(t_1,t_2)$as function of the univariate marginal survivor functions $S_1(t_1)$and $S_2(t_2)$.Within survival data, this is known as a copula approach referring to separate modelling of dependence and marginal distributions. This would correspond to what in a normal distribution repeated measurements model is known as a “population-average model.

5.6 Summary

The unit covers the fundamental concept of competing risks in survival analysis, where subjects face multiple, mutually exclusive failure events. It emphasizes the importance of understanding and modelling the probabilities associated with each potential event. The unit discusses Model Specification, which involves defining failure time and cause of failure, and introduces Cause-Specific Hazard Function and Cumulative Incidence Function. It explores the likelihood function for a sample, interpretations of probabilities in competing risk scenarios, and the Cox Proportional Hazard Model for modelling cause-specific hazards. Additionally, the unit discusses Cumulative Incidence Functions for comparing observed event incidences between groups, introduces Fine and Gray's Model, and explores Frailty models to account for unobserved heterogeneity or dependencies in survival analysis.

5.7 Self-Assessment Exercise

1. Consider a hypothetical study of the effect of a bone marrow transplant for leukaemia on leukaemia-free survival, where transplant failures can be of one of two types: relapse of leukaemia and non-relapse death (without prior relapse of leukaemia). Suppose that in hospital A, 100 patients undergo such a transplant and that within the first 4 years post-transplant, 60 die without relapse by year 2 and 20 relapse during year 4. Suppose that in hospital B, 100 patients undergo such a transplant but post-transplant, there are 20 non-relapse deaths by year 1, 15 relapses during year 2, 40 non-relapse deaths between years 3 and 4, and 5 relapses during year 4.
   a) What are the competing risks in this study?
   b) What is the proportion of initial patients in hospitals A and B, respectively, that have leukaemia relapse by 4 years?
2. What are the key factors that influence the occurrence of competing risk events in survival analysis? What are the essential components for specifying a model for competing risk data? How does the model specification differ when dealing with cause-specific hazards and cumulative incidence functions?
3. What role does the likelihood function play in the analysis of competing risk data? How should probabilities be interpreted in the context of competing risk events?
4. What are some effective regression modelling strategies for cause-specific hazards and cumulative incidence functions?

5. What is the primary purpose of frailty models in survival analysis? Provide examples of situations in which frailty models are notably advantageous. How is the hazard function modified in a frailty model to include the frailty effect?
6. Are there any correlations or dependencies between the different variables in the multivariate data for frailty models? What is the distribution of the univariate data?

5.8 References

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5.9 Further Reading

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MSc STAT – 403 (N)A/ MA STAT – 403(N)A
Survival Analysis and Reliability Theory

Block: 2 Reliability Analysis

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MScSTAT–403NA/MASTAT–403NA **SUEVIVAL ANALYSIS & RELIABILITY THEORY**

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Block & Units Introduction

The ***Block - 2 – Reliability Theory,*** is the second block of said self-learning material (SLM), which is divided into five units.

Unit – 6: Basic Concepts; This unit lays the groundwork for understanding reliability theory by defining key concepts and terms. It covers the basic ideas of reliability, failure rates, and the probability of failure. Topics include the definitions of reliability, maintainability, and availability, and their importance in system design. The unit also introduces the mathematical tools and statistical methods used to analyse reliability data, such as probability distributions and failure rate functions.

Unit – 7: Ageing; In this unit, the focus is on the phenomenon of ageing and its effects on system reliability. It explores the different stages of the life cycle of components and systems, including the infant mortality period, normal life, and wear-out phase. The unit discusses how ageing affects the failure rate and the reliability of components over time. It also covers models that describe ageing processes, such as the bathtub curve, and methods to assess and mitigate the impact of ageing on system performance.

Unit – 8: Reliability Estimation; This unit is dedicated to the techniques and methodologies for estimating the reliability of systems and components. It includes statistical methods for analysing reliability data, such as life data analysis, censored data analysis, and reliability testing. The unit also discusses the use of reliability block diagrams and fault tree analysis to model and estimate system reliability. Practical aspects of conducting reliability tests and interpreting the results to make informed decisions about system design and maintenance are also covered.

Unit – 9: Repairable Systems; This unit examines systems that can be repaired and restored to operational condition after experiencing failures. It covers the concepts of repair and maintenance, including preventive and corrective maintenance strategies. The unit introduces models for analysing the reliability of repairable systems, such as the renewal process, Markov chains, and availability models. It also discusses the impact of repair policies on system reliability and performance, and methods to optimize maintenance schedules to enhance system reliability.

Unit – 10: Growth Models and Accelerated Life Testing; In this unit, growth models and accelerated life testing techniques are explored. Growth models describe how the reliability of a system improves over time as a result of testing and corrective actions. The unit covers various reliability growth models, such as the Duane model and the AMSAA model. Accelerated life testing involves subjecting products to higher stress levels than normal to induce failures quickly and gather reliability data in a shorter period. The unit discusses the principles and methodologies of accelerated life testing, including stress testing, data analysis, and extrapolation of results to normal operating conditions.

At the end of every unit, the summary, self-assessment questions, and further readings are given.

UNIT - 6 BASIC CONCEPTS

Structure

6.1 Introduction

When we buy any product like electric lamp or car, we expect it to function properly for a reasonable period of time. When a new ceiling fan is floated in market, the customers would like to know about the average life of the fan. Another related queries may be about the failure pattern of units or about the probability of its satisfactorily functioning for a desired duration. Life testing experiments are conducted to answer such questions.

In a life testing experiment a number of units are subjected to test and the test is terminated according to the considered test plan. The data (sample) then consists of lifetimes of the units put to test. The sample is called a complete sample if the test is terminated after the failure of all the units put to test. However, sometimes due to time or cost constraints the experiment is terminated before the failure of all the units. Such experiment are termed as censored experiments and give rise to censored data. Consequently, the inferential procedures in reliability theory are developed in presence of censored data.

6.2 Objectives

After going through this unit, you will be able to know

- the reliability and related measures
- Basics of various system configurations
- Coherent systems
- Paths and cuts and their importance
- Modular decomposition of a system

6.3 Reliability Measures

The notion of reliability and some related concepts are defined as follows:

6.3.1 Reliability

Definition: Reliability of a unit at time t_0 is the probability that a unit will perform its intended function for a mission time t_0, under the stated operating conditions. Let X denote the lifetime of the unit, then the reliability of unit at time t_0 is given by

$$R(t) = P(X > t_0)$$

6.3.2 Hazard Rate

The hazard rate is the instantaneous rate of failure. We can think of the hazard function as an item's propensity to fail in the next short interval of time, given that the item has survived to time t. Mathematically, let the random variable T denote the lifetime of a unit and F be the distribution function of T, then the hazard rate h(t) can be expressed mathematically as follows

$$h(t) = \lim_{t \to 0} \frac{P(t \leq T < t + \Delta t)}{\Delta t} \tag{1}$$

1. Increasing failure rate (IFR): the instantaneous failure rate (hazard rate) increases as a function of time. We expect to see an increasing number of failures for a given period of time.
2. Decreasing failure rate (DFR): the instantaneous failure rate (hazard rate) decreases as a function of time. We expect to see a decreasing number of failures for a given period of time.
3. Bathtub failure rate (BFR): the instantaneous failure rate (hazard rate) begins high because of early failures ("infant mortality" or "burn-in" failures), levels off for a period of time ("useful life"), and then increases ("wear out" or "aging" failures).
4. Constant failure rate (CFR): the instantaneous failure rate (hazard rate) is constant for the observed lifetime. We expect to see a relatively constant number of failures for a given period of time.

Figure 1.1 shows four of the most common types of hazard functions.

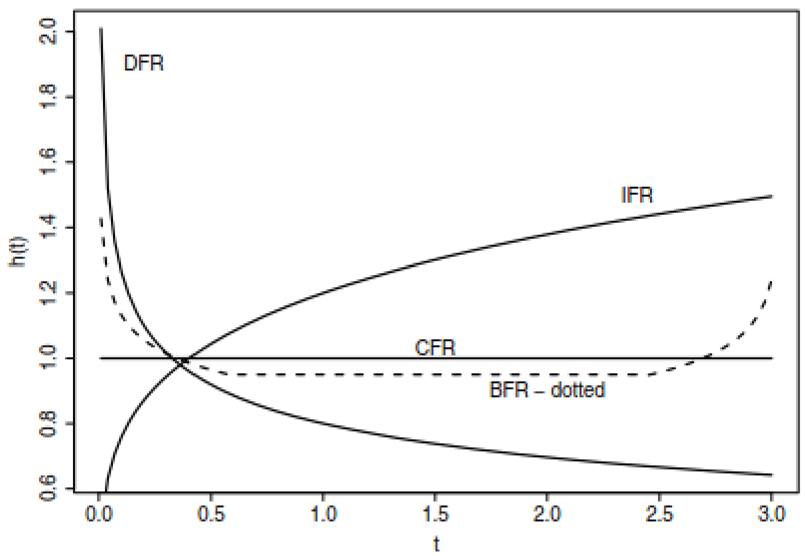


Fig. 1.1: Different plots of hazard rates. Dotted lines represent the bath-tub hazard function.

Relation between Hazard Rate and Reliability

The hazard function is defined as

$$h(t) = \lim_{t \to 0} \frac{P(t \leq T < t + \Delta t)}{\Delta t}$$

$$= \lim_{t \to 0} \frac{\mathrm{P}(\mathrm{t} \le \mathrm{T} < t + \Delta t\,,\ T > t)}{\mathrm{P}(\mathrm{T} > t)\ \Delta \mathrm{t}}$$

$$= \lim_{t \to 0} \frac{\mathrm{P}(\mathrm{t} \le \mathrm{T} < t + \Delta t)}{\mathrm{P}(\mathrm{T} > t)\ \Delta \mathrm{t}}$$

$$= \lim_{t \to 0} \frac{F(t + \Delta t) - F(t)}{1 - F(t)}$$

$$= \lim_{t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t\, R(t)}$$

$$\Rightarrow h(t) = \frac{f(t)}{R(t)} \qquad (2)$$

Further, integrating (2) from 0 to t, we have

$$\int_0^t h(u)du = \int_0^t \frac{f(u)}{R(u)} du$$

$$= \int_0^t \frac{f(u)}{1 - F(u)} du$$

Let, $1 - F(u) = z,\ \Rightarrow\ -f(u)du = dz,$

$$\int_0^t h(u)du = -\int_1^{1-F(t)} \frac{1}{z} dz$$

$$= -[logz]_1^{1-F(t)}$$

$$= -\log R(t) + 0$$

$$\Rightarrow R(t) = e^{-\int_0^t h(u)du} \qquad (3)$$

6.3.3 Mean Time to Failure (Average life)

$$E(T) = \int_0^t f(t)dt$$

$$= \int_0^\infty t dF(t)$$

$$= \int_0^\infty t d(1 - R(t))$$

$$= - \int_0^\infty t \, dR(t)$$

$$= [t\, R(t)]_0^\infty + \int_0^\infty R(t) dt$$

$$\Rightarrow E(T) = \int_0^\infty R(t) dt \qquad (4)$$

Example 1: The hazard rate of a unit is given by (i) $h(t) = \lambda$ (ii) $h(t) = \lambda t$, find the mean reliability function, *pdf f(t)* and mean life of the unit.

(i) We have from (3) that

$$R(t) = e^{-\int_0^t h(u) du}$$

$$= e^{-\int_0^t \lambda \, du}$$

$$= e^{-\lambda \, [u]_0^t}$$

$$= e^{-\lambda \, t}$$

Now, since

$$h(t) = \frac{f(t)}{R(t)}$$

or

$$f(t) = h(t). R(t)$$

We have

$$f(t) = \lambda e^{-\lambda \, t}$$

Further,

$$E(T) = \int_0^\infty R(t) \, dt$$

$$= \int_0^{\infty} e^{-\lambda t} \ dt$$

$$= \lambda$$

(ii) Again, we have from (3) that

$$R(t) = e^{-\lambda \int_0^t u.du}$$

$$= e^{\frac{-\lambda t^2}{2}}$$

$$h(t) = \frac{f(t)}{R(t)}$$

$$f(t) = h(t).R(t)$$

$$= \lambda t e^{\frac{-t^2}{2}}$$

Using (4), the mean life of the unit is given by

$$E(T) = \int_0^{\infty} R(t).dt$$

$$= \int_0^{\infty} e^{\frac{-\lambda t^2}{2}}.dt$$

$Let \ \frac{\lambda t^2}{2} = z$, that is $t = \sqrt{\frac{2z}{\lambda}}$, or $\frac{\lambda 2tdt}{2} = dz,$

therefore

$$E(T) = \int_0^{\infty} e^{-z} \frac{dz}{t\lambda}$$

$$= \frac{\sqrt{\lambda}}{\lambda} \int_0^{\infty} \frac{1}{(2z)^{1/2}} e^{-z} \ dz$$

$$= \frac{1}{\sqrt{2\lambda}} \int_0^{\infty} z^{\frac{1}{2}-1} e^{-z} \, d$$

$$= \frac{\Gamma(\frac{1}{2})}{\sqrt{2\lambda}}$$

$$= \sqrt{\frac{\pi}{2\lambda}}$$

Example 2: The hazard rate of a unit is given by $h(t) = \lambda t^{m-1}\,; m \geq 1$, find the mean reliability function, *pdf f(t)* and mean life of the unit. Using (3), the reliability function of the unit is

$$R(t) = e^{-\lambda \int_0^\infty t^{m-1} dt}$$

$$= e^{-\frac{\lambda t^m}{m}}$$

Now, using (4), the mean life of the unit is

$$E(T) = \int_0^\infty R(t)dt$$

$$= \int_0^\infty e^{-\frac{\lambda t^m}{m}} dt$$

$Let\ \frac{\lambda t^m}{m} = z$, that is $\frac{\lambda m t^{m-1}}{m} dt = dz$ or $dt = \frac{1}{\lambda t^{m-1}} dz$

$Since,\ t = \left(\frac{zm}{\lambda}\right)^{\frac{1}{m}}$, then $dt = \frac{1}{\lambda\left(\frac{mz}{\lambda}\right)^{\frac{m-1}{m}}} dz = \frac{1}{\lambda\left(\frac{mz}{\lambda}\right)^{1-\frac{1}{m}}} dz = \frac{1}{\lambda}\left(\frac{mz}{\lambda}\right)^{\frac{1}{m}-1} dz$

Finally, we have

$$E(T) = \int_0^\infty e^{-z} \frac{1}{\lambda^{1/m}} m^{\frac{1}{m}-1} z^{\frac{1}{m}-1}\, dz$$

$$= \frac{m^{\frac{1}{m}-1}}{\lambda^{1/m}}\, \Gamma(1/m)$$

Example-3: The hazard rate of a device is

$$h(t) = \begin{cases} 0, & t \leq u \\ \lambda, & t > u \end{cases}$$

Obtain the reliability function, lifetime pdf and mean life of the unit.

The reliability function of the unit can be obtained as follows:

$$R(t) = e^{-\int_0^t h(u).du}$$

$$= \begin{cases} e^{-\int_0^t 0.dt}, & t \leq u \\ e^{-\int_u^t \lambda.dt}, & t > u \end{cases}$$

$$= \begin{cases} 1, & t \leq u \\ e^{-\lambda(t-u)}, & t > u \end{cases}$$

$$f(t) = \begin{cases} \lambda e^{-\lambda(t-u)}, & t > u \\ 0, & otherwise \end{cases}$$

$$\text{Meanlife} = \int_0^\infty \text{R(t)dt}$$

$$= \int_0^u 1\, dt + \int_u^\infty e^{-\lambda(t-u)}\, dt$$

Let, $\text{t} - \text{u} = \text{x} \Rightarrow dt = dx$

$$\therefore MTTF = u + \int_0^\infty e^{-\lambda x}\, dx$$

$$= u + \frac{1}{\lambda}$$

Example-4: The hazard rate of a unit is

$(i)\ h(u) = a\, e^{-b\,u}$ (ii) $h(u) = \lambda$

Find the probability that a unit of age t will further be operative for a mission time x.

Answer:(i) We have

$R(t)$: A unit will be operative from 0 to t.

We have to obtain the conditional probability that unit will be operative from t to t+x given that it is operative at t.

$$R(x|t) = P(T > t + x|T > t)$$

$$= \frac{P(T > t + x,\ T > t)}{P(T > t)}$$

$$= \frac{P(T > t + x)}{P(T > t)}$$

$$= \frac{R(t + x)}{R(t)}$$

Now

$$R(t) = e^{-\int_0^t a\, e^{-b\,u}\, du} = e^{\frac{a}{b}[e^{-bu}]_0^t}$$

$$= e^{-\frac{a}{b}(1-e^{-b\,t})}$$

$$R(t + x) = e^{-\frac{a}{b}[1-e^{-b(\,t+x)}]}$$

Then

$$R(x|t) = e^{-\frac{a}{b}[1-e^{-b(\,t+x)}] - \frac{a}{b}(1-e^{-b\,t})}$$

$$= e^{\frac{-a}{b}e^{-\,bt}\,[1-\,e^{-b\,x}]}$$

(iii) Derive yourself.

6.4 System Configurations

In reliability analysis, we often model systems graphically. This provides a visual representation of the components and how they are configured to form a system. One of the most commonly used system representations in risk and reliability analysis is the reliability block diagram.

6.4.1 Series System

A system that functions if and only if all of its components are functioning is series system. Figure 1.2 below shows the reliability block diagram for a series system.

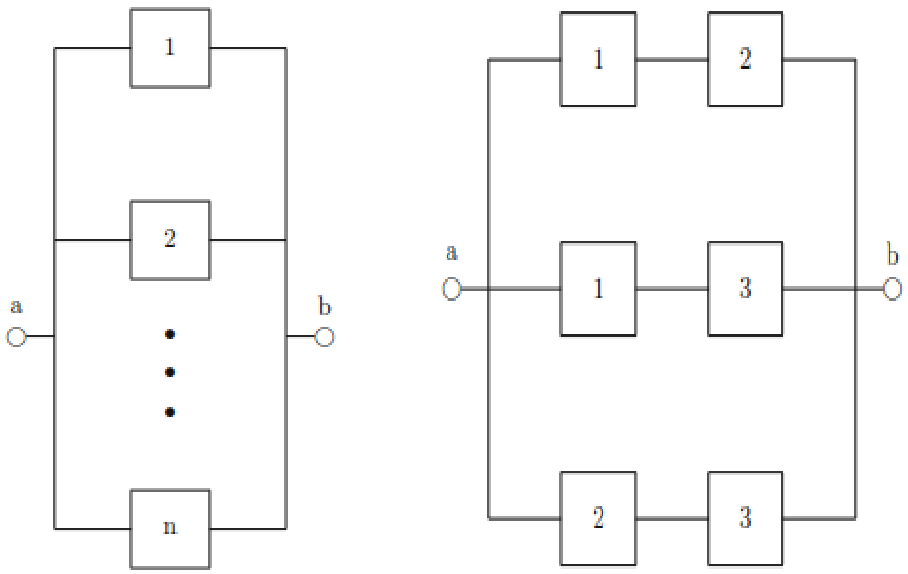
Fig. 1.2

6.4.2 Parallel System

A system that functions if at least one of its *n* components functions is called a parallel system. The first figure in Figure 1.3 shows the reliability block diagram for a parallel system.

6.4.2 k-of-n Systems

A *k*-of-*n* system functions if at least *k* of its *n* components function. Series and parallel systems are special cases of k-of-n systems. If k = n, it reduces to a series system; if *k* = 1, we have a parallel system. The second figure in Figure1.3 shows the reliability block diagram for a k-of-n system with k = 2 and n = 3.



Figures: Parallel and *k*-out of –*n* systems.

6.5 Coherent Systems

Here, we first define structure function of a system then coherent system.

6.5.1 Structure Function

Structure functions provide another way to summarize the relationships between components in a system. Consider a system with *n* components. For the i^{th} component and time t, define a random variable $x_i = X_i(t)$, so that

$$x_i = \begin{cases} 1 & \text{if the ith compenent is functioning} \\ 0 & \text{if the ith compenent is failed.} \end{cases}$$

Coherent System

A system is coherent if its structure function satisfies the following conditions:

1. φ(0, 0, . . . , 0) = 0,
2. φ(1, 1, . . . , 1) = 1,
3. φ(x) is nondecreasing in each argument.

Consider a series system, which functions if and only if all of its n components are functioning. Thus, ϕ(x) = 1 if $x_1 = x_2 = ... = x_n = 1$, and is 0 otherwise. We can write the following three equivalent expressions:

$$\phi(x) = \begin{cases} 1 \text{ if } x_i = 1 \text{ for all } i \\ 0 \text{ if } x_i = 0 \text{ for any } i, \end{cases}$$

$$= \min(x_1, x_2, \ldots, x_n),$$

$$= \prod_{i=1}^{n} x_i$$

A parallel system functions if at least one of its components is functioning. Thus, ϕ(x) = 0 if $x_1 = x_2 = ... = x_n = 0$, and is 1 otherwise. We can write the following three equivalent expressions:

$$\phi(x) = \begin{cases} 1 \text{ if } x_i = 1 \text{ for any } i \\ 0 \text{ if } x_i = 0 \text{ for all } i, \end{cases}$$

$$= \max(x_1, x_2, \ldots, x_n),$$

$$= 1 - \prod_{i=1}^{n}(1 - x_i).$$

A k-of-n system functions if k or more of its components function. We can write

$$\phi(x) = \begin{cases} 1 \; if \; \sum_{i=1}^{n} x_i \geq k \\ 0 \; if \; \sum_{i=1}^{n} x_i < k\,, \end{cases}$$

$$= \sum_j \left(\prod_{i \in A_j} x_i\right)\left[\prod_{i \in A_j^c}(1 - x_i)\right],$$

Where, A_j is any subset of {1, 2,...,n} with at least k elements, and the sum is over all such subsets. For example, the structure function for a 2-of-3 system is,

$$\phi(x) = \sum_j \left(\prod_{i \in A_j} x_i\right)\left[\prod_{i \in A_j^c}(1 - x_i)\right]$$

$$= x_1x_2(1 - x_3) + x_1x_3(1 - x_2) + x_2x_3(1 - x_1) + x_1x_2x_3$$

$$= x_1x_2 + x_1x_3 + x_2x_3 - 2x_1x_2x_3$$

6.5.2 Path and Cut of a Coherent System

In addition to reliability block diagrams and structure functions, we can use minimal path and cut sets to represent the structure of a system. We call any x for which φ(x) = 1 a path vector for the system, and any x for which φ(x) = 0 a cut vector for the structure. The set of component indices corresponding to the functioning (failed) components of a path vector (cut vector) is a path set (cut set). We denote by **y** < **x** if for all i, yi ≤ xi, and for some i, yi< xi, i = 1, . . . n.

Paths: A path vector, x, is a minimal path vector if for every y < x, φ(y) = 0. The minimal path set is the set of components in a minimal path vector that are functioning; that is, a minimal set of components such that if they are all functioning, the system is functioning, but if one of them fails (and all of the components outside the set have failed), then the system fails.

Cuts: A cut vector, x, is a minimal cut vector if for every y > x, φ(y) = 1. The minimal cut set is the set of components in a minimal cut vector that are failed, that is, a minimal set of components such that if they have all failed, the system has failed, but if one of them is functioning (and all of the components outside the set are functioning), then the system is functioning.

6.5.3 Structure function of a coherent system in terms of minimal paths and minimal cuts

We can determine the structure function of a coherent system from either its minimal path sets or its minimal cut sets. Suppose that $\{a_1, a_2, \ldots, a_m\}$is the collection of all minimal path sets of a coherent system, with x_i being the state variable of the ith component. The system is functioning if all of the components in one or more path sets are functioning. We can think of this as a parallel arrangement of m sets of components in series. In terms of the minimal path sets, the structure function of the system is

$$\varphi\ (x) = 1 - \prod_{i=1}^{m} \left[1 - \prod_{j \in a_j} x_j \right]$$

A similar result holds for cut sets. Let $\{b_1, b_2, \ldots, b_k\}$ be the collection of all minimal cut sets of a coherent system, with x_i being the state variable of the i^{th} component. The system fails if all of the components in one or more cut sets fail. We can think of this as a series arrangement of k sets of components in parallel. In terms of minimal cut sets, the structure function of the system is

$$\varphi\ (x) = \prod_{i=1}^{k} \left[1 - \prod_{j \in b_j} x_j \right]$$

Example 5.1: Using path sets and cut sets to determine a structure function. Consider the system in Fig. 5.5. The minimal path sets are $a_1 = \{1,2\}$, $a_2 = \{1,3\}$. Using Eq. 5.3, the structure function for the system is

$$\phi(x) = 1 - \prod_{j=1}^{2} \left(1 - \prod_{i \in a_j} x_i\right)$$

$$= 1 - (1 - x_1 x_2)(1 - x_1 x_3)$$

$$= x_1 x_2 + x_1 x_3 - x_1 x_2 x_3$$

The minimal cut sets for the system are $b_1 = \{1\}$ and $b_2 = \{2,3\}$. Using Eq. 5.4, the structure function for the system is

$$\phi(x) = \prod_{k=1}^{2}\left(1 - \prod_{i\in b_k}(1 - x_i)\right)$$

$$= \left(1 - (1 - x_1)\right)(1 - (1 - x_2)(1 - x_3))$$

$$= x_1(x_2 + x_3 - x_2x_3)$$

$$= x_1x_2 + x_1x_3 - x_1x_2x_3$$

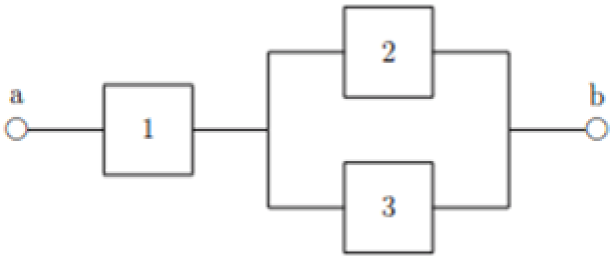


Fig. 5.5. System with minimal path sets $a_1 = \{1,2\}$ and $a_2 = \{1,3\}$.

6.5.4 Relative Importance of Components

For a given coherent system, some components are more important than others in determining whether system functions or not. For example, if a component is in series with rest of the system, then it would seem to be at least as important as any other component in the system.

First suppose we are given the state of each of the remaining components,

Then we would consider component *i* more important if $(*_j, \underline{x})$

$$\varphi(1_j, \underline{x}) - \varphi(0_j, \underline{x}) = 1 \qquad (5)$$

rather than

$$\varphi(1_j, \underline{x}) = 1 = \varphi(0_j, \underline{x}) \text{ or } \varphi(1_j, \underline{x}) = 0 = \varphi(0_j, \underline{x})$$

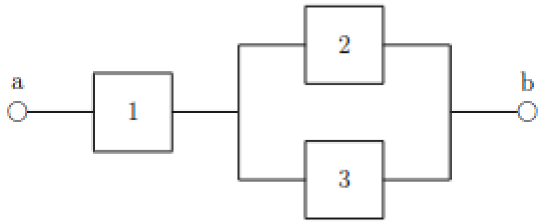
When (5) holds, we call $\varphi(1_j, \underline{x})$ a critical path vector for i^{th} component.

Let $\eta_\varphi(i) = \sum_{(\underline{x}|\underline{x}_j = 1)}[\varphi(1_j, \underline{x}) - \varphi(0_j, \underline{x})]$ denotes total number of critical paths.

Then, the relative importance of i^{th} component is

$$I_\emptyset(i) = \frac{1}{2^{n-1}}\eta_\varphi(i)$$

Example 5: Determine the importance of various components in following structure.



Here, the structure function of the system is

$$\varphi(\underline{x}) = x_1(x_2 v x_3)$$

Since, among four outcomes 100, 101, 110 and 111, there are three critical path vectors for component 1, given by (101, 110, and 111). Therefore, relative importance of component 1 is

$$I_{\emptyset}(1) = \frac{1}{2^2}3 = \frac{3}{4}$$

Similarly, for Component 2, we have

$$I_{\emptyset}(2) = \frac{1}{2^2}1 = \frac{1}{4}$$

since among four outcomes 010, 011, 110, 111 the only one critical path vector for component 2, is 110.

Here, it may be noted that *I(2)=I(3).*

6.6 Modular Decomposition

Definition: The coherent system (A, χ) is a module of the coherent system (c, φ) if

$$\varphi(\underline{x}) = \psi[\chi(\underline{x}^A), \underline{x}^A]$$

where ψ is a coherent structure function and *A⊂C* The set *A⊂C* is called modular set of (C, φ). Intuitively, a module (*A*, χ) of (*C*, φ) is a coherent sub-system that acts as if it were just a component. Knowing whether χ is 1 or 0 is as informative as knowing the value of x_i for each *i* in A, in determining the value of φ . In the usual performance diagram of a system, we can identify a module by the fact that it is a cluster of components with one wire leading into it and one wire leading out of it.

Example 6: Consider a coherent system (C, φ). having the structure function $\varphi(\underline{x}) = x_1(x_2 v\ x_3)(x_4 v\ x_5)$ and $C = \{1,\ 2,\ 3,\ 4,\ 5\}$. A module of (*C*, φ) is (*A*, χ) where $A = \{2,\ 3\}$and $\chi(\underline{x}^A) = (x_2 v\ x_3)$. We may write

$$\varphi(\underline{x}) = \psi[\chi(\underline{x}^A),\ \underline{x}^{A^c}] = x_1.\ \chi\ .(x_4 v\ x_5).$$

6.7 Summary

We have discussed various reliability measures, such as, reliability, hazard rate and meanlife of a unit. Interrelations between these functions were also obtained. Various types of hazard rates are described in detail. The notion of a coherent system is explained with example. Paths and cuts of a coherent system are discussed and representation of a coherent systems in terms of its paths and cuts is elaborated. Relative importance of components is formulated. Modullar decomposition is discussed.

6.8 Self-Assessment Questions

1. The hazard rate of a device is

$$h(t) = \begin{cases} 0, & t \leq a \\ \alpha t, & t > a \end{cases}$$

Obtain the reliability function, lifetime pdf and mean life of the unit.

2. $h(u) = \lambda$

Find the probability that a unit of age t will further be operative for a mission time x.

6.9 References

- Barlow, R.E. and Proschan, F. (1985): Statistical Theory of Reliability and Life Testing; Holt, Rinehart and Winston.
- Sinha, S.K.(1986): Reliability and Life Testing; Wiley Eastern Limited, New Delhi.

6.10 Suggested Readings

- Lawless, J.F. (1982): Statistical Models and Methods of Life Time Data; John Wiley.
- Nelson, W. (1982): Applied life Data Analysis; John Wiley.

UNIT - 7 AGEING

Structure

7.1 Introduction

By far we have built our understanding of the reliability concepts and their related measures, developed a comprehensive idea of system and components of coherent systems, understood cuts and paths, modular decomposition, discussed bounds on system reliability, and the structural and reliability importance of components. This unit will help us understand the concept of ageing, and explore how components and systems evolve, focusing on different ageing classes and their properties. These notions are essential for predicting failure rates, planning maintenance, and enhancing the overall reliability of systems.

7.2 Objectives

After going through this unit, you should be able to:

- Develop a clear understanding of ageing in reliability and its impact on system performance over time.
- Learn about various ageing classes, including IFR, IFRA, NBU, DMRL, and NBUE, and distinguish their characteristics.
- Examine the specific properties of each ageing class and their influence on failure rates and component reliability.
- Investigate the dual classes corresponding to each ageing class and understand their inverse relationships.

7.3 Concept of Ageing

Reliability finds its application in the field of engineering where units are exposed to wear, tear or shock. In this regard, the concept of ageing plays an important role in the selection of appropriate lifetime models. Before diving into the depths of understanding of ageing let us first develop the notion of the lifetime of a unit and the age of a unit in reliability analysis.

i) **Lifetime of a Unit:**

The lifetime of a unit, denoted as T, is a continuous, non-negative valued random variable that represents the total time duration for which the unit performs its appointed task satisfactorily until it transitions into a failed or dead state.

ii) Age of a Working Unit:

The age of a working unit is the elapsed time that the unit has been operating satisfactorily without failure

7.4 Some Basic Notions Related to Lifetime Random Variables

The probabilistic properties of the random variable are analysed using it's cumulative distribution function; $F(t)$, reliability function; $R(t)$, probability density function; $f(t)$ hazard function; $h(t)$, cumulative hazard function; $H(t)$ and mean residual life function at age x; $M_F(t)$.

i) **Cumulative Distribution Function:**

The cumulative distribution function (CDF) gives the probability that the event has occurred by time t. $F(t)$ is given by

$$F(t) = P[X \leq t],\ t \geq 0 \tag{7.1}$$

This function represents the probability that the event will have occurred by time t.

ii) **Reliability Function:**

The reliability function, represents the probability that the event has not occurred by time t, meaning the system or component is still functioning. The reliability function $R(t)$ is the complement of the CDF and is given by

$$R(t) = 1 - F(t) = P[X > t],\ t \geq 0 \tag{7.2}$$

iii) Probability Density Function:

The probability density function (PDF) provides the likelihood that the event occurs at a specific time t. It is denoted by $f(t)$ and is given as

$$f(t) = \frac{d}{dt}F(t) = -\frac{d}{dt}R(t) \tag{7.3}$$

(when it exists)

iv) Hazard Function (Failure Rate Function)

The hazard function, or failure rate function, represents the instantaneous rate of failure at any given time t, given that the system has survived up to that point. The hazard function $h(t)$ is given by

$$h(t) = \lim_{0<a\to 0} \frac{1}{a} P[\, t < T \leq t + a \mid T > t\,]$$

$$= \lim_{0<a\to 0} \frac{P[t > T] - P[t \geq t + a]}{aP[T > t]} \quad \text{[by the definition of conditional probability]}$$

$$= \lim_{0<a\to 0} \frac{R(t) - \mathrm{R}(t+a)}{a\mathrm{R}(t)}$$

$$= -\frac{1}{R(t)} \lim_{0<a\to 0} \frac{R(t+a) - \mathrm{R}(t)}{a} \quad [\because R(t) \text{ is independent of a}]$$

$$= \frac{1}{R(t)}\left(-\frac{d}{dt}R(t)\right) \tag{7.4}$$

$$= \frac{f(t)}{R(t)} \qquad \text{[from (7.3)] (7.5)}$$

The above equation [from (7.3)] (7.5) will be true provided $F(t) < 1$, and $f(t)$ exists.

Conversely, $h(t) = \frac{1}{R(t)}\left(-\frac{d}{dt}R(t)\right)$ shows that

$$h(t) = -\frac{d}{dt}\ln\left(R(t)\right)$$

integrating on both w.r.t t we obtain

$$\ln\left(R(t)\right) = -\int_0^t h(u)du$$

$$R(t) = \exp\left\{-\int_0^t h(u)du\right\} \qquad (7.6)$$

v) Cumulative Hazard Function

$H(t)$ is given by

$$H(t) = \int_0^t h(u)du,\ t \geq 0 \qquad (7.7)$$

Therefore using (7.7) in (7.6) we obtain

$$R(t) = \exp[-H(t)] \qquad (7.8)$$

vi) Mean Residual Life Function at Age x

Let a unit be of age x, i.e., it has been under "operating state" at time x. Since the unit has not yet failed it has a certain amount of residual lifetime. Let T_x be the residual lifetime and $\bar{F}_x$ be its survival function.

$$R_x(t) = P[T_x > t]$$
$$= P[T > x + t \mid T > x]$$
$$= \frac{P[T > x + t]}{P[T > x]} \qquad \text{[by the definition of conditional probability]}$$
$$= \frac{R(x + t)}{R(x)} \qquad (7.9)$$

Then the mean residual life function, $M_F(x)$is given by

$$\begin{aligned} M_F(x) &= E[T_x],\ x \geq 0 \\ &= \int_0^\infty R_x(u)du,\ x \geq 0 \\ &= \int_0^\infty \frac{R(x+u)}{R(x)}du,\ x \geq 0 \end{aligned} \tag{7.10}$$

This gives,

$$M_F(0) = E(T_0) = E(T) = \mu \tag{7.11}$$

And

$$h(t) = \frac{[1 + M'(x)]}{M(x)} \tag{7.12}$$

vii) Equilibrium Distribution Function

Suppose identical units are put into operation consecutively, i.e. a new unit is put in operation immediately after the failure of the one in operation. The lifetimes of these units are assumed to be independent identically distributed random variables (i.i.d.r.v.s), with distribution function F. Let us consider the residual lifetime of a unit in operation at time t as $t \to \infty$. The distribution function of this lifetime is called the equilibrium distribution function, say EQ_F. From renewal theory we have

$$EQ_F(t) = \frac{1}{\mu}\int_0^t R(u)du,\ \mu = E(T) = \int_0^\infty R(u)du \tag{7.13}$$

It can be verified that EQ_F is a proper distribution function. Let

$$\begin{aligned} h_{EQ}(t) &= \text{ failure rate of equilibrium distribution.} \\ &= \frac{R(t)}{\bar{EQ}_F(t)} \cdot \frac{1}{\mu} \end{aligned} \tag{7.14}$$

Then

$$h_{EQ}(0) = \frac{1}{\mu} \tag{7.15}$$

And

$$R(t) = \frac{h_{EQ}(t)}{h_{EQ}(0)} \exp\left\{-\int_0^t h_{EQ}(u)du\right\} \tag{7.16}$$

All the above functions clearly show one-to-one correspondence. A model uses the function which brings out the interesting properties most clearly.

7.5 Some Basic Lifetime Distributions

In this unit, we will explore some fundamental probability distributions that are widely used in reliability engineering and various other fields. These distributions—exponential, Weibull, gamma, and log-normal—are essential tools for modelling and analysing the lifetimes of systems and components. Each distribution has unique characteristics that make it suitable for different types of data and scenarios. Understanding these distributions will provide a solid foundation for analyzing the reliability and performance of systems, helping to predict failure rates, assess risks, and make informed decisions.

7.5.1 Exponential Distribution

The exponential distribution is a fundamental probability distribution used extensively in reliability theory and various fields of engineering and science. It is particularly significant for modelling the time until an event occurs, such as the failure of a component or system. The exponential distribution is characterized by its simplicity and its relationship with the Poisson process, where it models the time between events occurring independently and at a constant average rate.

The exponential distribution is unique in its memoryless property, which implies that the probability of an event occurring in the future is independent of the past. This property makes the exponential distribution particularly useful in situations where the time since the last event does not influence the likelihood of the next event.

(a) Probability Density Function (PDF)

A positive valued random variable X is said to follow exponential distribution if its probability density function is given by

$$f(x;\sigma) = \frac{1}{\sigma} exp\left\{-\frac{x}{\sigma}\right\}; \quad x > 0,\ \sigma > 0. \tag{7.17}$$

(b) Cumulative Distribution Function (CDF)

Using (7.17) in (7.1) $F(t)$ follows

$$F(t) = \int_0^t \frac{1}{\sigma} exp\left\{-\frac{x}{\sigma}\right\} dx$$

$$= \frac{1}{\sigma}\int_0^t \; exp\left\{-\frac{x}{\sigma}\right\} dx$$

$$= 1 - exp\left\{-\frac{t}{\sigma}\right\} \qquad (7.18)$$

(c) <u>Reliability Function</u>

Substituting the CDF for the exponential distribution in (7.2)

$$R(t) = \int_t^\infty \frac{1}{\sigma} exp\left\{-\frac{x}{\sigma}\right\} \; dx$$

$$= \int_{\frac{t}{\sigma}}^\infty e^{-u} \; du$$

$$= \lambda(e^{-u})_{\frac{t}{\sigma}}^{\infty} \qquad (7.19)$$

$$= e^{-\frac{t}{\sigma}}$$

Conversely on using (7.18) in (7.2) we have

$$R(t) = 1 - F(t) = 1 - \left(1 - e^{-\lambda t}\right) = e^{-\lambda t}$$

The reliability function shows an exponential decrease over time, reflecting the decreasing probability that the system or component will continue to function as time progresses.

(d) <u>Hazard Function</u>

For the exponential distribution, the hazard function $h(t)$ is derived using (7.17) and (7.19) in [from (7.3)] (7.5) as:

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\sigma} exp\left\{-\frac{t}{\sigma}\right\}}{exp\left\{-\frac{t}{\sigma}\right\}} = \frac{1}{\sigma}. \qquad (7.20)$$

Therefore h(t) is constant $= \frac{1}{\sigma}$ indicating that the failure rate remains the same over time. This constant hazard rate is a defining feature of the exponential distribution, making it suitable for modelling scenarios where the likelihood of an event occurring is consistent throughout the observation period.

$$Mean\ life = E(X) = \int_0^\infty x\ \frac{1}{\sigma} exp\left\{-\frac{x}{\sigma}\right\} dx$$
$$= \int_0^\infty y\ e^{-y} dy = \sigma.$$

Note: Sometimes, the exponential distribution is defined by the parameter, λ, known as the rate parameter. This parameter represents the rate at which events occur and is the reciprocal of the mean time to failure (MTTF). Mathematically, if the mean time between events (such as failures) is μ, then the rate parameter is given by $\lambda = \frac{1}{\mu}$.

7.5.2 Weibull Distribution

The Weibull distribution is a versatile probability distribution widely used in reliability engineering and lifetime data analysis. Introduced by Waloddi Weibull in 1951, it can model various types of failure rates—decreasing, constant, or increasing—depending on its shape parameter. This flexibility makes it ideal for analysing the life expectancy of products and systems, capturing behaviours like early-life failures (burn-in), random failures, and wear-out failures.

Its broad applicability across industries like manufacturing, aerospace, and electronics stems from this adaptability. By adjusting its parameters, the Weibull distribution can accurately reflect different real-world scenarios, making it a fundamental tool in predicting product lifespan and system reliability.

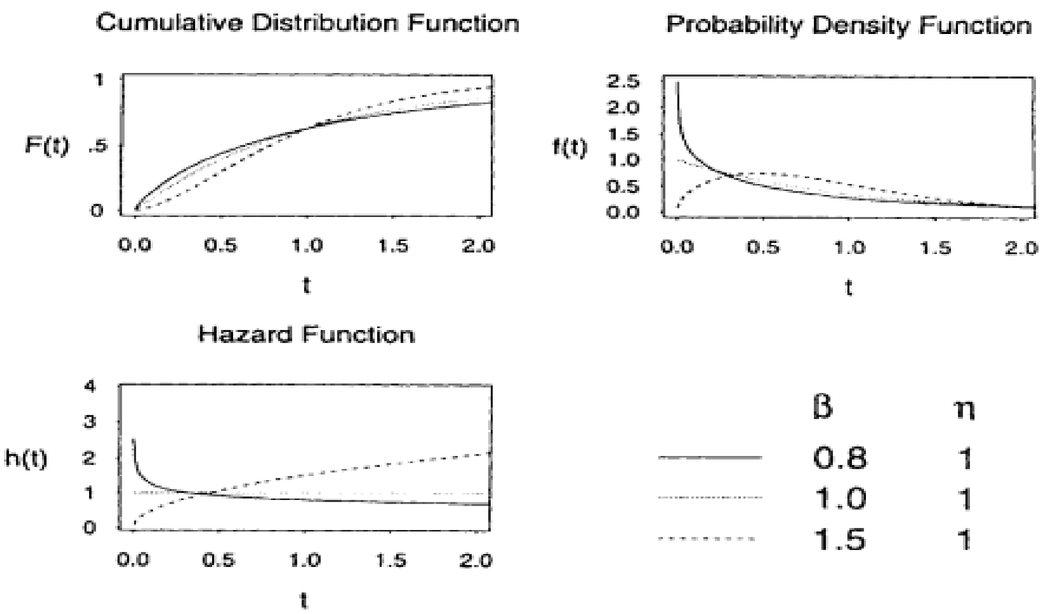


Fig. CDF, pdf and hazard rate functions of Weibull distribution ($p = \beta,\ \sigma = \eta$).

$$Mean\ life = E(X) = \int_0^{\infty} x\ \frac{p}{\sigma} x^{p-1}\, exp\left\{-\frac{x^p}{\sigma}\right\} dx$$

$$= \int_0^{\infty} \sigma^{1/p} y^{1/p}\ e^{-y} dy$$

$$= \sigma^{1/p} \int_0^{\infty} y^{\frac{p+1}{p}-1}\ e^{-y} dy$$

$$= \sigma^{1/p} \Gamma\left(\frac{p+1}{p}\right).$$

(a) <u>Probability Density Function (PDF)</u>

A positive valued random variable X is said to follow Weibull distribution if its probability density function is given by

$$f(x; p,\ \sigma) = \frac{p}{\sigma} x^{p-1}\, exp\left\{-\frac{x^p}{\sigma}\right\}; \quad x > 0,\ p,\ \sigma > 0. \qquad (7.21)$$

where p is shape parameter and σ is scale parameter.

(b) <u>Cumulative Distribution Function (CDF)</u>

To derive the CDF, integrate the PDF over the interval from 0 to t:

$$F(t) = \int_0^{t} f(x; p, \sigma\ )\, dx$$

$$= \int_0^{t} \frac{p}{\sigma} x^{p-1}\, exp\left\{-\frac{x^p}{\sigma}\right\} dx$$

Let $\frac{x^p}{\sigma} = y\ then\ \frac{px^{p-1}}{\sigma} dx = dy$. Thus (7.21)

$$F(t) = \int_0^{\frac{t^p}{\sigma}} e^{-y}\, dy = 1 - e^{-\frac{t^p}{\sigma}}.$$

(c) <u>Reliability Function</u>

Substituting the CDF (7.21) for the Weibull distribution in (7.2)

$$R(t) = e^{-\frac{t^p}{\sigma}} \qquad (7.22)$$

(d) <u>Hazard Function</u>

For the Weibull distribution, the hazard function $h(t)$ is derived using **Error! Reference source not found.** and (7.22) in [from (7.3)] (7.5) as:

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{p}{\sigma} t^{p-1} \, exp\left\{-\frac{t^p}{\sigma}\right\}}{exp\left\{-\frac{t^p}{\sigma}\right\}} = \frac{p}{\sigma} t^{p-1}. \tag{7.23}$$

Therefore:

When p>1, h(t) is an increasing function of t.

When p=1, h(t) is constant $= \frac{1}{\sigma}$.

When p<1, h(t) is a decreasing function of t.

This function helps in understanding the failure behaviour over time, which can either increase, decrease, or remain constant depending on the shape parameter α.

7.5.3 Gamma Distribution

The Gamma distribution is a continuous probability distribution that plays a significant role in various fields, including reliability engineering, queuing theory, and Bayesian statistics. It is particularly useful for modelling the time until the occurrence of an event, such as the failure of a system or the arrival of the nth event in a Poisson process. The distribution is characterized by its two parameters: the shape parameter (α) and the rate parameter (β). These parameters allow the Gamma distribution to model a wide range of behaviors, from exponential-like distributions (when $\alpha = 1$) to distributions that can account for varying rates of occurrence.

The Gamma distribution is flexible and can model scenarios where the event rate changes over time, making it ideal for representing life data and waiting times. In reliability engineering, it is often used to model systems where the failure rate changes as the system ages or undergoes different operational phases. The distribution's versatility and ability to adapt to various shapes make it an essential tool for analysing and predicting the lifespan and performance of components and systems across different industries.

(a) <u>Probability Density Function</u>

The PDF of the Gamma distribution is given by:

$$f(t; \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}, \qquad t \geq 0 \tag{7.24}$$

where:

- $\alpha > 0$ is the shape parameter,
- $\beta > 0$ is the rate parameter (inverse of the scale parameter),
- t is the time or variable of interest,
- $\Gamma(\alpha)$ is the Gamma function, defined as $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx$.

(b) <u>Cumulative Distribution Function</u>

To derive the CDF, integrate the PDF from 0 to t:

$$F(t) = \int_0^t f(u; \alpha, \beta) \, du$$
$$= \int_0^t \frac{\beta^{\alpha}}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u} \, du$$

Let $\beta u = k \implies \beta du = dk$ and the limits for k will be

for $u = 0; k = 0$ and $u = t; k = \beta t$

$$= \int_0^{\beta t} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\frac{k}{\beta}\right)^{\alpha-1} e^{-k} \frac{1}{\beta} dk$$
$$= \int_0^{\beta t} \frac{k^{\alpha-1}}{\Gamma(\alpha)} e^{-k} \, dk$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^{\beta t} k^{\alpha-1} e^{-k} \, dk$$
$$= \frac{\gamma(\alpha, \beta t)}{\Gamma(\alpha)} \tag{7.25}$$

where $\gamma(\alpha, \beta t)$ is the lower incomplete Gamma function, defined as $\gamma(\alpha, \beta t) = \int_0^{\beta t} k^{\alpha-1} e^{-k} \, dk$.

(c) <u>Reliability Function</u>

Substituting the CDF (7.26) for the Gamma distribution in (7.2)

$$R(t) = 1 - F(t) = 1 - \frac{\gamma(\alpha, \beta t)}{\Gamma(\alpha)}. \qquad (7.26)$$

(d) <u>Hazard Function</u>

For the Gamma distribution, the hazard function $h(t)$ is derived by substituting (7.24) and (7.26) in [from (7.3)] (7.5) as:

$$h(t) = \frac{f(t)}{R(t)} = \frac{\beta^{\alpha} t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha) - \gamma(\alpha, \beta t)} \qquad (7.27)$$

The hazard function of the Gamma distribution can take various shapes depending on the value of α, which makes it useful for modeling different types of time-to-failure data.

7.5.4 Lognormal Distribution

The Lognormal distribution is a continuous probability distribution widely used to model data that are positively skewed and cover several orders of magnitude. It arises when a variable is the product of many independent, positive factors, making it ideal for situations where the logarithm of the data follows a normal distribution. Common applications include modelling the distribution of income levels, stock prices, and the lifespan of products in reliability engineering.

This distribution is particularly valuable because it can capture the asymmetric nature of many real-world phenomena, where data is non-negative and exhibits a long right tail. For instance, in reliability engineering, the Lognormal distribution is used to model the time to failure of products, especially when failure rates increase initially and then stabilize. Its ability to represent such varied behaviors makes the Lognormal distribution a crucial tool in statistical analysis and risk assessment.

(a) <u>Probability Density Function</u>

The PDF of the Lognormal distribution is given by:

$$f(t; \mu, \sigma) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln t - \mu)^2}{2\sigma^2}\right), \qquad t > 0 \qquad (7.28)$$

where:

- μ is the mean of the logarithm of the variable,
- $\sigma > 0$ is the standard deviation of the logarithm of the variable,
- t is the time or variable of interest.

(b) Cumulative Distribution Function

The CDF of the Lognormal distribution is derived by integrating the PDF

$$F(t) = \int_0^t f(u;\mu,\sigma)\,du$$

$$= \int_0^t \frac{1}{u\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln u - \mu)^2}{2\sigma^2}\right) du$$

Let $\frac{\ln u - \mu}{\sigma\sqrt{2}} = k \implies \frac{1}{u\sigma\sqrt{2}} du = dk$ and the limits for k will be

for $u = 0; k = \infty$ and $u = t;\ k = \frac{\ln t - \mu}{\sigma\sqrt{2}}$

$$= \int_{\infty}^{\frac{\ln t-\mu}{\sigma\sqrt{2}}} \frac{1}{\sqrt{\pi}} \exp(-k^2)\,dk$$

$$= -\int_{\frac{\ln t-\mu}{\sigma\sqrt{2}}}^{\infty} \frac{1}{\sqrt{\pi}} \exp(-k^2)\,dk$$

$$= -\left(\frac{1}{2}\int_0^{\infty} \frac{2}{\sqrt{\pi}} \exp(-k^2)\,dk - \frac{1}{2}\int_0^{\frac{\ln t-\mu}{\sigma\sqrt{2}}} \frac{2}{\sqrt{\pi}} \exp(-k^2)\,dk\right)$$

We have an error function $\operatorname{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} \exp(-k^2)\,dk$, using this error function in above equation we have

$$= \frac{1}{2}\lim_{x\to\infty} \operatorname{erf}(x) + \frac{1}{2}\operatorname{erf}\left(\frac{\ln t - \mu}{\sigma\sqrt{2}}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{\ln t - \mu}{\sigma\sqrt{2}}\right) \quad (7.29)$$

(c) Reliability Function

Substituting the CDF (7.29) for the Lognormal distribution in (7.2)

$$R(t) = 1 - F(t) = \frac{1}{2} - \frac{1}{2}\text{erf}\left(\frac{\ln t - \mu}{\sigma\sqrt{2}}\right) \tag{7.30}$$

(d) <u>Hazard Function</u>

For the Gamma distribution, the hazard function $h(t)$ is derived by substituting (7.28) and (7.26) in [from (7.3)] (7.5) as:

$$h(t) = \frac{f(t)}{R(t)} = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln t - \mu)^2}{2\sigma^2}\right) / \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{\ln t - \mu}{\sigma\sqrt{2}}\right) \tag{7.31}$$

The Lognormal distribution's hazard function typically starts at zero, increases to a peak, and then decreases, making it useful for modelling scenarios where the risk of failure increases and then decreases, such as in the early wear-in period of a product followed by a stable operational phase.

7.6 Ageing Classes and their Properties

Ageing classes in reliability theory are essential to understand how a component or system's likelihood of failure changes over time. These classes can be broadly categorized into three main types: no ageing, positive ageing, and negative ageing. Each of these classes reflects a different relationship between the age of a component and its residual lifetime.

7.6.1 No Ageing

A component exhibits "no ageing" if the probability distribution of its residual lifetime is independent of its current age. In terms of hazard function, a component exhibits no ageing if its hazard function (failure rate) is constant, meaning that the component does not become more or less likely to fail as time progresses. Following are some properties of the no ageing class.

a) A mathematical way to describe no ageing would be to say that $T_x(t > 0)$ are identically distributed random variables. That is,

$$R(t) = R_x(t) \forall\, x, t \geq 0. \tag{7.32}$$

Or

$$R(t) = \frac{R(x+t)}{R(x)} \forall\, x, t \geq 0. \tag{7.33}$$

Or

$$R(t)R(x) \;=\; R(x\;+\;t). \tag{7.34}$$

This equation is commonly known as the **Cauchy functional equation**. It is well established that among continuous distributions, only the exponential distribution, satisfies this equation. This defining feature of the exponential distribution is also referred to as the "lack of memory" property. In the context of lifetime studies, this property is described as the "no ageing" property.

b) For the exponential distribution, the failure rate is constant which can be seen as follows, from [from (7.3)] (7.5) we know that

$$h(t) = \frac{f(t)}{R(t)}$$

$$= \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}}$$

$$h(t) = \lambda. \tag{7.35}$$

Based on the hazard function, the exponential distribution also expresses its no ageing property.

c) For exponential distribution, consider the mean residual life function:

$$M_F(t) = \int_0^\infty R_x(u)du$$

$$= \int_0^\infty e^{-\lambda u}du$$

$$= \left(e^{-\lambda u}\right)_0^\infty$$

$$= -\frac{1}{\lambda}\left(e^{-\lambda(\infty)} - e^{-\lambda(0)}\right)$$

$$M_F(t) = \frac{1}{\lambda} \quad \forall\, t > 0 \tag{7.36}$$

That is, the constant mean residual life also shows that exponential distribution can be categorised as no ageing distribution.

d) Yet another characterization of interest of the exponential distribution is in terms of its equilibrium distribution, defined as

$$H_F(t) = \frac{1}{\mu}\int_0^t R(u)du,\ \mu = E(T)$$

$$= \frac{1}{\mu}\int_0^t e^{-\lambda u}du \text{ where } \mu = \frac{1}{\lambda}$$

$$= 1 - e^{-\lambda t},\ t \geq 0$$

$$H_F(t) = F(t) \tag{7.37}$$

Similarly, the converse may be proved. Therefore, no ageing is equivalent to

$$H_F(t) \equiv F(t)$$

(e) Define

$$\text{TTT}_F(t) = \frac{1}{\mu}\int_0^{F^{-1}(t)} R(u)du,\ 0 \leq t \leq 1 \text{ and } \mu = E(T) \tag{7.38}$$

TTT_F is known as the scaled "total time on test" (TTT) transform of F provided F^{-1} exists and is unique. Trivially $H_F(t) = \text{TTT}_F(F(t))$

No ageing or the exponential distribution is characterized by

$$\text{TTT}_F(t) = t,\ 0 \leq t \leq 1 \tag{7.39}$$

for exponential distribution

$$\text{TTT}_F(t) = \lambda\int_0^{F^{-1}(t)} e^{-\lambda u}du = t,\ 0 \leq t \leq 1 \tag{7.40}$$

In short, NO AGEING can be described as the case where the following holds:

a) Cauchy functional equation is satisfied.

b) Failure rate is constant.

c) Mean residual life is constant.

d) Lifetime follows exponential distribution.

e) TTT is identity function.

Electronic items, such as light bulbs, often demonstrate the "no ageing" phenomenon. This means their performance doesn't degrade with use; instead, they fail suddenly when subjected to external factors like a surge in voltage. It can be shown that if these external shocks occur according to a Poisson process, the item's lifetime follows an exponential distribution.

Practical implications of no ageing:

- Since a used component is statistically as good as a new one, there is no benefit in replacing components that are still functioning, as planned replacement offers no advantage.
- In estimating metrics like mean life, percentiles, or survival functions, only the observed lifetimes and the number of failures are needed for data collection. The current age of the components being observed is irrelevant.

In the following sections, we will explore how deviations from the no ageing characteristic lead to various ageing behaviours.

7.6.2 Positive Ageing Class

The exponential distribution serves as a unique model for no ageing, characterized by a constant failure rate over time, meaning the likelihood of failure is independent of the unit's age. However, in most practical situations, positive ageing is prevalent, where the failure rate increases as the unit ages. This implies that as time passes, the unit becomes more prone to failure, and its remaining lifetime shortens.

To model positive ageing, distributions like the Weibull distribution are often used, as they allow the failure rate to increase over time, reflecting the impact of ageing. These models are crucial for accurately predicting system reliability and scheduling maintenance. Moreover, by adjusting these models, one can describe negative ageing scenarios, where the failure rate decreases with age, although this phenomenon is less common. Various notions of ageing are available under positive ageing scenario we will be discussing them now.

I. Increasing Failure Rate (IFR) Classes

Definition 1: A component or unit is said to have an Increasing Failure Rate (IFR) if the probability of failure increases as the unit ages or the reliability decreases as the age increases. Hence the unit's age adversely affects the conditional survival probability i.e. F is an IFR if $R(t|x)$ is decreasing $\forall\ t \geq 0$.

Definition 2: In terms of hazard rate, a unit is said to have IFR if its hazard rate $h(t)$ is increasing $\forall\ t \geq 0$.

The major difference between the two definitions is that the latter requires the existence of the density $f(t)$ whereas the former is always true.

Example: Consider a system with a lifetime T that follows an exponential distribution with a parameter $\lambda > 0$. Verify whether this distribution belongs to the IFR class.

Solution: For an exponential distribution, the failure rate $\lambda(t)$ is constant:

$$h(t) = \lambda$$

Since h(t) does not increase over time but remains constant, the exponential distribution does not belong to the IFR class. However, it is memoryless, meaning it does not exhibit positive ageing.

Example: Consider a system with a lifetime T that follows a Weibull distribution with shape parameter $\alpha = 2$ and scale parameter $\lambda > 0$ Verify if this distribution belongs to the IFR class.

Solution: For a Weibull distribution, the failure rate function is given by:

$$h(t) = \alpha\lambda(\lambda t)^{\alpha-1}$$

For $\alpha = 2$, this simplifies to:

$$h(t) = 2\lambda^2 t$$

Since $h(t)$ is a linear function of t and increases with time, the Weibull distribution with $\alpha > 1$ (in this case, $\alpha = 2$) is in the IFR class.

II. Increasing Failure Rate Average (IFRA) Classes

The failure rate average function is defined as

$$H_F(t) = \frac{1}{t} H(t)$$
$$= -\frac{1}{t} \log R(t) \qquad (7.41)$$

if the function $\bar{H}_F(t)$ is increasing, then the distribution F is said to possess the increasing failure rate average property and is said to belong to the IFRA class.

Theorem: A distribution F is IFRA if and only if

$$R(\alpha t) \geq [R(t)]^{\alpha} \text{ for } 0 < \alpha \leq 1 \& t \geq 0$$

Proof: F is IFRA

$$\Leftrightarrow \frac{1}{t}\int_0^t h_F(u)du \uparrow t \quad (7.42)$$

$$\Leftrightarrow -\frac{1}{t}\log\, R(t) \uparrow t \quad (7.43)$$

$$\Leftrightarrow R(t)^{1/t} \downarrow t \quad (7.44)$$

$$\Leftrightarrow [R(\alpha t)]^{1/\alpha t} \geq [R(t)]^{1/t}\ \forall t \geq 0 \text{ and } 0 < \alpha \leq 1 \quad (7.45)$$

$$\Leftrightarrow [R(\alpha t)] \geq [R(t)]^{\alpha} \forall t > 0,\ 0 < \alpha \leq 1 \quad (7.46)$$

It is obvious that IFR ⇒ IFRA as the average of an increasing function is increasing.

Remark : The classes IFR and IFRA are classes of progressive ageing. We shall now consider a weaker form of ageing which is different from progressive ageing.

Theorem: Suppose each of the independent component of a coherent system has an IFRA life distribution. Then the system itself has an IFRA life distribution.

Proof: Let F denotes the lifetime distribution of system and F_i denotes the lifetime distribution of i^{th}component.

$$R(\alpha t) = R[R_1(\alpha t), R_2(\alpha t), \cdots, R_n(\alpha t)] \quad (7.47)$$

Then for $0 \leq \alpha \leq 1$

Since F_i is IFRA from
(7.46) for i^{th} component we have

$$R_i(\alpha t) \geq R_i{}^{\alpha}(t)$$

Also R is increasing in each argument, thus

$$R(\alpha t) \geq R[R^{\alpha}{}_1(t), R^{\alpha}{}_2(t), \cdots, R^{\alpha}{}_n(t)] \quad (7.48)$$

But the following inequality holds

$$R[R^{\alpha}{}_1(t), R^{\alpha}{}_2(t), \cdots, R^{\alpha}{}_n(t)] \geq R([R_1(t), R_2(t), \cdots, R_n(t)])^{\alpha} \quad (7.49)$$

Using (7.48) and (7.49) we have

$$R(\alpha t) = R[R_1(\alpha t), R_2(\alpha t), \cdots, R_n(\alpha t)]$$

Example: Consider a system with a lifetime T that follows an exponential distribution with parameter $\lambda = 3$. Verify if this distribution belongs to the IFRA class.

Solution: The exponential distribution has a constant failure rate $h(t) = 3$. The average failure rate $H(t)$ is:

$$H(t) = \frac{1}{t}\int_0^t \lambda(u)du = 3$$

Since the average failure rate $H(t)$ is non-decreasing, the exponential distribution is in the IFRA class.

Example: Consider a system with a lifetime T that follows a Weibull distribution with shape parameter $\beta = 1.5$ and scale parameter $\eta = 2$. Determine if this distribution is IFRA.

Solution: The failure rate function is:

$$\lambda(t) = \frac{1.5}{2}\left(\frac{t}{2}\right)^{0.5} = \frac{0.75 \cdot t^{0.5}}{2^{0.5}}$$

Since $\lambda(t)$ increases with t, this distribution is IFR, and thus also IFRA.

Example: Consider a system with a lifetime T that follows a Gamma distribution with shape parameter $\alpha = 3$ and rate parameter $\lambda = 1$. Verify whether this distribution is IFRA.

Solution: The failure rate for a Gamma distribution with $\alpha > 1$ is increasing, which makes it IFR. Since IFR implies IFRA, the Gamma distribution with $\alpha = 3$ belongs to the IFRA class.

III. New Better than Used (NBU) Classes

Here, distribution of the lifetime of a new unit (say r.v. Y) is compared with the lifetime of a unit of age $x(> 0)$ [i.e. r.v. Y_x]. The distribution function of the two random variables are considered to be F and F_x respectively. F is said to have the "New Better than Used" property if

$$R(t) \geq R_x(t),\ \forall t, x > 0 \tag{7.50}$$

That is,

$$R(t)R(x) \geq \mathrm{R}(t + x),\ \forall t, x > 0 \tag{7.51}$$

This is a weaker form of ageing since in this criterion the comparison of the units is done at specific age.

Theorem: If F is IFRA then it implies F is NBU i.e.

IFRA ⇒ NBU.

Proof: We know that if $F\ is\ IFRA$ then

$$\Rightarrow R(\alpha t) \geq [R(t)]^{\alpha}, t > 0, 0 < \alpha < 1$$
$$\Rightarrow R((1-\alpha)t) \geq [R(t)]^{1-\alpha} \forall t > 0, 0 < \alpha < 1$$
$$\Rightarrow R(\alpha t) R[(1-\alpha)t] \geq R(t) \forall t > 0, 0 < \alpha < 1$$
$$\Rightarrow R(x)R(y) \geq R(x+y) \qquad (7.52)$$

where $t = x + y$ and $\alpha = \frac{x}{x+y}$. On comparing (7.52) with(7.51) one can infer that F is NBU.

IV. New Better than Used in Expectation (NBUE) Classes

A still weaker form of positive ageing than NBU is NBUE defined by the inequality

$$\int_0^{\infty} R(x)dx \geq \int_0^{\infty} R_t(x)dx \qquad (7.53)$$

Or

$$M_F(0) \geq M_F(t)\ \forall t > 0 \qquad (7.54)$$

It is obvious that NBU ⇒ NBUE.

It may be noted that for progressive ageing classes the comparison between units of different ages is possible. However, for NBU and NBUE classes the comparison is between brand new unit and a unit aged t. We shall now consider another progressive ageing class.

V. Decreasing Mean Residual Life (DMRL) Classes

Let $E(Y_x)$ denote the mean residual life time of a unit of age x. Then one can say that $E(Y_x) \downarrow x$ is also a way of describing progressive positive ageing. This is called the "Decreasing Mean Residual Life" (DMRL) property.

Theorem: If F possess IFR property $\Rightarrow F$ possess DMRL property.

Proof: For,

$$F \text{ is } IFR \Leftrightarrow R_{x_1}(x) \geq R_{x_2}(x) \ \forall \ x_1 < x_2$$

By integration, we get

$$\int_0^\infty R_{x_1}(x)dx \geq \int_0^\infty R_{x_2}(x)dx, \ \forall \ x_1 < x_2$$

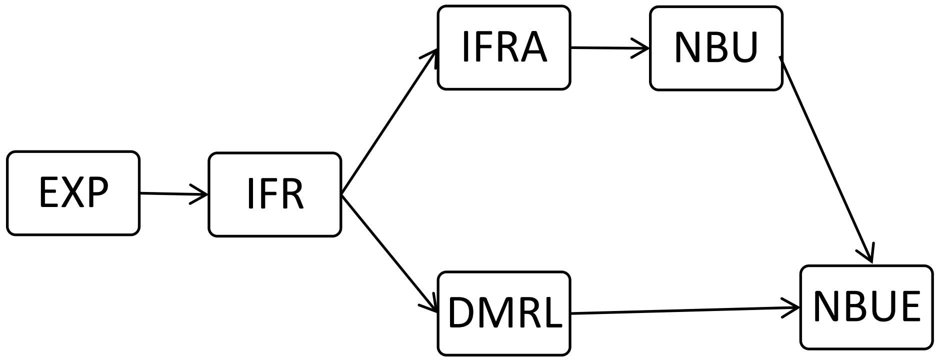
That is $E(Y_x) \downarrow x$ and hence F possess DMRL property

Theorem: If F possess DMRL property $\Rightarrow F$ possess NBU property.

Proof: This is seen by putting $x_1 = 0$ in the above.

The complete picture of implications of these ageing classes can be seen below

Fig. 1 Relation of implications between different classes of Positive Ageing



7.6.3 Negative Ageing Class as Dual of Positive Ageing Class

Negative ageing represents a scenario where the failure rate decreases over time, meaning that as a unit ages, it becomes less likely to fail. This phenomenon suggests that surviving components tend to "wear in" rather than "wear out," leading to an increased remaining lifetime as time progresses. Although less common, negative ageing is particularly relevant in situations where early-life defects or "infant mortality" are significant. During the initial period, units may experience higher failure rates due to inherent defects or weaknesses. However, as these defects are corrected or naturally eliminated, the remaining population of units becomes more robust, leading to a declining failure rate over time.

To model negative ageing, distributions such as the lognormal or mixtures of exponential distributions are often employed. These models effectively capture systems that improve with age or undergo a burn-in period, after which the likelihood of failure diminishes. Understanding negative ageing is crucial in contexts like manufacturing, where identifying and mitigating early failures can significantly enhance overall system reliability. By accurately modeling negative ageing, it becomes possible to optimize maintenance schedules and replacement strategies, ensuring that interventions are timed to maximize the longevity and performance of the system.

I. Decreasing Failure Rate (DFR) Classes

Definition 1: A component or unit is said to have an decreasing Failure Rate (DFR) if the probability of failure decreases as the unit ages or the reliability increases as the age increases. Hence the unit's age beneficially affects the conditional survival probability i.e. F is an DFR if $R(t|x)$ is increasing $\forall\, t \geq 0$.

Definition 2: In terms of hazard rate, a unit is said to have DFR if its hazard rate $h(t)$ is decreasing $\forall\, t \geq 0$.

The major difference between the two definitions is that the latter requires the existence of the density $f(t)$ whereas the former true in non-existence of $f(t)$. This is the dual class of IFR class.

II. Decreasing Failure Rate Average (DFRA) Classes

If the failure rate average function defined in (7.41) is decreasing, then the distribution F is said to possess the decreasing failure rate average property and is said to belong to the DFRA class.

It is obvious that DFR ⇒ DFRA as the average of an increasing function is increasing. This is the dual class of IFRA class.

Remark: The classes DFR and DFRA are classes of regressive ageing. We shall now consider a weaker form of ageing which is different from regressive ageing.

III. New Worse than Used (NWU) Classes

Here, distribution of the lifetime of a new unit (say r.v. Y) is compared with the lifetime of a unit of age $x(> 0)$ [i.e. r.v. Y_x]. The distribution function of the two random variables are considered to be F and F_x respectively. F is said to have the "New Worse than Used" property if

$$R(t) \leq R_x(t),\ \forall t, x > 0 \tag{7.55}$$

That is,

$$R(t)R(x) \leq \mathrm{R}(t + x),\ \forall t, x > 0 \tag{7.56}$$

This is a weaker form of ageing since in this criterion the comparison of the units is done at specific age. This is the dual class of NBU class.

IV. New Worse than Used in Expectation (NWUE) Classes

A still weaker form of positive ageing than NWU is NWUE defined by the inequality

$$\int_0^\infty R(x)dx \leq \int_0^\infty R_t(x)dx \tag{7.57}$$

Or

$$M_F(0) \leq M_F(t)\ \forall t > 0 \tag{7.58}$$

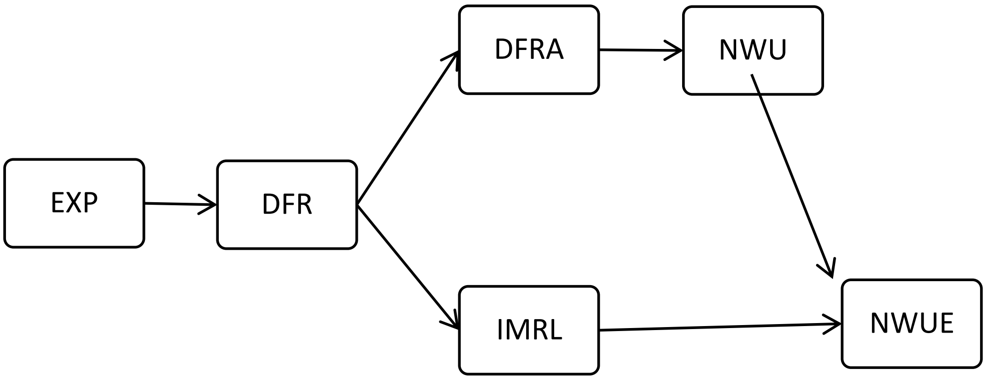
It is obvious that NWU ⇒ NWUE. This is the dual class of NBUE class.

It may be noted that for regressive ageing classes the comparison between units of different ages is possible. However, for NWU and NWUE classes the comparison is between brand new unit and a unit aged t. We shall now consider another regressive ageing class.

V. Increasing Mean Residual Life (IMRL) Classes

Let $E(Y_x)$ denote the mean residual life time of a unit of age x. Then one can say that $E(Y_x) \uparrow x$ is also a way of describing regressive negative ageing. This is called the "Increasing Mean Residual Life" (IMRL) property. This is the dual class of DMRL class.

Fig. 2 Relation of implications between different classes of Negative Ageing Classes



7.7 Self-Assessment Exercise

1) Given a system with a lifetime that follows a Rayleigh distribution, determine whether this distribution belongs to the IFR class. Justify your answer by examining the failure rate function.
2) A component's lifetime follows a distribution with an increasing failure rate (IFR). Explain why this implies that the component also belongs to the IFRA class. Provide a numerical example to support your explanation.
3) Consider a system with a lifetime that follows a uniform distribution over the interval [0,2] . Analyse whether this distribution belongs to the NBU class by checking the survival function's behaviour.
4) A device has a lifetime that follows a Gamma distribution with shape parameter α=2 and rate parameter λ=1. Determine whether this distribution belongs to the NBUE class by evaluating the mean residual life function.
5) Consider a system with a lifetime following a Weibull distribution with shape parameter β=0.8 and scale parameter η=3. Determine if this distribution belongs to the DMRL class and explain your reasoning.
6) Explain the relationship between the NBU, NBUE, and IFR classes. Can a distribution belong to one of these classes without belonging to the others? Provide an example to illustrate your explanation.

7.8 Summary

The unit explored the concept of ageing in reliability engineering, which examined how the likelihood of system failure evolves over time. Focussed on various ageing classes, including IFR(DFR), IFRA(DFRA), NBU (NWU), DMRL (IMRL), and NBUE(NWUE), each defined by specific patterns in failure rates and mean residual life functions. These classes help categorize and predict system reliability, with each class having its dual class. Understanding these concepts is crucial for assessing and managing the reliability and maintenance of systems over their lifetimes.

7.9 References

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7.10 Further Reading

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UNIT 8: RELIABILITY ESTIMATION

Structure

8.1 Introduction

Reliability estimation involves determining how likely a system, item, component, or product is to perform its intended function without failure over a specified period. This estimation is crucial in fields like engineering, manufacturing, and software development, where reliability directly impacts safety, performance, and customer satisfaction. Censoring occurs when the information about the survival time of an individual or component is incomplete. This incomplete information could result from a variety of situations, such as the end of the study period, loss of contact with study participants, or the removal of items before they fail. Despite this incomplete data, censoring enables analysts to make use of the available information, rather than discarding it.

Various quantities of Interest in Reliability Estimation:

1. **Failure Rate:** The frequency with which an engineered system or component fails. Often expressed as failures per unit of time (e.g., failures per hour).

2. **Mean Time to Failure (MTTF):** The average time a non-repairable system operates before failing. Used mainly for systems that are not repaired after failure.

3. **Mean Time Between Failures (MTBF):** Similar to MTTF but used for repairable systems, representing the average time between failures.

4. **Reliability Function (R(t)):** The probability that a system will perform without failure for a time period t.

The parametric reliability estimation includes the following steps:

Data Collection, Choice a suitable statistical model (e.g., exponential, Weibull) based on the nature of the failure data, Parameter Estimation and Reliability Analysis.

8.2 Objectives

After going through this unit, students should be able to:

- Understand the basic concepts of reliability estimation.
- Understand the uses and applications of censoring and stress- strength model
- Obtain the reliability estimate under Type-I and Type-II censoring schemes.
- Obtain the reliability estimates for stress strength model.

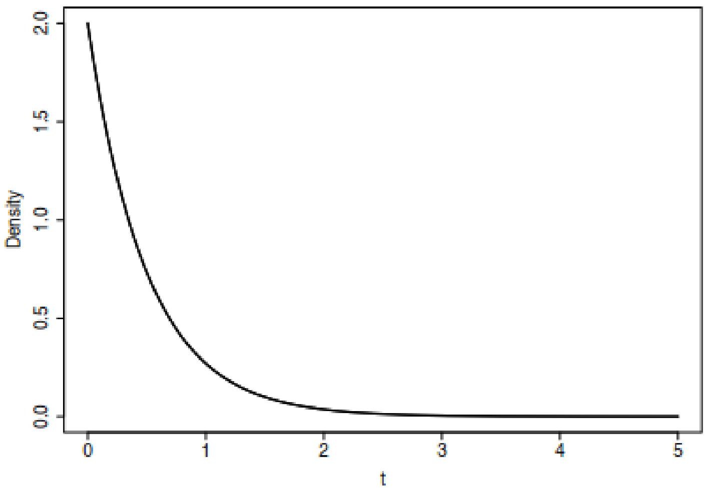
8.3 Reliability Estimation for some well-known Distributions

Here we consider some well-known distributions namely Exponential, Weibull, Gamma and Log Normal distributions and provide estimator of various functions for the same.

8.3.1 Exponential Distribution

A positive valued random variable X is said to follow exponential distribution if its probability density function is given by

$$f(x) = \lambda\, e^{-\lambda x}, \qquad x \geq 0$$



The probability density function of exponential random variable with $\lambda = 2$

Maximum Likelihood Estimation

Let $\underline{x} = (x_1, x_2, x_3 \dots \dots, x_n)$ are i.i.d. observations from exp(λ). The likelihood of λ given the observations $\underline{x}$ can be written as follows.

$$L(\lambda|\underline{x}) = \prod_{i=1}^{n} f(x_i)$$

Taking logarithm, we get

$$\ln L(\lambda) = nln\, \lambda - \lambda \sum_{i=1}^{n} x_i$$

To obtain MLE, we solve the likelihood equation

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = 0$$

That is

$$\frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0,$$

$$\frac{n}{\lambda} = \sum_{i=1}^{n} x_i$$

which gives the MLE of λ

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}$$

8.3.2 Weibull Distribution

Definition: A positive valued random variable X is said to follow Weibull distribution if its probability density function is given by

$$f(x; \lambda, \alpha) = \alpha\lambda(\lambda x)^{\alpha-1} e^{-(\lambda x)^{\alpha}}, x \geq 0$$

where:

- $\lambda > 0$ is the scale parameter,
- $\alpha > 0$ is the shape parameter,

Maximum Likelihood Estimation (Both parameters are unknown**):**

Let $\underline{x} = (x_1, x_2, x_3 \dots \dots, x_n)$ are iid observations from W(λ, α). The likelihood of λ, α given the observations $\underline{x}$ can be written as follows.

$$L(\lambda, \alpha|\underline{x}) = \prod_{i=1}^{n} f(x_i; \alpha, \beta)$$

$$L(\lambda, \alpha|\underline{x}) = \prod_{i=1}^{n} \left(\alpha\lambda(\lambda x_i)^{\alpha-1} e^{-(\lambda x_i)^{\alpha}}\right)$$

$$L(\lambda, \alpha|\underline{x}) = \alpha^n \lambda^{n\alpha} \prod_{i=1}^{n} \left((x_i)^{\alpha-1} e^{-(\lambda x_i)^{\alpha}}\right)$$

Taking logarithm of the likelihood function gives the log likelihood function, we get

$$\ln L(\lambda, \alpha) = \text{nln}\, \alpha + n\alpha \ln \lambda + (\alpha - 1) \sum_{i=1}^{n} ln(x_i) - \sum_{i=1}^{n} (\lambda x_i)^{\alpha}$$

We differentiate with respect to λ and α the likelihood equation and equal to 0

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + nln\,\lambda + \sum_{i=1}^{n} ln(x_i) - \sum_{i=1}^{n}(\lambda x_i)^{\alpha} ln(\lambda x_i)=0$$

$$\frac{\partial l}{\partial \lambda} = \frac{n\alpha}{\lambda} - \alpha\sum_{i=1}^{n} x_i{}^{\alpha}(\lambda)^{\alpha-1}=0$$

$$\frac{n\alpha}{\lambda} = \alpha\sum_{i=1}^{n} x_i{}^{\alpha}(\lambda)^{\alpha-1}$$

$$n = \lambda^{\alpha}\sum_{i=1}^{n} x_i{}^{\alpha}$$

$$\widehat{\lambda^{\alpha}} = \frac{n}{\sum_{i=1}^{n} x_i{}^{\alpha}}$$

And

$$\hat{\lambda} = \left(\frac{n}{\sum_{i=1}^{n} x_i{}^{\alpha}}\right)^{\frac{1}{\alpha}}$$

$$\frac{n}{\hat{\alpha}} + nln\,\hat{\lambda} + \sum_{i=1}^{n} ln(x_i) = \sum_{i=1}^{r}(\hat{\lambda}x_i)^{\hat{\alpha}} ln(\hat{\lambda}x_i)$$

This equation does not have a closed form solution. It can be solved through Newton-Raphson method.

8.3.3 Gamma Distribution

We consider the gamma distribution with pdf given by

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1}e^{-\beta x}, \qquad x \geq 0$$

Maximum Likelihood Estimation (Both parameters are unknown)

Suppose that n units are put to test and the lifetime of each unit follow gamma distribution with parameter α and β. Suppose that test is terminated after the failure of all the units. Let the sample $\underline{x} = (x_1, x_2, x_3 \ldots\ldots, x_n)$ is observed. The likelihood of α, β given the observations $\underline{x}$ can be written as follows.

$$L(\alpha,\beta|\underline{x}) = \prod_{i=1}^{n} f(x_i;\alpha,\beta)$$

Type equation here. $$L\left(\alpha,\beta|\underline{x}\right)=\prod_{i=1}^{n}\frac{\beta^{\alpha}}{\Gamma(\alpha)}{x_i}^{\alpha-1}e^{-\beta x_i}$$

$$L\left(\alpha,\beta\ |\underline{x}\right)=\frac{\beta^{n\alpha}}{\Gamma(\alpha)^{\mathrm{n}}}\prod_{i=1}^{n}{x_i}^{\alpha-1}e^{-\beta x_i}$$

Taking logarithm, we get

$$\ln L(\alpha,\beta\ )=n\ \alpha\ln\beta-nln\ \Gamma(\alpha)+(\alpha-1)\sum_{i=1}^{n}\ln\left(x_i\right)-\ \beta\sum_{i=1}^{n}x_i$$

To find the estimates of parameters, take the partial derivatives of the log likelihood function with respect to $\alpha\ and\ \beta$ and put them equal to 0.

$$\frac{\partial\ln L(\alpha,\beta\ )}{\partial\beta}=\frac{n\alpha}{\beta}-\sum_{i=1}^{n}x_i$$

$$\frac{n\alpha}{\beta}-\sum_{i=1}^{n}x_i=0$$

$$\frac{n\alpha}{\beta}=\sum_{i=1}^{n}x_i$$

$$\beta=\frac{n\alpha}{\sum_{i=1}^{n}x_i}$$

$$\beta=\frac{\alpha}{\frac{\sum_{i=1}^{n}x_i}{n}}$$

$$\hat{\beta}=\frac{\alpha}{\bar{x}}\qquad(*)$$

Note that, since α is unknown, $\frac{\alpha}{\bar{x}}$ is not MLE of β. With the help of (*), we solve the following equation to get MLE of α

$$\frac{\partial\ln L(\alpha,\beta\ )}{\partial\alpha}=n\ \ln\beta+\sum_{i=1}^{n}\ln(x_i)-n\frac{\acute{\Gamma}(\alpha)}{\Gamma(\alpha)}$$

$$n \; \ln\beta + \sum_{i=1}^{n} \ln(x_i) - n\frac{\acute{\Gamma}(\alpha)}{\Gamma(\alpha)} = 0$$

Now substituting the value of β from (*) , get

$$n \; \ln\alpha - nlog\bar{x} + \sum_{i=1}^{n} \ln(x_i) - n\frac{\acute{\Gamma}(\alpha)}{\Gamma(\alpha)} = 0$$

Solve this equation for α using numerically (e.g. Newton Raphson method). Using $\hat{\alpha}$, the MLE of α, we can obtain

$$\hat{\beta} = \frac{\alpha}{\bar{x}}.$$

8.3.4 Log-Normal Distribution

A random variable X is said to follow a log-normal distribution if ln(X) follows a normal distribution. The log-normal distribution is characterized by two parameters:

The probability density function (PDF) of the log-normal distribution is:

$$f(x; \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right), x>0$$

The cumulative distribution function (CDF) is:

$$F(x; \mu, \sigma) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

where Φ is the CDF of the standard normal distribution.

The reliability function R(x) (the probability that a component survives beyond time x) is:

$$R(x; \mu, \sigma) = 1 - F(x; \mu, \sigma) = 1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

Maximum Likelihood Estimation (Both parameters are unknown**):**

Suppose a sample $x_{1,}\; x_{2,} \dots, x_n$ from the log-normal distribution, the likelihood function L(μ, σ) is the joint probability of observed sample is:

$$L(\mu, \sigma|\underline{x}) = \prod_{i=1}^{n} \frac{1}{x_i \sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x_i - \mu)^2}{2\sigma^2}\right)$$

Taking log of likelihood function, we get

$$\ln L(\mu, \sigma|\underline{x}) = -nln\left(\sigma\sqrt{2\pi}\right) - \sum_{i=1}^{n} \ln(x_i) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln x_i - \mu)^2,$$

We can write it

$$\ln L(\mu, \sigma) = -\frac{2n}{2}\ln\sigma - \text{n}\ln(\sqrt{2\pi}) - \sum_{i=1}^{n}\ln(x_i) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln x_i - \mu)^2$$

Now again write

$$\ln L(\mu, \sigma) = -\frac{n}{2}\ln\sigma^2 - \text{n}\ln(\sqrt{2\pi}) - \sum_{i=1}^{n}\ln(x_i) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln x_i - \mu)^2$$

To obtain the estimate the parameters, we need to maximize the log likelihood function with respect to parameters and set it equal to zero

$$\frac{\partial \ln L(\mu,\sigma)}{\partial\mu} = \frac{2}{2\sigma^2}\sum_{i=1}^{n}(\ln x_i - \mu),$$

$$\frac{\partial \ln L(\mu,\sigma)}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(\ln x_i - \mu)=0,$$

$$\sum_{i=1}^{n}\ln x_i - n\,\mu = 0$$

We get

$$\hat{\mu} = \frac{\sum_{i=1}^{n}\ln\ x_i}{n}$$

Now we differentiate with respect to σ^2

$$\frac{\partial \ln L(\mu,\sigma)}{\partial\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(\ln x_i - \mu)^2=0$$

Solving this we obtain estimate of σ^2

$$\frac{1}{2\sigma^4}\sum_{i=1}^{n}(\ln x_i - \mu)^2 = \frac{n}{2\sigma^2},$$

$$\sum_{i=1}^{n}(\ln x_i - \mu)^2=\text{n}\sigma^2,$$

$$\widehat{\sigma^2} = \frac{\sum_{i=1}^{n}(\ln x_i - \hat{\mu})^2}{n}$$

Reliability estimate of Log normal Distribution is

$$R(t; \mu, \sigma) = 1 - \Phi\left(\frac{\ln t - \hat{\mu}}{\hat{\sigma}}\right)$$

8.4. Censoring

Censoring: Censoring is a common issue in survival analysis and reliability studies where the exact time to event (such as failure or death) is not fully observed for all subjects. Censoring occurs when the exact failure time is unknown for some subjects within the study period. There are several types of censoring:

8.4.1 Types of Censoring

1. **Right Censoring**: The most common type, where the event of interest has not occurred by the end of the observation period for some subjects. For example, in a clinical trial, patients who are still alive at the end of the study are right-censored.

2. **Left Censoring**: Occurs when the event has already occurred before the observation period starts. For example, if the exact time of an event is known to be before a certain point but the exact time is not known, it is left-censored.

3. **Interval Censoring**: Occurs when the exact time of the event is unknown, but it is known to have occurred within a specific time interval. For example, if a patient visits a doctor at irregular intervals and an event (such as disease onset) is detected during a visit, the exact time between visits is unknown, leading to interval censoring.

Two popular forms of right censoring, frequently considered in reliability estimation, are

Type I Censoring (Time Censoring): The study is terminated at a pre-fixed time. Suppose r items have failed by this time and the remaining n_c = n-r items remain operative. These are called the censored items. Not that, in this case number of failures are random variable.

For Example- Suppose a manufacturer company conducts a reliability test in which 10 power supplies are operated over the same duration. The manufacturer company decides that test is terminating after 60000 hrs. Suppose 7 power supplies failed during predetermined time interval. Then remaining three is type I censored.

Type II Censoring (Failure Censoring): The study continues until a predetermined number of failures have occurred. **For Example-**Ten semiconductors are subjected to a life test and the test is terminated a predetermined number five failures.

8.4.2 Reliability Estimation of Exponential Distribution under Type I Censoring

For an exponential distribution with unknown parameter λ, the PDF is:

$$f(x;\lambda) = \lambda\, e^{-\lambda x}, \qquad x \geq 0$$

The CDF is:

$$F(x;\lambda) = 1 - e^{-\lambda x}$$

The reliability function R(t) is:

$$R(t;\lambda) = e^{-\lambda t}$$

Type I Censoring

In Type I censoring, the study is terminated at a fixed time t_0 . Any component that has not failed by time t_0 is considered censored**.** let us suppose $x_1, x_2, \ldots, x_n$ be a random sample from exponential distribution with scale parameter λ and t_0 be the pre fixed time at which test is terminate. We observe that $t_1, t_2, \ldots, t_n$ where $t_i = x_i$ if $x_i \leq t_0$ and t_0 if $x_i > t_0$.

Estimating the Scale Parameter θ Using Maximum Likelihood Estimation (MLE)

1. Likelihood Function:

$$L(\lambda|\underline{x}) = \frac{n!}{n-r!}\prod_{i=1}^{r} f(x_i)[1 - F(t_0)]^{n-r}, \quad 0 < x_{(1)} < \ldots < x_{(r)} \leq t_0 < \infty$$

After simplifying, we get

$$L(\lambda|\underline{x}) = \frac{n!}{n-r!}\lambda^{r} e^{-\lambda \sum_{i=1}^{r} x_i}\left(e^{-\lambda\, t_0}\right)^{n-r},$$

2. Log-Likelihood Function:

The log-likelihood function is:

$$\ln L\,(\lambda) \propto r \ln \lambda - \lambda \sum_{i=1}^{r} x_i - (n-r)\lambda\, t_0$$

3. MLE for λ :

To find the maximum likelihood estimate $\hat{\lambda}$, differentiate the log-likelihood function with respect to λ and set it to zero:

$$\frac{d}{d\lambda} \ln L(\lambda) = \frac{r}{\lambda} - \sum_{i=1}^{r} x_i - (n-r)\, t_0 = 0$$

$$\frac{r}{\lambda} = \sum_{i=1}^{r} x_i + (n-r)\, t_0$$

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{r} x_i + (n-r)\, t_0}$$

Reliability Estimation

Once you have the MLE $\hat{\lambda}$, you can estimate the reliability function for any time t .

1. Reliability Function with Estimated λ :

$$\hat{R}(t) = e^{-\hat{\lambda} t}.$$

Example Calculation

Assume the following data:

- $n = 10$ components, t_0=10.
- $r = 5$ failures observed at times 2, 4, 5, 7, 9

1. Calculate the MLE for λ :

   Sum of failure times= 2 + 4 + 5 + 7 + 9 = 27

$$\hat{\sigma} = \frac{5}{77} = 0.065$$

2. ML estimate of reliability at $t = 6$:

$$\hat{R}(6) = e^{-6*0.065} \approx e^{-0.39} \approx 0.6770$$

8.4.3 Reliability Estimation of Weibull Distribution under Type I Censoring

Let us suppose $x_1, x_2, \ldots, x_n$ be a random sample from Weibull distribution with scale parameter λ and shape parameter α, t_0 be the pre fixed time at which test is terminate. The Weibull distribution is characterized by:

The probability density function (PDF) is:

$$f(x;\lambda,\alpha) = \alpha\lambda(\lambda x)^{\alpha-1}e^{-(\lambda x)^{\alpha}}, x \geq 0$$

Where Shape parameter (α) and Scale parameter (λ)

The cumulative distribution function (CDF) is:

$$F(x) = 1 - e^{-(\lambda x)^{\alpha}}$$

The reliability function R(t) is:

$$R(t) = 1 - F(t) = e^{-(\lambda t)^{\alpha}}$$

Maximum Likelihood Estimation (MLE)

To estimate the parameters $\lambda \; and \; \alpha$ of the Weibull distribution. Similar to exponential distribution, we derive for Weibull distribution.

1. Likelihood Function:

$$L(\lambda,\alpha|\underline{x}) = \frac{n!}{n-r!}\prod_{i=1}^{r} f(x_i)[1-F(t_0)]^{n-r}, \quad 0<x_{(1)}<\ldots< x_{(r)}\leq t_0< \infty$$

The likelihood function is:

$$L(\lambda,\alpha|\underline{x}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}\left(\alpha\lambda(\lambda x_i)^{\alpha-1}e^{-(\lambda x_i)^{\alpha}}\right)\left(e^{-(\lambda t_0)^{\alpha}}\right)^{n-r}$$

$$L(\lambda,\alpha|\underline{x}) = \frac{n!}{(n-r)!}\alpha^{r}\lambda^{r\alpha}\prod_{i=1}^{r}\left((x_i)^{\alpha-1}e^{-(\lambda x_i)^{\alpha}}\right)\left(e^{-(\lambda t_0)^{\alpha}}\right)^{n-r}$$

$$L(\lambda,\alpha|\underline{x}) = \frac{n!}{(n-r)!}\alpha^{r}\lambda^{r\alpha}\prod_{i=1}^{r} x_i^{\alpha-1}e^{-\lambda^{\alpha}(\sum_{i=1}^{r} x_i^{\alpha}+(n-r)t_0^{\alpha})}$$

2. Log-Likelihood Function is:

$$\ln L(\lambda,\alpha) \propto r\ln\alpha + r\alpha\ln\lambda + (\alpha-1)\sum_{i=1}^{r} ln(x_i) - \lambda^{\alpha}(\sum_{i=1}^{r} x_i^{\alpha} + (n-r)t_0^{\alpha})$$

Now we differentiate log likelihood with respect to parameters

$\frac{\partial l}{\partial \alpha} = \frac{r}{\alpha} + r ln\, \lambda + \sum_{i=1}^{r} ln(x_i) - \lambda^{\alpha} \ln(\lambda) \left(\sum_{i=1}^{r} x_i{}^{\alpha} + (n-r) t_0{}^{\alpha}\right) - \lambda^{\alpha} [\sum_{i=1}^{r} x_i{}^{\alpha} \ln(x_i) + (n-r) t_0{}^{\alpha} \ln(t_0)] = 0$

$\frac{r}{\alpha} + r ln\, \lambda + \sum_{i=1}^{r} ln(x_i) - \lambda^{\alpha} [\ln(\lambda) \left(\sum_{i=1}^{r} x_i{}^{\alpha} + (n-r) t_0{}^{\alpha}\right) + \sum_{i=1}^{r} x_i{}^{\alpha} \ln(x_i) + (n-r)\, t_0{}^{\alpha} \ln(t_0)] = 0$

$$\frac{\partial l}{\partial \lambda} = \frac{r\alpha}{\lambda} - \alpha \lambda^{\alpha-1} \left(\sum_{i=1}^{r} x_i{}^{\alpha} + (n-r) t_0{}^{\alpha}\right) = 0$$

$$\frac{r}{\lambda} = \lambda^{\alpha-1} \left(\sum_{i=1}^{r} x_i{}^{\alpha} + (n-r) t_0{}^{\alpha}\right)$$

$$r = \lambda^{\alpha} \left(\sum_{i=1}^{r} x_i{}^{\alpha} + (n-r) t_0{}^{\alpha}\right)$$

$$\lambda^{\alpha} = \frac{r}{\left(\sum_{i=1}^{r} x_i{}^{\alpha} + (n-r) t_0{}^{\alpha}\right)}$$

$$\hat{\lambda} = \left(\frac{r}{\left(\sum_{i=1}^{r} x_i{}^{\alpha} + (n-r) t_0{}^{\alpha}\right)}\right)^{\frac{1}{\alpha}}$$

Solving $\frac{\partial l}{\partial \alpha} = 0$. Using numerical approximation technique, we can obtain MLEs of parameters.

After obtain the estimate of parameter, we can evaluate reliability estimate

$$R(t) = e^{-(\hat{\lambda} t)^{\hat{\alpha}}}$$

8.5.1 Reliability Estimation of Exponential Distribution under Type II Censoring

When the scale parameter λ of the exponential distribution is unknown, and you have Type II censoring, you need to estimate λ from the data before estimating reliability. Suppose $x_1, x_2, \ldots, x_n$ be a random sample from exponential distribution with scale parameter λ and r be the pre fixed number of failure, at which test is terminate.

Step 1: Estimate the Parameter λ

We use Maximum Likelihood Estimation (MLE) to estimate λ . The likelihood function for the exponential distribution, considering r (fixed before start to test) failures and n - r censored observations, is:

1. Likelihood Function:

$L(\lambda|\underline{x}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_i)[1-F(x_{(r)})]^{n-r}$; $0< x_{(1)}<\ldots< x_{(r)}< \infty$

After substituting the expressions of pdf and cdf, we get

$$L(\lambda|\underline{x}) = \frac{n!}{(n-r)!}\left(\prod_{\mathrm{i}=1}^{\mathrm{r}} \lambda\, e^{-\lambda x_i}\right)\left(e^{-\lambda x_{(r)}}\right)^{n-r}$$

$$L(\lambda|\underline{x}) = \frac{n!}{(n-r)!}\lambda^{\mathrm{r}} e^{-\lambda\sum_{i=1}^{r} x_i}\left(e^{-\lambda\, x_{(r)}}\right)^{n-r}$$

2. Then the Log-Likelihood function is

$$\mathrm{Ln}\, L(\lambda) \propto r \ln\lambda - \lambda\sum_{i=1}^{r} x_i - (n-r)\lambda\, x_{(r)}$$

3. MLE for λ:

To find the maximum likelihood estimate of λ, take the derivative of the log-likelihood function with respect to λ and equals it to zero:

$$\frac{d}{d\lambda}\ln L(\lambda) = \frac{r}{\lambda} - \sum_{i=1}^{r} x_i - (n-r)\, x_{(r)} = 0$$

Which implies

$$\frac{r}{\lambda} = \sum_{i=1}^{r} x_i + (n-r)\, x_{(r)}$$

or

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{r} x_i + (n-r)\, x_{(r)}}$$

Step 2: Calculate the Reliability Function

After estimating $\hat{\lambda}$ and using invariance property if MLE, we can estimate the reliability function at any time t .

1. Reliability Function with Estimated $\hat{\lambda}$:

$$\hat{R}(t) = e^{-\hat{\lambda}t}$$

Example Calculation

Assume the following data:

- n = 10 components
- r = 6 failures observed at times 2, 4, 5, 7, 8, 9.
- t = 10 (censoring time for the remaining n - r = 4 components)

1. Calculate the MLE for $\hat{\lambda}$:

Sum of failure times = 2 + 4 + 5 + 7 + 8 + 9 = 35

$$\hat{\lambda} = \frac{6}{75} = 0.08$$

2. Reliability Calculation for t = 5 :

$$\hat{R}(5) = e^{-5*0.08} \approx e^{-0.4} \approx 0.6703.$$

8.5.2 Reliability Estimation of Weibull Distribution under Type II Censoring

The Weibull distribution is widely used in reliability engineering and survival analysis due to its flexibility in modelling various types of failure rates. Reliability estimation under Type II censoring involves determining the reliability function when the study is terminated at a fixed number of failures.

Weibull Distribution

The Weibull distribution is characterized by two parameters:

- Shape parameter (α)
- Scale parameter (λ)

The probability density function (PDF) is:

$$f(x; \lambda, \alpha) = \alpha\lambda(\lambda x)^{\alpha-1}e^{-(\lambda x)^{\alpha}}, x \geq 0$$

The cumulative distribution function (CDF) is:

$$F(x;\lambda,\alpha) = 1 - e^{-(\lambda x)^{\alpha}}$$

The reliability function R(t) (the probability that a component survives beyond time t) is:

$$R(t) = 1 - F(t) = e^{-(\lambda t)^{\alpha}}$$

Maximum Likelihood Estimation (MLE)

To estimate the parameters $\lambda \; and \; \alpha$ under Type II censoring, we use the maximum likelihood estimation method. The likelihood function L for r failures and n - r censored observations is:

$$L(\lambda,\alpha|\underline{x}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}(f(x_i))\left(1 - F(x_{(r)})\right)^{n-r}$$

$$L(\lambda,\alpha|\underline{x}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}\left(\alpha\lambda(\lambda x_i)^{\alpha-1}e^{-(\lambda x_i)^{\alpha}}\right)\left(e^{-(\lambda x_r)^{\alpha}}\right)^{n-r}$$

$$L(\lambda,\alpha|\underline{x}) = \frac{n!}{(n-r)!}\alpha^{r}\lambda^{r\alpha}\prod_{i=1}^{r}\left((x_i)^{\alpha-1}e^{-(\lambda x_i)^{\alpha}}\right)\left(e^{-(\lambda x_r)^{\alpha}}\right)^{n-r}$$

2. **Log-Likelihood Function is:**

$\ln L(\lambda,\alpha) \propto \text{r}\ln\alpha + r\alpha\ln\lambda + (\alpha-1)\sum_{i=1}^{r} ln(x_i) - -\lambda^{\alpha}(\sum_{i=1}^{r} {x_i}^{\alpha} + (n-r){x_{(r)}}^{\alpha})$

Now we differentiate log likelihood with respect to parameters

$\frac{\partial l}{\partial\alpha} = \frac{r}{\alpha} + rln\,\lambda + \sum_{i=1}^{r} ln(x_i) - \lambda^{\alpha}\ln(\lambda)\left(\sum_{i=1}^{r} {x_i}^{\alpha} + (n-r){x_{(r)}}^{\alpha}\right) - \lambda^{\alpha}\left[\sum_{i=1}^{r} {x_i}^{\alpha}\ln(x_i) + (n-r){x_{(r)}}^{\alpha}\ln(x_{(r)})\right]=0$

$$\frac{r}{\alpha} + rln\,\lambda + \sum_{i=1}^{r} ln(x_i) - \lambda^{\alpha}\left[\ln(\lambda)\left(\sum_{i=1}^{r} {x_i}^{\alpha} + (n-r){x_{(r)}}^{\alpha}\right) + \sum_{i=1}^{r} {x_i}^{\alpha}\ln(x_i) + (n-r)\,{x_{(r)}}^{\alpha}\ln(t_r)\right]$$

=0

$$\frac{\partial l}{\partial\lambda} = \frac{r\alpha}{\lambda} - \alpha\lambda^{\alpha-1}\left(\sum_{i=1}^{r} {x_i}^{\alpha} + (n-r){x_{(r)}}^{\alpha}\right)=0$$

$$\frac{r}{\lambda} = \lambda^{\alpha-1}\left(\sum_{i=1}^{r} x_i{}^{\alpha} + (n-r)x_{(r)}{}^{\alpha}\right)$$

$$r = \lambda^{\alpha}\left(\sum_{i=1}^{r} x_i{}^{\alpha} + (n-r)x_{(r)}{}^{\alpha}\right)$$

$$\lambda^{\alpha} = \frac{r}{\left(\sum_{i=1}^{r} x_i{}^{\alpha} + (n-r)x_{(r)}{}^{\alpha}\right)}$$

$$\hat{\lambda} = \left(\frac{r}{\left(\sum_{i=1}^{r} x_i{}^{\alpha} + (n-r)x_{(r)}{}^{\alpha}\right)}\right)^{\frac{1}{\alpha}}$$

To find the MLEs for α and λ, we need to solve the system of equations obtained by setting the partial derivatives of the log-likelihood function with respect to α and λ to zero. This typically requires numerical methods.

Reliability Estimation: Once α and λ are estimated, the reliability function R(t) can be estimated using the invariance property of MLE which is given by:

$$R(t) = e^{-(\hat{\lambda}t)^{\hat{\alpha}}}$$

8.6 Stress-Strength Reliability

The stress-strength model is a reliability model used to evaluate the probability that a system or component will perform its intended function under given stress conditions. It is commonly applied in engineering and reliability analysis. In the stress-strength model, the reliability of a component is determined by comparing the random variables representing the stress, applied to the component, and the strength of the component. Let *Y* be the random variable representing the stress and *X* be the random variable representing the strength.

Reliability Calculation

The reliability R of the component is the probability that the strength X exceeds the stress Y:

$$R = P(X > Y)$$

8.6.1 Application in Reliability Engineering

In reliability engineering, the Stress-Strength model helps in evaluating the likelihood that a component or system will perform reliably under expected stress conditions. It provides insights into how well a product will withstand operational stresses and helps in designing products that meet safety and performance standards. Reliability Prediction: The model predicts the probability of failure by calculating the likelihood that the applied stress will exceed the strength of the component. This prediction helps in assessing the overall reliability of the system under typical operating conditions.

8.6.2 Application in Survival Analysis

In survival analysis, the Stress-Strength model is used to understand the relationship between the duration until an event occurs (e.g., failure, death) and the underlying factors that influence survival. This model helps in evaluating the effects of different stress factors on the survival time of subjects.

(i) Medical Research

(i) Patient Survival: In medical research, the Stress-Strength model can be applied to predict patient survival times based on factors such as disease severity, treatment efficacy, and patient characteristics. The model helps in understanding how stressors (e.g., disease progression) affect patient survival.

(ii) Treatment Efficacy: Researchers use the model to evaluate the effectiveness of treatments by comparing the strength of the treatment effect against the stress imposed by the disease. This helps in determining the likelihood of treatment success and patient recovery.

(ii) Reliability of Medical Devices

(i) Device Performance: In the context of medical devices, the Stress-Strength model assesses the reliability of devices under operational conditions. This includes evaluating how well a device performs under stress conditions like prolonged use or varying environmental conditions.

(ii) Failure Analysis: The model helps in analysing the likelihood of device failure and understanding the factors that contribute to it. This information is crucial for improving device design and ensuring patient safety.

8.6.3 Applications

Example 1: Reliability Engineering

(i) Scenario: An engineer is designing a new mechanical component that must withstand a maximum operational stress of 1500 psi. The component is tested under various conditions, and its strength follows a normal distribution with a mean of 1600 psi and a standard deviation of 100 psi.

(ii) Application: Using the Stress-Strength model, the engineer calculates the probability that the component will fail under the maximum stress condition. This probability helps in assessing whether the component meets the reliability requirements.

Example 2: Survival Analysis in Medicine

(i) Scenario: A study aims to evaluate the impact of a new drug on patient survival rates for a specific type of cancer. The stress (e.g., cancer progression) and strength (e.g., drug efficacy) are modelled using survival data.

(ii) Application: The model helps in estimating the probability that patients will survive beyond a certain time frame, given the drug's efficacy and the severity of the cancer. This information is used to make decisions about the drug's effectiveness and treatment recommendations.

Summary

(i). Reliability Engineering: The Stress-Strength model helps in designing, manufacturing, and testing products by comparing the strength of materials/components with the applied stress. It predicts reliability, assesses risk, and estimates component life.

(ii). Survival Analysis: The model is used to understand and predict survival times based on stress factors and treatment strengths. It is applied in medical research, epidemiology, and evaluating medical devices.

The Stress-Strength model is a powerful tool in both fields, providing valuable insights into how stress and strength interact to influence performance and survival.

8.6.4 Stress Strength Model with Examples

Stress-Strength Model

The stress-strength model is used in reliability engineering to assess the probability that a system will function properly under varying stress levels. The model compares the distribution of the stress applied to a system with the distribution of the system's strength. The primary goal is to evaluate the reliability of the system by determining the probability that the stress does not exceed the strength.

Key Components

The Stress Distribution is represented by Y with cumulative distribution function (CDF) is $F_Y(y)$ and strength Distribution is represented by X with CDF $F_X(x)$.

The reliability of the system R, is the probability that the strength is greater than the stress:

$$R = P(X>Y)$$

If Y and X are independent, the reliability can be calculated as:

$$R = P(X > Y) = \int_0^{\infty} \int_y^{\infty} f_X(x)\, \mathrm{f}_\mathrm{Y}(y)\, dxdy$$

where $f_X(x)$is the probability density function (PDF) of X , and $\mathrm{f}_\mathrm{Y}(y)$ is the CDF of S .

Examples

1. Normal Stress and Strength

- Stress Distribution: $Y \sim \mathrm{N}(\mu_Y, \sigma_Y^2)$
- Strength Distribution: $X \sim \mathrm{N}(\mu_X, \sigma_X^2)$

For normal distributions, the reliability function can be calculated using:

$$R = P(S > X) = P\left(\frac{X - \mu_X}{\sigma_X} > \frac{Y - \mu_Y}{\sigma_Y}\right)$$

This can be simplified using the cumulative distribution function of the standard normal distribution:

$$R = 1 - \Phi\left(\frac{\mu_Y - \mu_X}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right)$$

Example Calculation:

- Stress: $\mu_Y = 5, \sigma_Y = 1$

- Strength: $\mu_X = 7, \sigma_X = 2$

Reliability:

$$R = 1 - \Phi\left(\frac{5-7}{\sqrt{1^2+2^2}}\right)$$

$$R = 1 - \Phi\left(\frac{-2}{\sqrt{5}}\right)$$

$$R = 1 - \Phi(-0.894) \approx 1 - 0.185 = 0.815$$

8.6.4.1 Reliability Estimation of Exponential Distributions for Stress Strength Model

Reliability estimation involves calculating the probability that the strength exceeds the stress. The probability density function (PDF) of an exponential distribution with parameter λ (rate parameter) is:

$$f(x;\lambda) = \lambda\, e^{-\lambda x}, \qquad x \geq 0$$

The cumulative distribution function (**CDF**) is:

$$F(x;\lambda) = 1 - e^{-\lambda x}$$

Let X be the random variable representing the strength of a component, and Y be the random variable representing the applied stress. Both X and Y follow exponential distributions with parameters λ_x and λ_y , respectively. Our aim is to calculate the reliability R, which is the probability that the strength X exceeds the stress Y. Mathematically:

$$R = P(X > Y)$$

Calculation

Since X and Y are independent random variable, the reliability R can be computed as:

$$R = P(X > Y) = \int_0^{\infty} \int_y^{\infty} f_X(x)\, f_Y(y)\, dx dy$$

Solving inner integral, we get

$$R = \int_0^{\infty} e^{-\lambda_X y} \;\; \lambda_Y e^{-\lambda_Y y}\, dy$$

$$R = \lambda_Y \int_0^{\infty} e^{-(\lambda_X+\lambda_Y)y}\, dy$$

The integral is the standard form of an exponential distribution's CDF:

$$\int_0^{\infty} e^{-(\lambda_X+\lambda_Y)y}\, dy = \frac{1}{(\lambda_X + \lambda_Y)}$$

The stress-strength reliability when both X and Y follows exponential distributions is:

$$\text{R}=\frac{\lambda_Y}{(\lambda_X+\lambda_Y)}$$

Using the ML estimators of parameters, the ML estimator of stress strength reliability of exponential distribution

$$\text{R}=\frac{\widehat{\lambda_Y}}{(\widehat{\lambda_X}+\widehat{\lambda_Y})}$$

Example Calculation

Let's compute the reliability for specific Exponential parameters.

Given:

- Stress Distribution: $Y \sim \text{Exponential}(\lambda_Y = 1)$
- Strength Distribution: $X \sim \text{Exponential}(\lambda_X = 2)$

Calculate Reliability:

$$\text{R}=\frac{\lambda_Y}{(\lambda_X+\lambda_Y)}$$

$$R = \frac{1}{1+2}$$

$$R = \frac{1}{3}$$

$$R = 0.33$$

8.6.4.2 Reliability Estimation of Weibull Distribution for Stress Strength Model

Suppose stress and strength both follow Weibull distributions; reliability estimation involves comparing the Weibull distributions of stress and strength to determine the probability that the strength will exceed the stress.

A Weibull distribution is defined by two parameters:

The probability density function (PDF) for a Weibull distribution is:

$$f(x;\lambda,\alpha) = \alpha\lambda(\lambda x)^{\alpha-1}e^{-(\lambda x)^{\alpha}}, x \geq 0$$

- Where Shape Parameter (α) and Scale Parameter (λ)

The cumulative distribution function (CDF) is:

$$F(x;\lambda,\alpha) = 1 - e^{-(\lambda x)^{\alpha}}$$

Let X be the random variable representing the strength of a component, and Y be the random variable representing the applied stress. Both X and Y follow Weibull distributions with parameters λ_x, α_x and λ_y, α_y, respectively. Our aim is to calculate the reliability R, which is the probability that the strength X exceeds the stress Y. Mathematically:

$$R = P(X > Y)$$

Given stress and strength distributions:

- Strength Distribution: $X \sim$ Weibull(λ_x, α_x)
- Stress Distribution: $Y \sim$ Weibull(λ_y, α_y)

The aim is to calculate the reliability R . Mathematically:

$$R = P(X > Y)$$

Given that X and Y are independent, the reliability function R is:

$$R = P(X > Y) = \int_0^\infty \int_y^\infty f_X(x)\, \mathrm{f_Y}(y)\, dxdy \qquad (*)$$

where $f_X(x)$is the pdf of X and $\mathrm{f_Y}(y)$ is the pdf of Y. Further, we have

$$\int_y^\infty \alpha_x\lambda_x(\lambda_x x)^{\alpha_x-1}e^{-(\lambda_x x)^{\alpha_x}}\, dx = \alpha_x{\lambda_x}^{\alpha_x}\int_y^\infty (t)^{\alpha_x-1}e^{-(\lambda_x t)^{\alpha_x}}\, dx$$

$$= e^{-(\lambda_x y)^{\alpha_x}} \qquad (**)$$

Finally from (*) and (**), the reliability function becomes:

$$R = \int_0^{\infty} \alpha_y \lambda_y (\lambda_y y)^{\alpha_y - 1} e^{-(\lambda_y y)^{\alpha_y}} e^{-(\lambda_x y)^{\alpha_x}} \, dy$$

$$R = \alpha_y {\lambda_y}^{\alpha_y} \int_0^{\infty} y^{\alpha_y - 1} \, e^{-(\lambda_y y)^{\alpha_y} - (\lambda_x y)^{\alpha_x}} dy$$

This integral generally does not have a closed-form solution and often requires numerical methods to evaluate.

8.7 Summery

This unit covers the fundamental concepts and methodologies for estimating the parameters of well-known distributions such as Exponential, Weibull, Gamma, and Log-Normal. It also explores how to evaluate reliability for complete failure data as well as censored data, providing a comprehensive understanding of reliability assessment. Additionally, the units explain the stress-strength model, which is crucial for assessing system reliability based on the comparison between applied stress and material strength.

8.8 Self-Assessment Exercises

1. A life test was conducted on 14 identical components, and the following failure times (in hours) were recorded: 5, 7, 12, 15, 20, 22, 30, 35, 40, 45, 50, 55, 60, and 80. Assume the failure times follow an exponential distribution. Estimate the failure rate (λ) using the Maximum Likelihood Estimation (MLE) method, estimate the Mean Time to Failure (MTTF) and also Estimate the reliability at 25 hours using the estimated λ.

2. A life test was conducted on 10 identical components, and the recorded failure times (in hours) were as follows: 15, 20, 30, 35, 50, 55, 60, 75, 80, and 100. Assume that the data follows a Weibull distribution. Estimate the parameters of Weibull distribution using the Maximum Likelihood Estimation (MLE) method and also estimate the reliability at 40 hours using the estimated parameters.

3. Consider a scenario where the lifetimes of a particular type of component are follow a log-normal distribution. Suppose a sample of 10 components is tested, and their

lifetimes (in hours) are recorded as follows: 100, 120, 150, 200, 250, 300, 350, 400, 450, and 500 hours. Estimate the parameters of the log-normal distribution: the mean μ and standard deviation σ of the natural logarithms of the lifetimes.

4. A life test is performed on 8 identical components. The test is terminated after 5 failures, and the recorded failure times (in hours) are: 10, 15, 20, 25, and 30. The test is stopped after the fifth failure, and the remaining 3 components are censored. Assuming the failure times follow an exponential distribution, estimate the failure rate λ and the reliability at 20 hours.

5. A life test is conducted on 10 identical components, with the test ending after 50 hours. The failure times (in hours) of the components are recorded as follows: 12, 18, 25, 30, and 40. The remaining 5 components did not fail by the end of the test and are therefore censored at 50 hours. Estimate the reliability of the components at 40 hours.

6. In a reliability test, the strength X and stress Y of a material are modelled by exponential distributions. The strength of the material has a rate parameter $\lambda_X = 0.01$ failures per hour, and the stress has a rate parameter $\lambda_Y = 0.015$ failures per hour. (i) Calculate the system reliability $R = P(X > Y)$.(ii) If the rate parameter for the stress λ_Y were to increase to 0.02 failures per hour, recalculate the system reliability.

8.9 References

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8.10 Further Readings

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UNIT 9: REPAIRABLE SYSTEMS

Structure

9.1 Introduction

As discussed in earlier chapters, a system is an attachment of components that work together to perform a specific function. It is obvious that during performing a given function the system will experience failure at a time point which is called failure time. Once a system fails then the choice is either it can be replaced by the new one or it can be repaired. There are various advantages to repairing a system instead of replacing it such as repair cost likely to be lower than a new system, less time to repair than replace the entire system setup, better resource optimization and many more.

In reliability theory, a repairable system is a chain which, after it has failed to perform properly, can be restored to satisfactory performance by any method except replacement of the entire system. The repairable systems are designed with the expectation that they will experience failure/breakdowns over time, but their useful life can be extended through part replacement, maintenance, or adjustments. Repairable systems are common in industries where replacing entire units would be costly or impractical. In many industries such as manufacturing, automotive, electronics, and infrastructure repairable systems are utilized.

A real-life example of a repairable system in the electronics industry is the smartphone. Once a smartphone experiences issues such as a cracked screen, faulty battery, or malfunctioning charging port, it can be repaired instead of being completely replaced. For instance, if the screen is damaged, a technician can replace just the screen, restoring the device to working condition. Similarly, if the battery loses its capacity, it can be swapped out for a new one, extending the overall lifespan of the smartphone.

The study of repairable systems involves understanding failure patterns, optimizing maintenance strategies, and balancing repair costs with system reliability to ensure long-term performance. In the following sections our aim is to introduce some statistical methods to deal with the repairable system.

9.2 Objectives

After going through this unit, you will be able to

- Know about basic concepts of repairable systems
- Maintenance and Replacement Policies for repairable systems
- Non-homogeneous Poisson process modeling
- Availability of repairable systems
- Preventive Maintenance Policy for Repairable Systems

9.3 Maintenance and Replacement Policies

The purpose of a reliability system is to create a sustainable environment, where a system and its components perform at their best. To achieve this, it becomes necessary to define the maintenance and replacement policies. For an organization, these policies serve as essential guidelines for managing the maintenance and eventual replacement of equipment, assets, or systems. These policies ensure reliability, minimize downtime (not available for use due to maintenance), and optimize costs over time.

9.3.1 Maintenance Policy

A maintenance policy outlines the procedures and frequency of servicing or repairing assets to ensure they operate efficiently and safely. The major types of maintenance policy include:

A. **Preventive Maintenance:** This is a proactive approach to maintaining a system by performing regular, scheduled tasks to prevent expected failure. This type of maintenance is carried out before issues arise. It is based on time intervals, usage, or condition-based triggers, to ensure that equipment remains in good working order and to avoid unexpected breakdowns. The examples of preventive maintenance are such as lubrication, cleaning, and replacing worn-out parts, routine inspections.

B. **Predictive Maintenance:** This is also a proactive maintenance strategy that uses real-time data and advanced diagnostic tools to predict when equipment or systems are likely to fail, allowing for maintenance to be performed just before a failure occurs. This approach is based on monitoring and analyzing key performance indicators, such as vibration, temperature, and wears patterns, to assess the condition of equipment.

C. **Corrective Maintenance:** Corrective maintenance is a reactive approach to maintaining equipment or systems that involves performing repairs or taking corrective actions after a failure or malfunction has occurred. It focuses on addressing and fixing the root cause of the issue to ensure that the system or equipment can resume its intended function.

D. **Routine Maintenance:** It refers to the regular, often scheduled, tasks performed to keep components or systems in good working condition and to prevent potential problems. Routine maintenance helps to ensure that the system operates smoothly. These tasks are typically performed on a daily, weekly, or monthly basis and include activities such as cleaning, lubricating, adjusting, and inspecting.

The above-mentioned maintenance policies can be effectively implemented in an organization through the following practices: conducting frequent checks and inspections, assigning maintenance tasks to responsible personnel, maintaining proper documentation of all maintenance activities, and allocating an appropriate budget for maintenance tasks and supplies. These practices help to ensure that maintenance is carried out systematically and efficiently.

9.3.2 Replacement Policy

As a system ages, the costs and risks associated with maintenance and repairs may increase rapidly and replacement becomes a better option for ensuring reliability. A well-defined replacement policy helps to avoid unexpected system failures, reduce downtime, optimize resource allocation, and plan for capital expenditures. So the purpose of a replacement policy is to ensure that systems are replaced at the right time before they become unreliable, or too costly to maintain. The major types of replacement policy include:

A. **Age-Based Replacement:** This is a maintenance strategy in which systems or components are replaced after reaching a prefixed time, regardless of their current condition or performance. This approach assumes that the probability of failure increases with age. Such replacement is often used for systems with predictable life time, such as batteries, filters, etc..

B. **Condition-Based Replacement:** This is a maintenance strategy in which systems or components are replaced based on their actual condition at inspection/monitoring time rather. When the condition of a system falls below a certain level or shows signs of possible failure, it is replaced to prevent breakdowns. For example, in data centers, where server hard drives are monitored using Self-Monitoring technique and when the system detects signs of declining performance, the hard drive is proactively replaced before it fails.

C. **Usage-Based Replacement:** This is a maintenance strategy where systems or components are replaced based on the amount of usage they have experienced, rather than on a fixed time schedule or their current condition. This approach relies on tracking the actual operational usage, such as hours of operation, cycles, or distance traveled, to determine when an item should be replaced. For example, in vehicles, machinery, or aircraft, where components like tires, engines, or filters are replaced after a certain number of miles, cycles, or operating hours.

D. **Failure-Based Replacement:** In this maintenance strategy, systems or components are replaced only after they have failed. This approach is reactive, focusing on addressing issues that arise due to actual breakdowns. However, this strategy may lead to higher downtime and increased maintenance costs due to unexpected breakdowns.

So, replacement policies provide guidelines on when to replace a system so the repair cost does not exceed a proportion of the replacement cost. This is planning for the costs of replacements over time.

A good strategy is to use both maintenance and replacement policies together to keep systems function satisfactorily. By figuring out if a system needs more regular maintenance or should be replaced sooner. This helps them understand whether it's cheaper to keep fixing the system or to replace it, making sure they make the best choice for both cost and performance. For example, regularly updating and taking care of computers can help them last longer, but after about 5-7 years, they often need to be replaced because they become outdated or slow. Similarly, for vehicles, regular oil changes can keep a vehicle running smoothly, but after a certain number of distance or if it starts to wear out, replacing it with a new vehicle might be more cost-effective than constantly fixing the old one.

9.4 Availability of Repairable Systems

Repairable systems offer significant benefits across various industries by allowing organizations to maintain and extend the life of their equipment. They help save costs by enabling repairs instead of complete replacements, making them a cost-effective solution for managing expensive machinery and technology. Regular maintenance and repairs also reduce downtime, increase reliability, and ensure operational efficiency, which is crucial for continuous productivity. Additionally, repairable systems contribute to environmental sustainability by reducing waste, as parts are fixed or upgraded rather than discarded. Overall, repairable systems are essential for maximizing the value of investments, maintaining smooth operations, and supporting sustainable practices. Repairable systems are used in many different industries, and they are designed to be fixed and maintained rather than replaced entirely. Here are a few areas where such system available:

A. Manufacturing: In factories, equipment like conveyor belts, CNC machines, and industrial robots are built to be repaired. If something breaks down or wears out, parts can be replaced or fixed to keep everything running smoothly. This helps ensure that production doesn’t stop and that the factory continues to operate efficiently.

B. Aerospace and Aviation: Airplanes have important parts like engines, landing gear, and navigation systems that can be repaired. Regular maintenance and repairs are crucial to keep aircraft safe and reliable for many years of flying.

C. Automotive: Cars, trucks, and buses have many parts that can be repaired, such as engines, transmissions, and brakes. When these parts wear out or break, they can be fixed or replaced to keep the vehicle running safely and effectively.

D. Electronics: Consumer gadgets like smartphones, laptops, and home appliances often have parts that can be repaired, such as screens, batteries, and hard drives. If these parts fail, they can be serviced or replaced, which helps extend the life of the device.

E. Energy and Utilities: Equipment used to generate power, like turbines and generators, as well as infrastructure like pipelines and transformers, are designed to be repaired. Regular maintenance ensures they continue to work efficiently and supply energy without interruptions.

F. Healthcare: Medical devices, including MRI machines, ventilators, and diagnostic equipment, can be repaired to ensure they keep functioning correctly. This is important for providing accurate diagnoses and effective patient care.

G. Railway and Public Transportation: Trains, buses, and trams have components like engines, brakes, and electronics that can be repaired. Keeping these parts in good condition is essential for safe and reliable transportation.

Repairable systems are designed to be maintained and fixed as needed. This approach helps to keep things running well, extends the life of the equipment, and manages costs effectively.

9.5 Modeling a Repairable System

The objective of this chapter is to study about the repairable system. So in this section, few simple statistical methods are introduced for a repairable system under reliability data. So let's consider a repairable system where $T_1, T_2, T_3, \ldots$ represent the times at which the system fails. Let X_i for $i = 1,2,3, \ldots$ be the time between the failure $(\mathrm{i}-1)^{\mathrm{th}}$ and the i^{th} failure, with T_0 defined as zero. Both T_i and X_i are random variables. The corresponding observed values are t_i andx_i, respectively. Additionally, let $N(t)$ denote the number of failures that occur within the time interval $(0, t]$. In repairable system reliability, the random variable X is a variable of interest. For such systems, the occurrence of dependence or autocorrelation between the X is quite obvious.

One can understand that in component reliability, dealing with the time until the first failure of several similar components, it's common to assume that the lifetimes of these components are independent and identical. However, in repairable systems, you're looking at the times between repeated failures of the same system. In these cases, it's the differences from the independent and identical assumption that matter. So, it's not surprising that the statistical methods used for repairable systems are different from those used for component reliability.

In a repairable system, the behaviour can be termed as "happy" and "sad". When the time between failures gets longer, a system is said to be a happy system and for a shorter time it is called a sad one. It becomes important to identify and fix "sad" systems to prevent them from causing problems in production processes.

9.5.1 Rate of Occurrence of Failure

The next step is to design a mathematical formula to express whether a system is "happy" or "sad." To do this, we will consider the Rate of Occurrence of Failure (ROCOF), which can be defined as follows:

$$v(t) = \frac{\mathrm{d}}{\mathrm{dt}}\mathrm{E}[\mathrm{N(t)}].$$

A happy system will have a decreasing ROCOF, while a sad system will have an increasing ROCOF. However, it is important to note that just because a system is happy (or sad) does not necessarily mean that, in practical terms, it is satisfactory (or unsatisfactory). For example, a system with a very low ROCOF may be perfectly satisfactory for its intended lifespan, even if its ROCOF is increasing.

It is also important to distinguish between ROCOF and the hazard function, as both are sometimes referred to as the failure rate. Interestingly, it is possible for a system to have a non-decreasing hazard function while still experiencing a decreasing ROCOF. A natural estimator of$v(t)$ is$\hat{v}(t)$, given by

$$\hat{v}(t) = \frac{\text{no.of failures in (t,t+}\delta\text{t)}}{\delta\text{t}} \qquad (9.1)$$

Here, δt represents a suitable time interval. The choice of δt is arbitrary, the goal is to highlight the main features of the data while smoothing out the noise.

One can visualize the behaviour of a repairable system by a simple informative graph comprising the cumulative number of failures versus the cumulative time. In such graphs departures from linearity are indicative of the fact that the X, are not IID. In particular this plot is useful for detecting the presence of a trend. To gain an indication as to the form of the trend the estimated ROCOF may be calculated and/or plotted against time.

For example, in Table 9.1 21 observations are the failure time (in days) corresponding to the failed component of a smartphone before it is replaced by the owner. Figure 9.1 shows the behaviour of the repairable system. In the plot it can be seen that failure pattern is almost linear so it is expected that ROCOF is also approximately constant.

Table 9.1: Failure time data of a smartphone with component of failure.

Failure Number	Failure Time	Inter-Failure Inter val	Component
1	112	112	Battery
2	251	139	Camera
3	262	11	Camera
4	289	27	Screen
5	381	92	Camera
7	402	21	Screen
8	425	23	Processor
9	486	61	Battery
10	642	156	Screen
11	660	18	Screen
12	781	121	Camera
13	783	2	Camera
14	800	17	Battery
15	852	52	Camera
16	882	30	Processor
17	891	9	Screen
18	903	12	Processor
19	978	75	Processor
20	981	3	Processor
21	1135	154	Battery

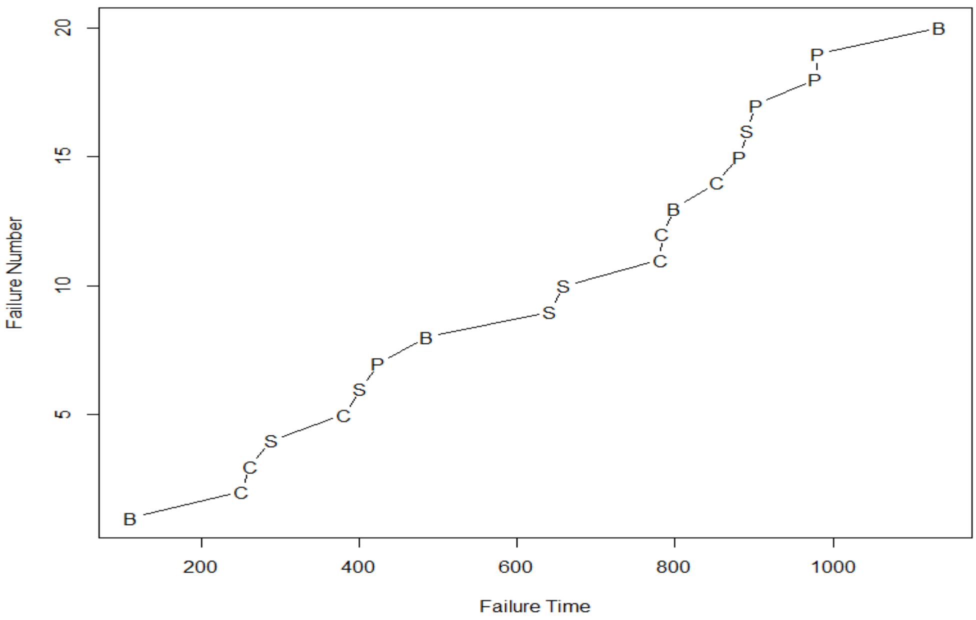

Figure 9.1: Plot of failure number versus failure time in days for the smartphone data.

9.5.2 Non-Homogeneous Poisson Process Model

A stochastic point process can be thought of as a series of events happening in a continuous space, like time, where the events follow a certain probability pattern. If we think of the continuous space as time and the events as failures, this idea fits well with the study of systems that can be repaired after failing. While there are different models, we could use to describe how failures happen in a repairable system, we will focus on the Non-Homogeneous Poisson Process (NHPP). This model is easy to understand, can represent both systems that improve over time and those that get worse, and has well-developed statistical methods that are simple to use. The assumptions for a NHPP are as for the Poisson process except that the ROCOF varies with time rather than being a constant.

Consider a NHPP with time dependent ROCOF$\nu(t)$ (this is sometime called intensity function), then the numbers of failures in the time interval $(t_1, t_2]$ has a Possion process with mean $\int_{t_1}^{t_2} \nu(t)dt$. Thus, the probability of number of failures in time interval (t_1, t_2) is $\exp\left\{-\int_{t_1}^{t_2} \nu(t)dt\right\}$. Here it is easy to see that if $\nu(t) \equiv \nu$ for all t, then this will give a homogenous process with constant ROCOF ν. By choosing a suitable parametric form for$\nu(t)$, a flexible model for the failures of a repairable system in a 'minimal repair' setup can be obtained.

Suppose we observe a system for the time interval $[0, t_0]$ with failures occurring at $t_1, t_2, t_3, \dots t_n$. The likelihood may be obtained as follows. The probability of observing no failures in$(0, t_1)$, one failure in$(t_1, t_1 + \delta t_1)$, no failures in$(t_1 + \delta t_1, t_2)$, one failure in$(t_2, t_2 + \delta t_2)$ and so on up to no failures in$(t_n + \delta t_n, t_n)$, (for small$\delta t_1, \delta t_2, \dots \delta t_n$) is

$$\left\{exp\left(-\int_0^{t_1} \nu(t)dt\right)\right\} \nu(t_1)\delta t_1 \left\{exp\left(-\int_{t_1+\delta t_1}^{t_2} \nu(t)dt\right)\right\} \nu(t_2)\delta t_2 \dots \left\{exp\left(-\int_{t_n+\delta t_n}^{t_0} \nu(t)dt\right)\right\}$$

Dividing throughout by$\delta t_1, \delta t_2, \dots \delta t_n$ and letting $\delta i \to 0$, $(i = 1,2 \dots n)$ gives the likelihood function

$$L = \{\prod_{\nu=1}^{n} \nu(t_i)\}\, exp\left(-\int_0^{t_0} \nu(t)dt\right) \quad (9.2)$$

and the log likelihood is thus,

$$l = \sum_{i=1}^{n} log\, \nu\,(t_i) - \int_0^{t_0} \nu(t)dt \quad (9.3)$$

In some cases, the exact times of the failures aren't recorded, and only the number of failures within separate, non-overlapping time periods is known. For example, if

$n_1, n_2, n_3, \dots n_m$ failures have been recorded in non-overlapping time intervals $(a_1, b_1], (a_2, b_2] \dots, (a_m, b_m]$, then the likelihood function is

$$L = \left\{ exp\left(-\int_{a_1}^{b_1} v(t)dt\right) \frac{\left(\int_{a_1}^{b_1} v(t)dt\right)^{n_1}}{n_1!} \right\} \left\{ exp\left(-\int_{a_1}^{b_1} v(t)dt\right) \frac{\left(\int_{a_1}^{b_1} v(t)dt\right)^{n_1}}{n_1!} \right\} \dots\dots\dots$$

$$\dots\dots\dots\dots\dots \left\{ \left\{ exp\left(-\int_{a_m}^{b_m} v(t)dt\right) \frac{\left(\int_{a_m}^{b_m} v(t)dt\right)^{n_m}}{n_m!} \right\} \right\}$$

$$= \left\{ exp\left(-\sum_{i=1}^{m} \int_{a_1}^{b_1} v(t)dt\right) \prod_{i=1}^{n} \frac{\left(\int_{a_i}^{b_i} v(t)dt\right)^{n_i}}{n_i!} \right\}$$

Thus, the log-likelihood (apart from additive constant) is

$$l = \sum_{i=1}^{m} \left\{ n_i \log \int_{a_i}^{b_i} v(t)dt - \int_{a_i}^{b_i} v(t)dt \right\} \tag{9.4}$$

Therefore, once $v(t)$ has been specified, it is straight forward to use the likelihood-based method to obtain ML estimates for any unknown parameters inherent in the specification of$v(t)$. Here the two obvious choices of $v(t)$ that give monotonic ROCOF are considered for inference purpose.

(a) Log-Linear ROCOF $\quad v_1(t) = exp(\beta_0 + \beta_1 t)$

(b) Weibull Process $\quad v_2(t) = \gamma\delta t^{\delta-1}; \quad \gamma > 0, \delta > 0$

Both of these widely used due to their applicability in the real-life problems. Other complicated models are also available in the literature which can be utilised as per the requirements.

9.5.3 NHPP with Log-Linear ROCOF

A log-linear ROCOF model, $v_1(t) = exp(\beta_0 + \beta_1 t)$ is a simple model to describe a happy system($\beta_1 < 0$) or a sad system($\beta_1 > 0$). Also, if the parameter $\beta_1$is near zero$v_1(t)$ approximate a linear trend in ROCOF over short time period. Here we discuss some likelihood based statistical methods for fitting a NHPP with$v_1(t)$ to a set of repairable system data.

Putting the value of $v(t)$ in (9.3), we have

$$l_1 = \sum_{i=1}^{n} (\beta_0 + \beta_1 t_i) - \int_0^{t_0} e^{\beta_0 + \beta_1 t_i} dt$$

$$=n\beta_0 + \beta_1 \sum_{i=1}^{n}(t_i) - e^{\beta_0} \int_0^{t_0} e^{\beta_1 t_i}\, dt$$

$$=n\beta_0 + \beta_1 \sum_{i=1}^{n}(t_i) - e^{\beta_0} \frac{(e^{\beta_1 t_i - 1})}{\beta_1}$$

To obtain ML estimate of β_0 and β_1, we have

$$\frac{\partial l_1}{\partial \beta_0} = n - \frac{1}{\beta_1} e^{\beta_0}\left(e^{\beta_1 t_0} - 1\right) \quad (9.5)$$

$$\frac{\partial l_1}{\partial \beta_0} = \sum_{i=1}^{n} t_i - e^{\beta_0}\left[\frac{\beta_1 e^{\beta_1 t_0} t_0 - (e^{\beta_1 t_0} - 1) * 1}{\beta_1^2}\right] \quad (9.6)$$

The ML estimates of parameters can be obtained by equating (9.5) and (9.6) to zero. From the equation (9.5), by putting the value of $e^{\beta_0} = n\beta_1/\left(e^{\beta_1 t_0} - 1\right)$ in (9.6), we get

$$\sum_{i=1}^{n} t_i - \frac{n\beta_1}{(e^{\beta_1 t_0} - 1)}\left[\frac{t_0 \beta_1 e^{\beta_1 t_0} - \left(e^{\beta_1 t_0} - 1\right)}{\beta_1^2}\right] = 0$$

$$\sum_{i=1}^{n} t_i - n t_0 \frac{e^{\beta_1 t_0}}{(e^{\beta_1 t_0} - 1)} + n\beta_1^{-1} = 0$$

$$\sum_{i=1}^{n} t_i - n t_0 \{1 - e^{\beta_1 t_0}\}^{-1} + n\beta_1^{-1} = 0$$

Any simple iterative method (i.e. Bisection method) is sufficient to obtain the estimate $\hat{\beta}_1$ of$\beta_1$and by putting the value $\hat{\beta}_1$ we can get

$$\hat{\beta}_0 = log \frac{n\hat{\beta}_1}{\left(e^{\hat{\beta}_1 t_0 - 1}\right)} \quad (9.7)$$

The observed information matrix, evaluated at the MLE, has entries

$$\frac{-\partial^2 l_1}{\partial \beta_0^2} = n$$

$$\frac{-\partial^2 l_1}{\partial \beta_0 \partial \beta_1} = \sum_{i=1}^{n} t_i$$

$$\frac{-\partial^2 l_1}{\partial \beta_1^2} = \widehat{\beta_1^{-1}}\left\{\sum_{i=1}^{n} t_i \left(\widehat{\beta_1} t_0 - 2\right) + n t_0\right\}$$

Inverting the information matrix gives the variance-covariance matrix for $(\hat{\beta}_0, \hat{\beta}_1)$. In particular the standard error of $\hat{\beta}_1$ is

$$se(\hat{\beta}_1) = \left\{\hat{\beta}_1^{-1}\left\{\sum_{i=1}^{n} t_i(\hat{\beta}_1 t_0 - 2) + nt_0\right\} - n^{-1}\left(\sum_{i=1}^{n} t_i\right)^2\right\}^{-1/2} \qquad (9.8)$$

The maximized log-likelihood may be obtained by substituting $\hat{\beta}_0$ and $\hat{\beta}_1$, giving

$$l_1(\hat{\beta}_0, \hat{\beta}_1) = n\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{n} t_i - n$$

A natural hypothesis to test in repairable system reliability is that the ROCOF is constant; that is $\beta_1 = 0$. If β_1 is set to zero, then

$$\frac{\partial l}{\partial \beta_0} = n\beta_0 - t_0 \exp(\beta_0) = 0$$

$$\Rightarrow \beta_0 = \log \frac{n}{t_0}$$

This is maximized when β_0 is $log(n/t_0)$. So the maximized log-likelihood when $\beta_1 = 0$ is

$$n\, log(n/t_0) - n$$

Hence using the result

$$W = 2\{l(\hat{\beta}_0, \hat{\beta}_1) - l(\hat{\beta}_0, **)\}$$

We get

$$W = 2\left\{n\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{n} t_i - n\, log(n/t_0)\right\} \qquad (9.9)$$

has an approximate $\chi^2(1)$ distribution when $\beta_1 = 0$. Large values of W supply evidence against the null hypothesis. Note that it is necessary to evaluate the MLEs in order to perform the test based on W. A more commonly used test of $\beta_1 = 0$ is **Laplace's test.** This is based on the statistic

$$U = \frac{\sum_{i=1}^{n} t_i - \frac{1}{2} n t_0}{t_0 (n/12)^{1/2}}. \qquad (9.10)$$

The statistics U has approximately a standard normal distribution under the null hypothesis. If the alternative hypothesis is $\beta_1 \neq 0$, then large value of $|U|$ supply evidence against the null hypothesis in favour of the general alternative. If the alternative is that $\beta_1 > 0$, a large value of U supplies evidence against the null hypothesis in favour of this alternative. If the alternative is $\beta_1 < 0$, a large value of –U supplies evidence against the null hypothesis in favour of alternative. Laplace's test is asymptotically equivalent to the test based on *W*, but it does not require explicit evaluation of the MLEs. For further details see Cox and Lewis (1966).

In the case in which the repairable system is observed until the n^{th} failure, Laplace's test statistic *U* must be modified slightly. In (9.4) $t_0$must be replaced byt_n, and n must be replace by *(n-1)*.

When only numbers of failures within non-overlapping time intervals are available, then

$$\sum_{i=1}^{n} n_i \left\{\beta_0 + log\left(\frac{e^{\beta_1 b_i - e^{\beta_1 a_i}}}{\beta_1}\right)\right\} - e^{\beta_0} \sum_{i=1}^{m} \left(\frac{e^{\beta_1 b_i - e^{\beta_1 a_i}}}{\beta_1}\right) \quad (9.11)$$

In the special case in which the intervals are contiguous from time a_1 up to time a_1 so that$b_1 = a_2, b_2 = a_3, \dots, b_m = a_{m+1}$ the final term in (9.11) simplifies to

$$-e^{\beta_0}\left(\frac{e^{\beta_1 a_{m+1} - e^{\beta_1 a_i}}}{\beta_1}\right).$$

In this case the MLEs for$\beta_0$and$\beta_1$may be found first by solving

$$\sum_{i=1}^{m} n_i \left(\frac{a_{i+1}e^{\beta_1 a_{i+1}} - a_i e^{\beta_1 a_i}}{e^{\beta_1 a_{i+1}} - e^{\beta_1 a_i}}\right) - n\left(\frac{a_{m+1}e^{\beta_1 a_{m+1}} - a_1 e^{\beta_1 a_1}}{e^{\beta_1 a_{m+1}} - e^{\beta_1 a_i}}\right) = 0,$$

(9.12) where$n = n_1 + n_2 + \cdots + n_m$. Use any iterative method to solve (9.12) to obtain$\hat{\beta}_1$. Then$\hat{\beta}_0$ can be obtain by

$$\hat{\beta}_0 = log\left(\frac{n\beta_1}{e^{\beta_1 a_{m+1}} - e^{\beta_1 a_1}}\right).$$

9.5.4 NHPP with Weibull Process

In this section, let develop the estimation method for the ROCOF, $\nu_2(t) = \gamma\delta t^{\delta-1}$ then putting. So if this ROCOF is used in equation (9.3), the log-likelihood function will be as follows:

$$l_2 = \sum_{i=1}^{n} \log\left(\gamma\delta t_i^{\delta-1}\right) - \int_0^{t_0} \gamma\delta t^{\delta-1}\, dt$$

$$= nlog\gamma + nlog\delta + (\delta-1)\sum_{i=1}^{n} logt_i - \frac{\gamma\delta t_0^{\delta-1}}{\delta}$$

$$= nlog\gamma + nlog\delta + (\delta-1)\sum logt_i - \frac{\gamma\delta t_0^{\delta-1}}{\delta}$$

$$= nlog\gamma + nlog\delta + (\delta - 1)\sum_{i=1}^{n} logt_i - \gamma t_0^\delta$$

To obtain the parameter estimates, let differentiate the above derived log-likelihood function with respect to parameters. The

$$\frac{\partial l_2}{\partial \gamma} = \frac{n}{\gamma} - t_0^\delta = 0$$

$$\frac{\partial l_2}{\partial \gamma} = \frac{n}{\delta} + \sum \log t_i - \gamma t_0^\delta \log t_0 = 0$$

Now equating these equations to zero, the estimated of the parameters will be

$$\hat{\gamma} = \frac{n}{t_0^\delta}$$

$$\hat{\delta} = \frac{n}{nlogt_0 - \sum_{i=1}^{n} logt_i}$$

For information matrix, let first calculate the second order derivative of the log-likelihood function as follows:

$$-\frac{\partial^2 l_2}{\partial \gamma^2} = \frac{n}{\gamma^2}$$

$$-\frac{\partial^2 l_1}{\partial \gamma \partial \delta} = t_0^\delta logt_0$$

$$-\frac{\partial^2 l_1}{\partial \delta^2} = \frac{n}{\delta^2} + \text{n}\,(logt_0)^2$$

Let for ease of mathematics if choose $t_0 = 1$then the estimates of the above parameters will be as follows:

$$\hat{\gamma} = n; \hat{\delta} = \frac{-n}{\sum_{i=1}^{n} log\, t_i}.$$

In addition, the information matrix will become

$$-\frac{\partial^2 l_2}{\partial \gamma^2} = \frac{1}{n}$$

$$-\frac{\partial^2 l_2}{\partial \gamma \partial \delta} = 0$$

$$-\frac{\partial^2 l_1}{\partial \delta^2} = \frac{n}{\delta^2}$$

Further the standard error of $\hat{\delta}$ is, $se(\hat{\delta}) = \frac{\hat{\delta}}{\sqrt{n}}$.

Let consider the case of general t_0. The most commonly used test of constant ROCOF relative to the power law model is to test the null hypothesis that $\delta = 1$ using

$$V = 2\sum_{i=1}^{n} log\left(\frac{t_0}{t_i}\right)$$

Under the null hypothesis, V has a $\chi^2(2n)$ distribution. Large values of V supply evidence against the null hypothesis in favor of reliability growth $(\delta < 1)$. Small values of V are indicative of reliability deterioration$(\delta > 1)$. When the system is observed up to the n^{th} failure, the statistic V should be modified by replacing n by n-1, and t_0 by t_n with $\chi^2(2n-1)$ distribution.

9.6 Preventive Maintenance Policy for Repairable Systems

Preventive maintenance is when you regularly check and repair equipment before it breaks down. This is different from waiting until something breaks and then fixing it. The goal of preventive maintenance is to keep things running smoothly and avoid unexpected problems. There are several importance of preventive maintenance for the reliability. It avoids breakdowns, save money of an organization and extends the system life. The preventive maintenance has the following steps:

List All Equipment: Make a list of all the repairable systems that need preventive maintenance. Rank them by how important they are to the operation.

A. **Find Potential Problems:** Think about how each system could fail and what would happen if it did. This helps you decide what maintenance tasks are necessary.

B. **Set Maintenance Schedules:** Decide how often maintenance should be done for each system. This can be based on the manufacturer's recommendations or past experience.

C. **Assign Resources:** Make sure you have the people, tools, and parts ready for the maintenance work. This could mean having spare parts on hand.

D. Track Maintenance Work: Keep records of when and what maintenance was done. This helps you know when it's time for the next check-up and can improve future planning.

Preventive maintenance can be significantly enhanced through the strategic use of data, condition monitoring, and a focus on critical systems. By analyzing historical maintenance records and failure data, organizations can make more precise decisions regarding when maintenance is necessary, thereby avoiding redundant tasks while effectively preventing breakdowns. Moreover, advanced condition monitoring technology facilitates real-time oversight of equipment health, allowing for maintenance interventions only when truly warranted. This approach not only optimizes resource allocation but also minimizes operational downtime. Prioritizing maintenance on essential systems ensures that key equipment remains functional, thereby maintaining smooth operations and mitigating the potential effects of any failures.

9.7 Self-Assessment Questions

1. Describe the difference between a happy system and a sad system in terms of the Rate of Occurrence of Failures (ROCOF).
2. Explain the concept of a Non-Homogeneous Poisson Process (NHPP) in the context of repairable systems.
3. Provide an example of a situation where a repairable system might have an increasing ROCOF but still be considered satisfactory.
4. An automobile company is analyzing the reliability of its brake system over time based on the following failure data.

   Failure Time (miles): 10,000, 20,000, 30,500, 35,000, 40,200, 50,000, 55,300, 60,100, 70,000, 80,000, 85,000, 90,500, 95,000, 100,000, 110,000

   Component: Brake Pad, Brake Disc, Brake Fluid, Brake Pad, Brake Pad, Brake Disc, Brake Pad, Brake Pad, Brake Pad, Brake Disc, Brake Fluid, Brake Pad, Brake Pad, Brake Pad, Brake Disc

   Then determine whether the brake system is "happy" (decreasing failure rate) or "sad" (increasing failure rate) using the Rate of Occurrence of Failure (ROCOF) as the key indicator.

   Remark: The company has collected the following failure data for the brake system over a period of time.

9.8 Summary

In this unit we discuss the modelling of data from repairable systems. The Rate of Occurrence of Failures (ROCOF) is defined. We have employed Non-Homogeneous Poisson Processes (NHPP) models which allows the ROCOF to vary over time. Likelihood-based methods for Log-Linear and Weibull processes, are used to estimate parameters of these models and assess system reliability.

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UNIT 10: GROWTH MODELS AND ACCELERATED LIFE TESTING

Structure

10.1 Introduction

Reliability engineering is one of the crucial fields of research that ensures products or systems perform their intended functions without failures over a specified period of time. Under this discipline, reliability growth models and accelerated life testing (ALT) are two principal techniques that enhance the durability and dependability of systems. Reliability growth models are statistical tools used during the development and operational phases of systems to improve the reliability of systems. These models allow engineers to identify design flaws, manufacturing defects, and operational inefficiencies. By orderly analysis of data on failures and implementing corrective measures, the reliability growth models provide direction for continuous improvement in the reliability of systems. ALT is also an important technique that helps to estimate the life characteristics of products and systems in less time than under normal operating conditions. It significantly shortens the time needed to identify any ongoing or potential failure modes and mechanisms of a system by applying elevated stress levels to them.

In addition to these topics, probability plotting, a graphical technique used to assess whether a given data set follows a specified distribution, will also be discussed. In reliability analysis, probability plots are useful for visualizing and interpreting failure data. Examples used here in probability plots include the exponential, Weibull, and normal plots for portraying an underlying pattern, trend, or departure from assumed distributions to engineers. The plots aid reliability practitioners in making decisions regarding model selection and parameter estimation by graphically understanding the data. It will introduce one-to-one probability plotting techniques, together with their step-by-step procedures and illustrative examples.

Apart from these models and visualization, this chapter also covers tests for Exponentiality and tests for HPP vs. NHPP with Repairable System. Tests for Exponentiality cover two tests: the Hollander-Proschan test and the Deshpande test. These tests can be beneficial for testing whether failure data follows an exponential distribution. Tests for HPP vs. NHPP with repairable systems are beneficial for differentiating between HPP and NHPP, which provides accurate reliability analysis and maintenance planning. Both tests are important for accurate reliability analysis and maintenance planning.

10.2 Objectives

This unit will largely be on reliability growth models and accelerated life testing. After going through the chapter, readers will be equipped with the knowledge and tools to:

- Understand the principles and applications of reliability growth models.
- Utilize probability plotting techniques for reliability analysis.
- Conduct and interpret tests for exponentiality.
- Differentiate between HPP and NHPP in repairable systems.
- Design and implement accelerated life testing experiments.

10.3 Reliability Growth Models

Reliability Growth Models assist in improving the performance of a system and predicting future reliability based on the view that the system has met its intended reliability targets. More description is as follows:

10.3.1 Definition and Purpose

Reliability growth models are statistical tools used for assessing and improving system reliability over time. Especially in complex systems, such as software, hardware, and aerospace systems, these models are very useful. In the reliability analysis of such a system, it becomes essential to understand reliability growth since the improvement in reliability may happen due to various reasons at any stage of life. For instance, in software development and testing, defects are detected early to identify and rectify bug-related issues. In the case of hardware, the failures of the components are analysed to enhance the reliability of the system. In aerospace systems, the highest level of reliability is essential, as system failure can lead to catastrophic consequences. In all these cases, reliability growth models provide information on how reliability can be improved during the development and operational phases of the system. Such models also help to identify design flaws, manufacturing defects, and operational inefficiencies through failure data analysis and implementation of corrective measures. The overall objective of these models is to assist in improving the performance of a system and predicting future reliability based on the view that the system has met its intended reliability targets.

10.3.2 Duane Model

The Duane model, introduced by J.T. Duane in 1964, is a cornerstone methodology for analysing reliability growth during testing and development. The Duane model is an empirical model that describes reliability growth through a power-law relationship between cumulative operating time and cumulative failures. It is one of the earliest models used for reliability growth analysis. The Duane model is based on empirical observations and assumes a power law relationship between cumulative test time and cumulative number of failures. It can be interpreted as either a decrease in the failure rate or an increase in the mean time between failures (MTBF).

Cumulative Failure Rate (λ)

In the Duane model, we consider the cumulative failure rate as a power-law model, given as

$$\lambda_c = \frac{k}{T^m}, \tag{10.1}$$

where λ_c denotes the cumulative failure rate at time t, k represents the initial failure rate at T=1, T denotes the cumulative test time, and m represents the slope of the log-log plot of cumulative failures vs cumulative test time, as taking the logarithm of both sides gives

$$log\, \lambda_c = log\, k - m\, log\, T. \tag{10.2}$$

It can be observed from equation (8.2) that there exists a linear relationship between $log\, \lambda_c$ and $log\, T$, and m represents the slope of log-log plot. Notice that here, the linear model coefficient can be obtained by performing the ordinary least square linear regression method.

Further, suppose at start of the test, that is $T = T_0$, failure rate if λ_0. Using this, another form of growth model can be obtained as follows

$$\lambda_c = \lambda_0 \left(\frac{T}{T_0}\right)^{-m}.$$

Cumulative MTBF

Cumulative MTBF can be used as an alternative to failure rate. It can be expressed as

$$Cumulative\ MTBF = \frac{Total\ time\ of\ test}{Total\ number\ of\ failures}.$$

In the Duane growth model, as an alternative to failure rate, MTBF can also be used, given as

$$\theta_c = T^m / k , \tag{10.3}$$

where θ_c denotes the cumulative MTBF at time T. Taking the logarithm of both sides in equation (8.3), the following relation can be observed

$$log\, \theta_c = m\, log\, T - log\, k. \tag{10.4}$$

In certain cases, if someone prefers to work with MTBF instead of failure rate, then they can use a least-squares regression model to derive the model's coefficients. Note that it's important to update the regression model as new data is collected periodically.

Suppose at the initial time $T = T_0$, MTBF is obtained as θ_0 . This value helps us to derive following relation

$$\theta_c = \theta_0 \left(\frac{\mathrm{T}}{T_0}\right)^m \tag{10.5}$$

From equation (8.5), the following relation can be obtained

$$T = T_0 \left(\frac{\theta_c}{\theta_0}\right)^{1/m} \tag{10.6}$$

Using equation (8.6), the total time of test required to meet specific reliability requirements can be evaluated. At this point, I believe the reader can start thinking about the following question

How can MTBF be obtained at the starting time t_0?

The MTBF at the time t_0 can be obtained in three ways.

I. Using the assessment of past data.
II. When the system is new, there can be no past data, or there might be very little information about past data. In this case, the failure modes and corresponding causes can be analysed using tools such as cause-and-effect diagrams, Failure mode, effects, and criticality analysis (FMECA), and Fault tree analysis (FTA).
III. The short reliability demonstration test can also estimate MTBF at the starting point. If any failure occurs in this process, components can be replaced without considering it a design improvement.

Instantaneous Failure Rate and MTBF

The cumulative MTBF sometimes underestimates the current MTBF of the system, as it is evaluated using total failure and total time. For this reason, instantaneous MTBF might be a better indicator for identifying and solving the problem. Note that, instantaneous MTBF can be derived using the cumulative MTBF. By definition, cumulative MTBF can be derived as the ratio of total time and total number of failures, given as

$$\theta_c = \frac{T}{n}, \tag{10.7}$$

where T is the total time and n denotes the number of failures.

Note that here $\frac{dn}{dT}$ represents the instantaneous failure rate λ_i, whereas $\frac{dT}{dn}$ denotes the instantaneous MTBF θ_i. Using (10.7), following relation holds

$$n = \frac{T}{\theta_c},$$

Furthermore, using (10.5), n can be derived as

$$n = \frac{T}{\theta_0 \left(\frac{T}{T_0}\right)^m} = \frac{1}{\theta_0} \frac{T_0^m}{T^{m-1}}.$$

It gives,

$$\frac{dn}{dT} = (1-m)\frac{1}{\theta_0}\left(\frac{T_0}{T}\right)^m$$

or,

$$\frac{dn}{dT} = \frac{(1-m)}{\theta_c}. \tag{10.8}$$

Based on equation (10.8), we can also write

$$\theta_i = \frac{dT}{dn} = \frac{\theta_c}{(1-m)}, \tag{10.9}$$

Using (10.9), we can write

$$\lambda_i = (1-m)\lambda_c.$$

Example:

An aerospace company has tested a new aircraft engine over multiple phases. The testing phases and corresponding failure data are as follows:

Phase 1: 100 hours, 2 failures

Phase 2: 200 hours, 2 failures

Phase 3: 300 hours, 3 failures

Phase 4: 500 hours, 2 failures

Phase 5: 700 hours, 2 failures

Given the cumulative data up to Phase 5:

a) Calculate the cumulative MTBF θ_c.
b) Estimate the slope m.
c) Calculate the instantaneous MTBF θ_i at the end of Phase 3.

Solution

a) ***Calculate Cumulative MTBF :***

Total Operational Time (T):

$$T=100+200+300+500+700=1800 \text{ hours.}$$

Total Number of Failures (n):

$$n=2+2+3+2+2=11 \text{ failures}$$

Cumulative MTBF (θ_c):

$$\theta_c=T/n=1800/11\approx163.64 \text{ hours}$$

b) Estimate slope m:

To estimate m, we use the reliability growth model. Let's assume the Duane Model is applicable and use the cumulative data to estimate m.

Cumulative Data:

Table for evaluation of log T and log θ_c

Time	**Number of failures**	**T**	$\boldsymbol{\theta_c}$	**log T**	**log $\boldsymbol{\theta_c}$**
100	2	100	50.00	2.00	1.70
200	4	300	75.00	2.48	1.88
300	7	600	85.71	2.78	1.93
500	9	1100	122.22	3.04	2.09
700	11	1800	163.64	3.26	2.21

Using ordinary least square regression estimation technique for

$$log\, \theta_c \ = m\, log\, T - log\, k,$$

we obtain

$$m=0.39.$$

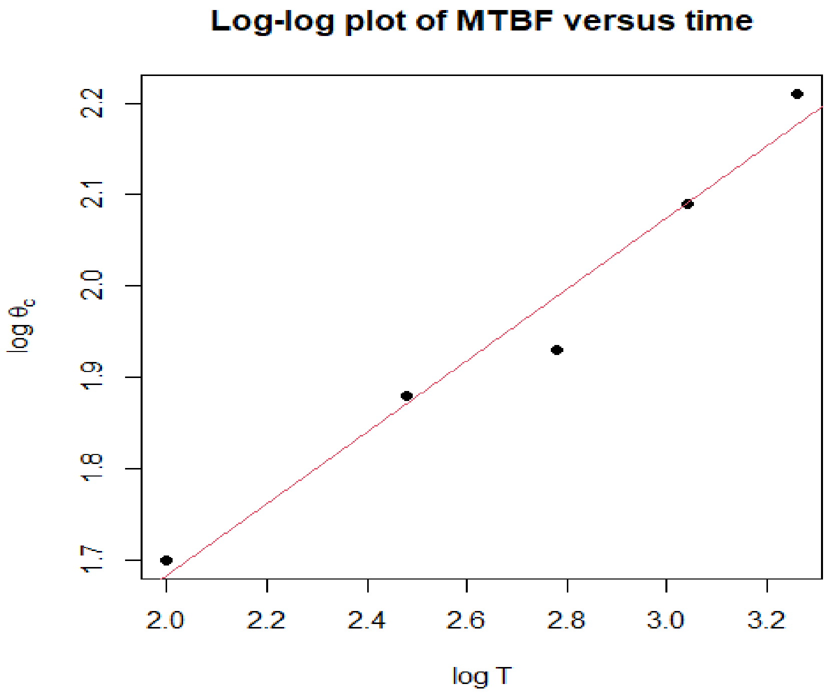

Figure 10.2 log-log plot of MTBF vs time

Note that the slope of the reliability growth curve reflects the rate at which problems can be identified and solved. Here, m for aerospace company is observed as 0.39, which can be considered as good.

c) Calculate Instantaneous MTBF (θ_i)

Using the relation:

$$\theta_i = \frac{\theta_c}{(1-m)} = \frac{163.64}{1-0.39} = 268.26$$

The instantaneous MTBF of 268.26 hours is a better indicator of the current reliability of the aircraft engine compared to the cumulative MTBF of 163.64 hours. This suggests that the reliability growth program is effective, and the engine's reliability is improving over time.

10.3.3 Latent or Dormant Failures

Sometimes, reliability growth tests might take a long time to observe any failure. In these cases, the concept of latent or dormant failures can be useful. Latent failures occur when potential failures have not occurred, or there is not enough planned time. There can be various reasons behind it. For instance, when the test duration is very small compared with a component's MTTF, it is highly unlikely that the component will actually fail during testing. Another instance might be the test conditions. If the component's failure is triggered by a specific environment that cannot be replicated in the reliability growth test, it is less likely to fail during testing. However, these failures will show up with sufficient test time under appropriate test conditions. Consequently, for repairable products, when the MTTF for latent failures is less than the useful life of the product, these extremely long-life components must be considered in developing reliability goals and in determining test time.

How to set a target for reliabilities growth testing under Dormant Problems?

The setting of test targets for reliability growth testing should take the presence of dormant problems into account. The instantaneous MTBF and instantaneous failure rate will provide the most reliable estimates of MTBF and system failure rate, respectively. These values would be observed if you stop the reliability growth test at any given point and run a reliability demonstration test in the values where failures are fixed, but no other design improvements are made. System failure rate and MTBF can be evaluated using equations (8.13) and (8.14). Another point to note is that if there are problems of dormant failures with MTTFs significantly longer than the system MTBFs and you do not plan to test long enough for those long-life components to fail, then it is necessary to adjust the failure rate or MTBF to account for these dormant components.

Example: Suppose a product is designing to meet a MTBF goal of 400 hours. Besides the critical components that contribute the majority to system MTBF, there are three long-life, non-repairable components in the system. From supplier data and component life testing it has been determined that the MTTFs for the components are 1000, 1500, and 2000 hours. The components may yield to change out and are likely to fail in useful life of the product.

After 250 hours of initial testing to establish a baseline MTBF, two failures were observed at an initial MTBF of 150 hours. A historical data collected for the reliability growth testing of the same product indicated that the log-log plot of cumulative MTBF versus cumulative test time should show a reasonable slope of 0.50. Now using the Duane Growth Model, we first estimate the test time and test goal required to attain system MTBF.

Here, the instantaneous MTBF for the system can be calculated by using

$$\frac{1}{400} = \frac{1}{\theta_i} + \frac{1}{1000} + \frac{1}{1500} + \frac{1}{2000}.$$

This gives, $\theta_i = 3000$ hours. Further, we obtain

$$\theta_c = (1 - m)\theta_i = 0.5 \times 3000 = 1500\ hours$$

The total test time can be calculated using the (8.6)

$$T = T_0 \left(\frac{\theta_c}{\theta_0}\right)^{1/m} = 250 \left(\frac{1500}{150}\right)^{1/0.5} = 25000\ hours$$

10.3.4 Drivers of Higher Reliability Growth Rate in Product Development

Some major elements that drive higher reliability growth rates in product development are those that support problem-solving efficiency and overall system optimization. Responsive behaviours for fast problem solving engage teams in creative and wise use of time and resources. This makes sure that the product is field-ready for use by customers. Excellence in cross-functional teamwork is also critical because project teams shall use the best capabilities of the organization to optimize system-level performance. The cross-functional teams, with their development and test facilities, enable engineers to quickly respond to issues correctly for increased reliability. There is a standardized corrective action system to ensure a robust project management system that will support problem analysis and resolution with clear responsibilities and transparent tracking. Management's expectation of an effective and consistent corrective action process assures predictable progress, while a documented system for failure reporting and corrective actions assures that solutions are understood correctly and applied by production and service operations to benefit future product development.

Effective risk management processes prioritize problem-solving investments by their potential impacts on the customer and ensure that critical issues are handled promptly. Enough time, labour, and capacity for multiple iterations of test planning, execution, failure analysis, and corrective actions introduce reliable market entry by identification and solution of problems efficiently. Stressful and accelerated test plans find out the problems in a quicker way to provide superior solutions. Workforce training in problem-solving methods, such as six sigma, provides uniformity of process across development teams. Rapid prototyping speeds up the build-test-fix cycle. Agile development processes incorporate changes into the designs even in the later stages of work. The independence of testing and verification responsibilities ensures that solutions indeed solve the problems by giving the highest priority to customer satisfaction.

10.4 Probability Plotting Techniques

Probability plotting is one of the basic tools of statistical analysis that can be used for graphical examination of the likelihood that a given data set may be representative of some stipulated theoretical distribution. Such plots help to diagnose deviations from the expected distribution and can, therefore, be very useful in guiding the choice of appropriate statistical models. We will here present an overview of some important probability plotting techniques, including applications and interpretations.

10.4.1 Quantile-Quantile (Q-Q) Plot

The quantile-quantile plot is a graphical technique for checking the quantiles of a sample distribution against the quantiles of another theoretical distribution. One of the main uses of this plot is to verify the normality of a dataset.

Steps to Create a Q-Q Plot:

- Order the Data: Rank the data from smallest to largest.
- Calculate Theoretical Quantiles: Determine the theoretical quantiles from the specified distribution (e.g., normal distribution).
- Plot the Points: Plot the ordered data points (sample quantiles) against the theoretical quantiles.

From the plot, if the data points lie approximately along a straight line, the sample distribution matches the theoretical distribution. If we observe deviations from the straight line, it indicates departures from the theoretical distribution.

Example: Suppose we have the following failure times (in hours) for a sample of mechanical components:

400,500,600,150,200,240,300, 700,850,1000.

Using a Q-Q plot, we can evaluate whether a dataset of failure times follows Weibull ($\beta = 1.5, \eta = 600$), where β represents shape and η denotes scale .

- Order the Data: 150,200,240,300,400,500,600,700,850,1000
- Calculate Theoretical Quantiles: Assume the failure times follow Weibull (β, η). The theoretical quantiles can be computed using the Weibull cumulative distribution function (CDF):

$$F(t) = 1 - exp\left(-(t/\eta)^{\beta}\right).$$

Using inverse CDF, the quantile can be obtained for the given probability p_i as

$$Q_i = \eta(-log(1 - p_i))^{1/\beta}.$$

Here, p_i is the probability position usually assumed as

$$p_i = \frac{i - 0.5}{n}.$$

where i denotes the rank of the data point, and n represents the total number of data points. For our dataset:

$$p_i = \frac{i - 0.5}{10}.$$

Further, theoretical quantiles Q_i can be obtained using $\beta = 1.5$ and $\eta = 600$ from Weibull distribution.

Table for Q-Q plot

	Failure Times	Ordered Data	Plotting Positions	Theoretical Quantiles
1	400	150	0.05	82.83
2	500	200	0.15	178.68
3	600	240	0.25	261.47
4	150	300	0.35	342.23
5	200	400	0.45	425.80
6	240	500	0.55	516.42
7	300	600	0.65	619.77
8	700	700	0.75	745.97
9	850	850	0.85	919.49
10	1000	1000	0.95	1246.87

- Plot the Points: Plot the ordered data points (sample quantiles) against the theoretical quantiles.

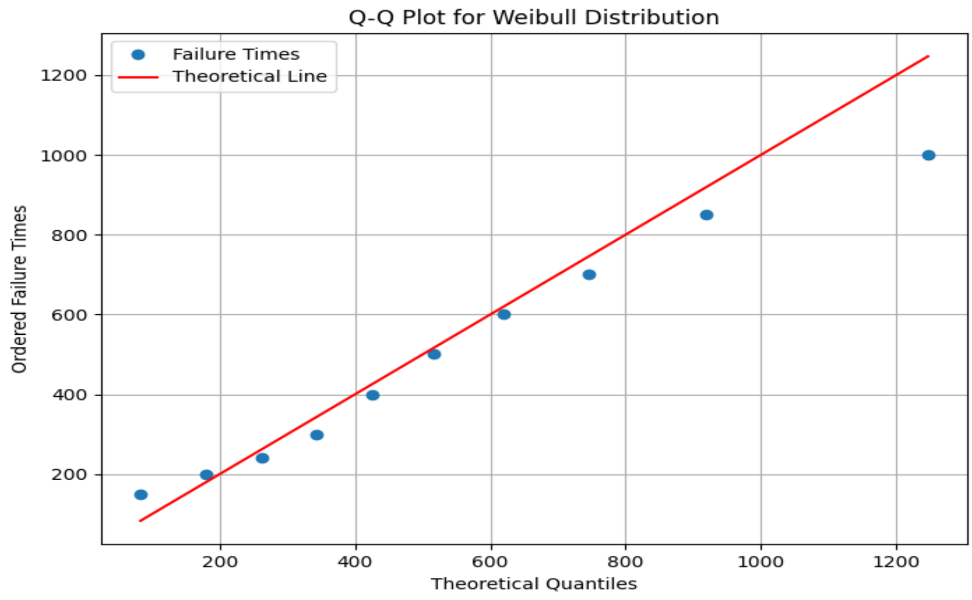

Figure 10.2 Q-Q plot for Weibull distribution

From Figure 10.2, the linear trend line indicates that the data fits the Weibull distribution.

10.4.2 Probability-Probability (PP) Plot

The Probability-Probability (P-P) plot is a technique similar to a Q-Q plot, but it compares the cumulative distribution functions (CDFs) of the sample data with the theoretical distribution.

Steps to Create a P-P Plot:

1. Order the Data: Rank the data from smallest to largest.

2. Calculate Empirical CDF: Compute the empirical CDF for the sample data using (i-0.5)/n or (i+05)/n. In the case of tied observations, the empirical CDF is plotted against only time with the largest i.

3. Plot the Points: Plot the empirical CDF values against the time or its function. The linear relationship can be established from the assumed distribution.

Similar to the Q-Q plot, a 45-degree line suggests that the sample distribution fits the theoretical distribution well, and deviations from this line highlight discrepancies between the sample and theoretical distributions.

Example: Consider a new set of failure times (in hours) for a different sample of mechanical components:

408, 357, 422, 220, 37, 124, 303, 572, 32, 1259

Check whether data follows an exponential distribution.

The CDF of exponential distribution given by

$$F(t) = 1 - exp(-t/\lambda)$$

It gives,

$$1 - F(t) = exp(-t/\lambda)$$

Or,

$$t = -\lambda \, log\big(1 - F(t)\big) \qquad (10.10)$$

From (10.10), we observed a linear relation between t and $log\big(1 - F(t)\big)$. For F(t), we choose $p = \frac{i-0.5}{n}$. Further, we will use this relationship in the probability plot given in Figure 3. This figure confirms data follows an exponential distribution.

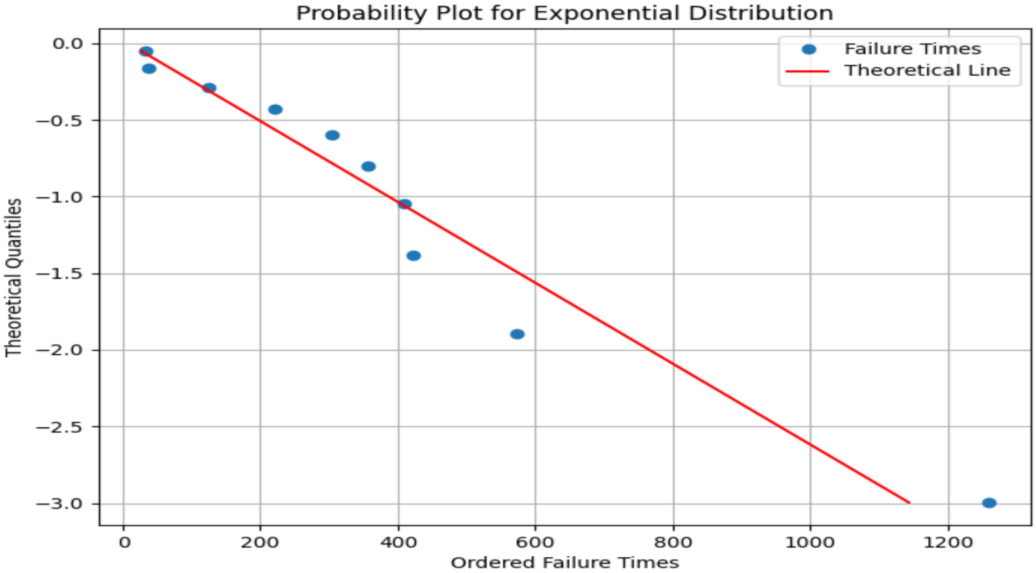

Figure 10.3: Probability Plot of Exponential Distribution

Example: Consider the following time recorded from a test of electrical system

1.47 ,2.87, 0.77, 0.84, 1.81, 1.14 ,1.91 ,1.59 ,0.50, 1.09 ,1.81, 2.31, 1.54, 1.28, 0.90, 1.92, 1.54, 0.66, 2.09, 1.12

Check whether the data follows the Weibull distribution using a probability plot.

Let us first assume data follows Weibull distribution with CDF

$$F(t) = 1 - exp\left(-(t/\eta)^{\beta}\right)$$

It gives,

$$1 - F(t) = exp\left(-(t/\eta)^{\beta}\right)$$

Or,

$$(t/\eta)^{\beta} = -\,log\left(1 - F(t)\right)$$

Taking the logarithm of both sides one more time, we get

$$\log t = \frac{1}{\beta}\log\left(-\,log\left(1 - F(t)\right)\right) + log\,\eta. \tag{10.11}$$

From (10.11), we observe a linear relation between $\log t$ and $log\left(-\,log\left(1 - F(t)\right)\right)$. We utilized this relation to get the probability plot in Figure 10.4.

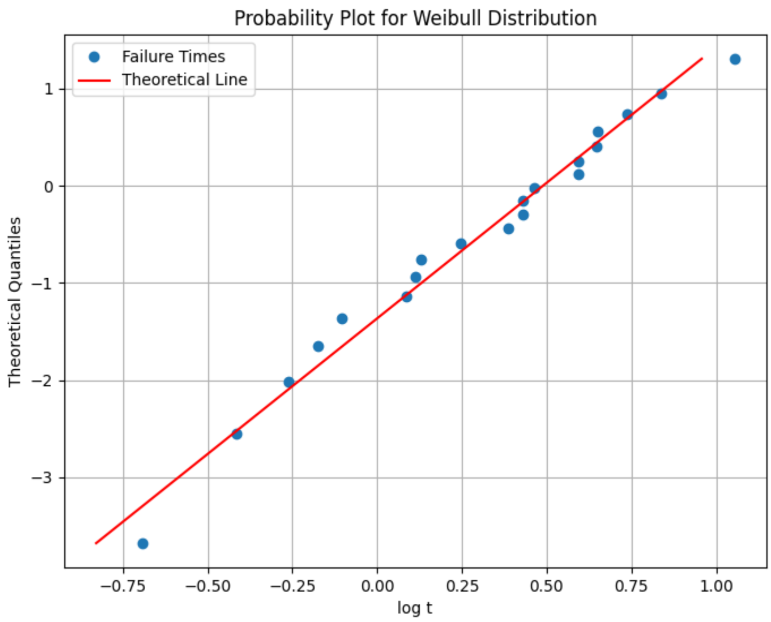

Figure 10.4: Probability plot of Weibull distribution

Example: Consider a manufacturing company that produces specified electronic components, which are then assembled into high-performance equipment. The firm is interested in analysing the failure times of these components to understand their reliability. The company believes that these failure times are log-normally distributed and wishes to check this assumption using the log-normal probability plot. The recorded time of failures are

150, 200, 250, 300, 400, 600, 800, 1200, 1600, 2000, 2500, 3000

The cumulative distribution function (CDF) of a log-normal distribution is given by:

$$F_X(x) = P(X \leq x) = P(\ln(X) \leq \ln(x)) = \Phi\left(\frac{\ln(x) - \mu}{\sigma}\right)$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

To derive the linear relationship used in log-normal probability plotting, we take the inverse of the CDF:

For a given probability p, the inverse relationship is:

$$\frac{\ln(x) - \mu}{\sigma} = \Phi^{-1}(p)$$

Rearranging this, we get:

$$\ln(x) = \mu + \sigma\Phi^{-1}(p)$$

The above equation is now in the form of a linear equation:

$$(\text{log-transformed data}) = \mu + \sigma \times (\text{standard normal quantile})$$

We use this equation to show that the data is log-normally distributed. In this process, we plot the logarithm of the data $\ln(x)$ against the standard normal quantiles $\Phi^{-1}(p)$, the points should approximately form a straight line with a slope σ and intercept μ if the data is log-normally distributed. For p, we choose $p = \frac{i-0.5}{n}$. Following these operations, we obtain the probability plot in Figure 5, showing that data follow a log-normal distribution.

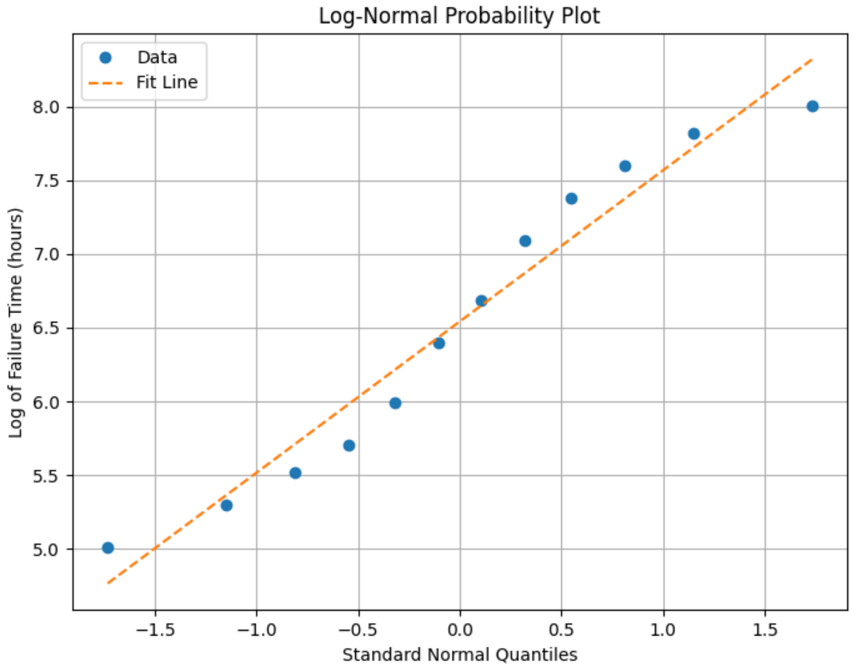

Figure 10.5: Probability plot of log-normal distribution

10.5 Tests for Exponentiality

The exponential distribution is important for reliability engineering and several other fields. Perhaps the most common application of the exponential distribution concerns modeling times between events in a Poisson process. Thus, it is one of the basic elements in the theory of analysis of lifetimes and failure rates. The exponential distribution is often used in reliability engineering for several important reasons.

- **Memoryless property:** The exponential distribution follows memoryless property; that is, the future probability of an event's occurrence does not depend upon the past. This makes the analysis of systems and processes very easy.
- **Modelling Failures:** This is commonly used for modelling times to failure of components and systems, mainly when the failure rate is constant over time.
- **Easy Calculations:** The mathematical simplicity of the exponential distribution allows simple calculations with straightforward interpretations for reliability studies.

Due to the most frequent use of exponential distribution, it becomes important to perform a test of exponentiality.

10.5.1 Hollander-Proschan Test

The Hollander-Proschan test represents a non-parametric test specially designed to answer the question of whether a given data set follows an exponential distribution. This test is based on the total time on the test concept and is particularly devised to detect deviation from exponentiality in increasing or decreasing failure-rate forms. In the Hollander-Proschan test, we assumed that the sample data represents the lifetimes of independently and identically distributed (i.i.d.) components, and the test is sensitive to monotonic trends in the hazard function.

Steps in Conducting the Hollander-Proschan Test

1. Calculate TTT: Compute the total time on test for the given sample data.
2. Plot TTT: Create a TTT plot by plotting the cumulative TTT against the number of failures.
3. Analyze the Plot: Compare the TTT plot to the 45-degree line (which represents exponentiality). Deviations from this line indicate departures from exponentiality.

If the plot of TTT is always above the 45-degree line, it will show the decreasing failure rate; the meaning is that the component or the system is improving with time. If the plot of TTT always lies below the 45-degree line, then one would know that the failure rate is increasing; thus, the component or the system is getting worse with time. Otherwise, it confirms the data according to an exponential distribution.

Example: Let us consider the following data representing the lifetimes (in hours) of 10 identical components:

120, 180, 250, 300, 330, 400, 480, 550, 610, 700

Perform the Hollander-Proschan test to determine if these lifetimes follow an exponential distribution.

Step 1: Calculate TTT:

First, we sort the data in ascending order, which is already sorted in this example. Then, we calculate TTT for each component. TTT can be defined as the cumulative sum of ordered lifetimes, divided by number of components. The TTT corresponding to i^{th}component, TTT_i, is given by

$$TTT_i = \frac{1}{n}\sum_{j=1}^{i} X_j,$$

where n is the number of components and X_j represents the jth ordered lifetime.

No. of failure	1	2	3	4	5	6	7	8	9	10
Cumulative TTT	12	20	55	85	118	158	206	261	322	392

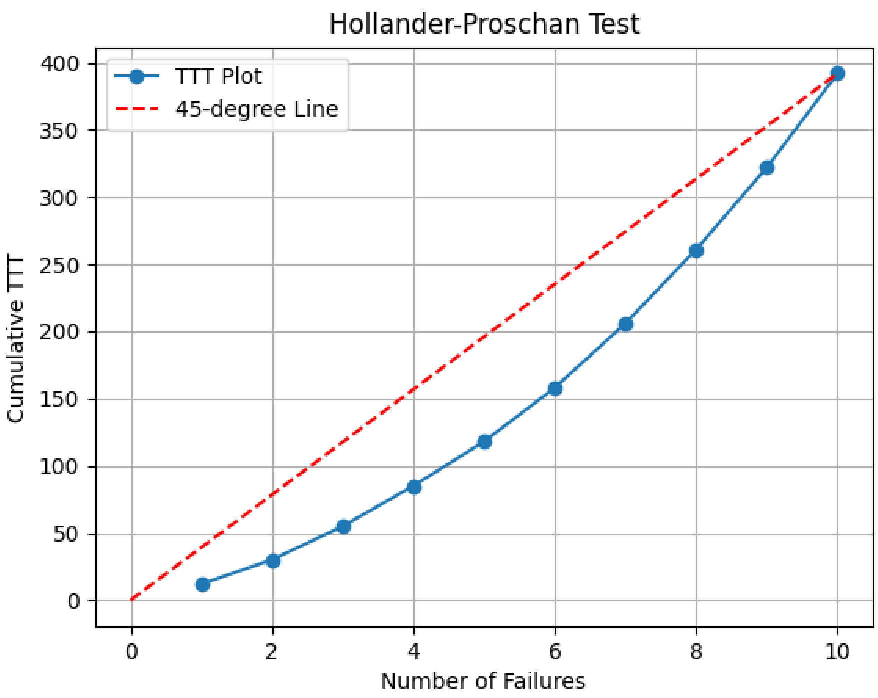

Figure 10.6: Hollander-Proschan test

In Figure 10.6, points are below the 45-degree line, indicating an increasing failure rate (deteriorating system).

10.5.2 Deshpande Test

The Deshpande test checks exponentiality in a given dataset. This test is based on ratios of pairs of data points. The underlying idea is that certain properties should characterize the ratios of data points in an exponentially distributed dataset. In this test, we assume that the sample data consists of IID lifetimes, and the test is particularly effective in detecting deviations from exponentiality due to non-constant failure rates.

Hypotheses:

Null Hypothesis (H_0): The underlying distribution is exponential, i.e., $\left(F(x) = 1 - e^{-\lambda x}\right)$.

Alternative Hypothesis (H_1): The distribution is increasing failure rate average (not exponential), where (F(x)) satisfies $(F(bx) = [F(x)]^b)$ for all (x > 0).

Steps in Conducting the Deshpande Test

- Calculate the Pairwise Ratios: Compute all possible pairwise ratios $(X_i/X_j) for\ all (i \neq j)$, where Xi and Xj are data points.
- Significance Level: Set a significance level α (e.g., α=0.05).
- Choose a value for b: Select a threshold value b (commonly, b=0.5).
- Compute the Test Statistic J:

$$J = \frac{1}{n(n-1)} \sum_{i \neq j} I\left(X_i > bX_j\right)$$

where I(·) is an indicator function that equals 1 if the condition inside is true and 0 otherwise. Notice that, this statistic counts the proportion of pairwise comparisons where Xi is greater than bXj.

- Degrees of Freedom: After calculating the test statistic J, it is compared against a critical value obtained from the chi-squared distribution with n(n−1)/2 degrees of freedom, where *n* is the number of observations.
- Determine the Critical Value: Determine the critical value from the chi-squared distribution table for the given degrees of freedom and significance level α.
- Decision Rule: If J exceeds the critical value, we should reject the null hypothesis, meaning that the data do not follow an exponential distribution. If J does not exceed the critical value, then we fail to reject the null hypothesis, indicating that the data may follow an exponential distribution.

Example: Let's consider a real-world example using medical data related to renal cancer patients. Suppose we have a dataset that represents the survival times (in months) of patients after diagnosis.

12, 18, 24, 30, 36, 42, 48, 54, 60, 66

We want to assess whether the survival times follow an exponential distribution.

1. Step-by-Step Deshpande Test:

- Calculate the pairwise ratios: $(X_i/X_j) for\ all (i \neq j)$.

- Consider a level of significance as $\alpha = 0.05$.

- Compute the test statistic J:

$$J = \frac{1}{n(n-1)} \sum_{i \neq j} I\left(X_i > bX_j\right)$$

- Choose a value for b = 0.5.

- Compare the observed J with the critical value from the chi-squared distribution with n(n-1)/2 degrees of freedom.

- If J exceeds the critical value, we reject the null hypothesis (non-exponential distribution).

2. Calculation:

- For our dataset, let's compute the pairwise ratios:

$$\frac{X_i}{X_j} = \left\{\frac{12}{18}, \frac{12}{24}, \dots, \frac{66}{60}\right\}.$$

Compare this with b, or

- Using b = 0.5, we find:

$$J = \frac{1}{10 * 9}\left(I(12 > 0.5 * 18) + I(12 > 0.5 * 24) + \cdots + I(66 > 0.5 * 60)\right)$$
$$= 0.7778$$

Here, the critical value at df=45 is 61.65 based on the chi-square distribution table. It can be noticed that J does not exceed the critical value; therefore, we fail to reject the null hypothesis. That means data follows an exponential distribution.

4. Conclusion:

-We fail to reject the null hypothesis, which suggests that survival times follow an exponential distribution.

10.6 Tests for HPP vs. NHPP with Repairable Systems

Several statistical tests and model comparison techniques can be used to compare HPP and NHPP models applied to repairable systems. Here are some common methods:

10.6.1 Likelihood Ratio Test (LRT)

This test compares the likelihoods of the HPP and NHPP models.

The likelihood ratio statistic is given by:

$$\lambda = -2(log\, L_{HPP} - log\, L_{NHPP})$$

where L_{HPP} and L_{NHPP} are the likelihoods of the HPP and NHPP models, respectively.

Under the null hypothesis (HPP), λ follows a chi-square distribution with degrees of freedom equal to the difference in the number of parameters between the NHPP and HPP models.

Example: Consider a dataset of failure times for a repairable system. Suppose we have 10 failure times over a period of 1000 hours.

HPP Model: Estimate (λ):

$$\hat{\lambda} = \frac{10}{1000} = 0.01 \text{ failures per hour}$$

Compute the log-likelihood:

$$\log L_{HPP} = 10 \log(0.01) - 0.01 \times 1000 = -46.0517.$$

NHPP Model: Suppose we fit a Weibull process with parameters (β) and (η). After estimation, let's say the log-likelihood comes out as:

$$\log L_{NHPP} = -40.1234$$

Therefore, the LRT Statistic will be:

$$\lambda = -2(-46.0517 + 40.1234) = 11.8566$$

Notice that here, the degree of freedom will be 2-1=1.

Critical Value: For χ^2distribution with 1 degree of freedom and at 5 % level of significance, the critical value cab be calculated as 3.84. Now, since 11.8566 > 3.84, we reject the null hypothesis and conclude that the NHPP model better fits the data, indicating a time-varying failure rate.

10.6.2 Cox-Lewis Test

The Cox-Lewis test is, in fact, a goodness-of-fit test designed to examine how well the observable failure data from a repairable system can be fitted with an HPP. In this respect, it is quite a useful test for differentiating between the fittings of HPP and NHPP. The main idea behind the Cox-Lewis test lies in the graphical examination of the cumulative number of failures against time for trends or patterns that may indicate non-compliance with the assumptions of HPP. This test uses two key concepts

- **Interarrival Times:** The times between successive failures, inter-arrival times, for an HPP must be exponentially distributed. This implies that the failures must occur at random in time with a constant rate λ.
- **Cumulative failures:** Considering an HPP, the cumulative number of failures against time will follow a straight line. Any prominent deviation from this linear trend would suggest a trend in the failure rate, and hence, an NHPP may be present.

Test Procedure:

- **Data Collection and Preparation:** Collect the failure times $t_1, t_2, \dots, t_n$ of the repairable system.
- **Calculate Interarrival Times:** Compute the interarrival times $\Delta t_i = t_i - t_{i-1}$

for $i=2,\ldots,n$. For an HPP, these interarrival times should follow an exponential distribution.

- **Cumulative Plot:** Plot the cumulative number of failures N(t) against time t. For an HPP, the relation between the two should be a straight line with a slope equal to the failure rate λ.
- **Interpretation:** If the cumulative plot is considerably nonlinear, or if there are patterns in the residuals from the linear regression, then this would be indicative of a trend in the failure rate, hence indicative that the assumptions of the HPP model are not correct. If the plot is approximately linear and there is no presence of any systematic pattern in the residuals, it may be concluded that the HPP model is reasonable.

Example

- Data Collection and Preparation: Suppose we have the following failure times (in hours) for a repairable system:

$$t=[5, 10, 14, 19, 23, 28, 35, 39, 45, 50]$$

- Calculate Interarrival Times:

$$\Delta t=[5, 5, 4, 5, 4, 5, 7, 4, 6,5]$$

- Cumulative Plot: Construct the cumulative number of failures N(t) against time t:

$$N(t)=[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$$

- Plot Cumulative Failures Against Time

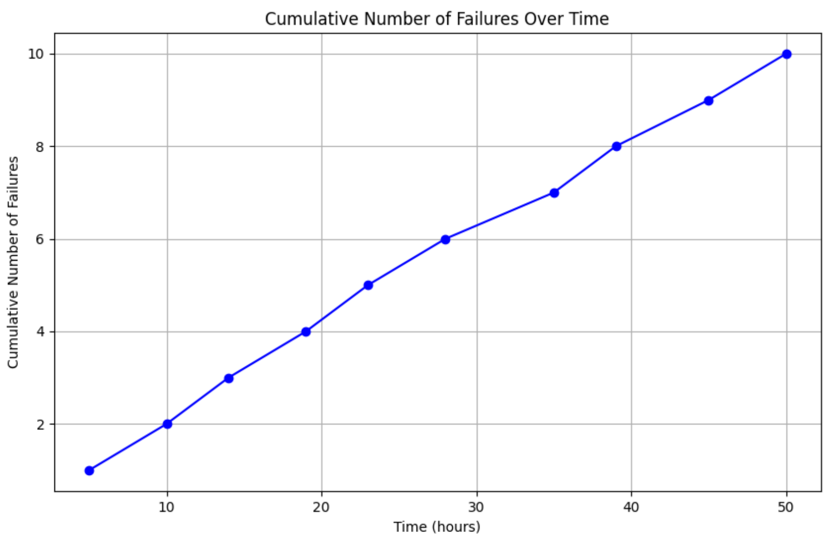

Figure 10.7: the cumulative number of failures N(t) against the failure times t.

From Figure 10.7, one can see that the cumulative plot is linear. Hence, we conclude that the HPP model is adequate for the data.

10.7 Basic Ideas of Accelerated Life Testing

Accelerated life testing (ALT) is recognized as one of the important techniques of reliability engineering and product lifecycle analysis. We discuss some basic concepts related to ALT.

10.7.1 Definition and Purpose

Accelerated life testing (ALT) is recognized as one of the important techniques of reliability engineering and product lifecycle analysis. The technique involves increasing the level of stress on a product to make it fail at a quicker rate than it would do with regular usage. The main aim of ALT is to estimate the basic characteristics of a product's life related to its reliability and durability within a considerably short time. This method offers an opportunity for life prediction of a product and identification of possible failure mechanisms without waiting for it in real time.

To illustrate the concepts and mathematical framework of ALT, consider a case in which a device has an exponential failure distribution under a constant application of one stress. The exponential failure distribution is given by:

$$f(t, \lambda_i) = \lambda_i e^{-\lambda_i t},$$

where λ_i is constant hazard rate under stress V_i. If $\theta_i = 1/\lambda_i$, then θ_i will be the mean time to failure under the stress V_i.

We may have the following relationship between $\lambda_i$and V_i.

1. **The Power Rule Model:**

   The Power Rule Model was derived from kinetic theory and considerations of activation energy and is applicable to most paper-impregnated dielectric capacitors. In $V_i's$model, we have

$$\frac{1}{\lambda_i} = \frac{c}{V_i^p}; c > 0$$

$$\Rightarrow \lambda_i = \frac{1}{c} V_i^p$$

where V_i is voltage and p and c are estimated.

2. **The Reaction Rate Model**:

This model, often used for semiconductors, relates the hazard rate λ_i to temperature stress V_i as follows:

$$\lambda_i = exp\left(A - \frac{B}{V_i}\right)$$

where V_i is temperature stress, and A and B are parameters need to be estimated.

3. **The Eyring Model:**

The Eyring Model describes the hazard rate λ_i as a function of temperature V_i and another stressor, such as voltage or pressure:

$$\lambda_i = V_i\, exp\left(A - \frac{B}{V_i}\right)$$

where V_i is temperature.

10.7.2 Conducting Accelerated Life Test

Suppose that k values of stress $V_i, i = 1,2 \cdots, k,$ are chosen randomly and are to be applied on a device. Suppose the device under stress V_i has an exponential failure distribution with scale parameter $\lambda_i = 1/\theta_i$. Further, suppose that while applying each stress V_i, n_i devices are put to the test, and the test is terminated after the failure of r_i items, with time of failure $t_{1i}, \dots, t_{r_i i}$. This way, we observe k such time of failure for each stress V_i. Apart from data, we also obtain $\left(V_i, n_i, r_i, \hat{\theta}_i\right)$ from k life test of data, where $\hat{\theta}_i$ is ML estimator of θ. It also comes as UMVUE of θ_i, $\hat{\theta}_i$is given as

$$\hat{\theta}_i = \frac{\sum_{j=1}^{r_i} t_{ji} + (n_i - r_i) t_{r_i i}}{r_i}$$

Since t_{ji} follows the exponential distribution; therefore, the pdf of $\hat{\theta}_i$ comes out as a gamma density with shape r_i and scale θ_i/r_i, that is

$$g\left(\hat{\theta}_i\right) = \frac{1}{\Gamma r_i}\left(\frac{r_i}{\theta_i}\right)^{r_i}\left(\hat{\theta}_i\right)^{r_i - 1} exp\left(-\frac{r_i \hat{\theta}_i}{\theta_i}\right)$$

Estimation under power rule model:

Under power rule mode, we have

$$\theta_i = \frac{c}{\left({V_i}/{\bar{V}}\right)^p}$$

where $\bar{V} = \prod_{i=1}^{k} V_i^{\,r_i} / \sum_{i=1}^{k} r_i$ is the weighted geometric mean of V_i's.

The likelihood function having parameters C and P comes out as

$$L(C, P|\hat{\theta}) = \prod_{i=1}^{k} \frac{1}{\Gamma r_i} \left[\frac{r_i}{C}\left(\frac{V_i}{\bar{V}}\right)^P\right]^{r_i} \widehat{\theta_c}\, exp\left[-\frac{r_i \hat{\theta}_i}{C}\left(\frac{V_i}{\bar{V}}\right)^P\right],$$

where $\hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k]$. Taking the logarithm of both sides

$$\ln L\,(C, P|\hat{\theta}) = \ln\left(\prod_{i=1}^{k} \frac{1}{\Gamma r_i} \left[\frac{r_i}{C}\left(\frac{V_i}{\bar{V}}\right)^P\right]^{r_i} \widehat{\theta_c}\, \exp\left[-\frac{r_i \hat{\theta}_i}{C}\left(\frac{V_i}{\bar{V}}\right)^P\right]\right)$$

$$\ln L\,(C, P|\hat{\theta}) = \sum_{i=1}^{k}\left(\ln\left(\frac{1}{\Gamma r_i}\right) + \ln\left(\left[\frac{r_i}{C}\left(\frac{V_i}{\bar{V}}\right)^P\right]^{r_i}\right) + \ln(\widehat{\theta_c}) + \ln\left(\exp\left[-\frac{r_i \hat{\theta}_i}{C}\left(\frac{V_i}{\bar{V}}\right)^P\right]\right)\right)$$

$$= \sum_{i=1}^{k}\left(\ln\left(\frac{1}{\Gamma r_i}\right) + r_i \ln(r_i) - r_i \ln(C) + r_i P \ln\left(\frac{V_i}{\bar{V}}\right) + \ln(\widehat{\theta_c}) - \frac{r_i \hat{\theta}_i}{C}\left(\frac{V_i}{\bar{V}}\right)^P\right)$$

Taking the derivative with respect to C, we get

$$\frac{d}{dC}\ln L\,(C, P|\hat{\theta}) = \sum_{i=1}^{k}\left(\frac{d}{dC}(-r_i \ln(C)) + \frac{d}{dC}\left(-\frac{r_i \hat{\theta}_i}{C}\left(\frac{V_i}{\bar{V}}\right)^P\right)\right)$$

$$= \sum_{i=1}^{k}\left(-\frac{r_i}{C} + \frac{r_i \hat{\theta}_i}{C^2}\left(\frac{V_i}{\bar{V}}\right)^P\right)$$

Here, $\frac{d}{dC}\ln L\,(C, P|\hat{\theta}) = 0, gives$

$$\sum_{i=1}^{k}\left(-\frac{r_i}{C} + \frac{r_i \widehat{\theta_\iota}}{C^2}\left(\frac{V_i}{\bar{V}}\right)^P\right) = 0$$

which gives

$$\hat{C} = \frac{\sum_{i=1}^{k} r_i \hat{\theta}_i \left(\frac{V_i}{\bar{V}}\right)^P}{\sum_{i=1}^{k} r_i} \tag{10.12}$$

Further,

$$\frac{\partial \ln L}{\partial P} = \sum_{i=1}^{k} r_i \left[log\left(\frac{V_i}{\bar{V}}\right)\right] - \sum_{i=1}^{k} \frac{r_i \hat{\theta}_i}{C} \left(\frac{V_i}{\bar{V}}\right)^P log\left(\frac{V_i}{\bar{V}}\right)$$

$$= C\sum_{i=1}^{k} r_i \, log\left(\frac{V_i}{\bar{V}}\right) - \sum_{i=1}^{k} r_i \hat{\theta}_i \left(\frac{V_i}{\bar{V}}\right)^P log\left(\frac{V_i}{\bar{V}}\right) = 0$$

Substituting the value of $\widehat{C}$ from (10.2), we get

$$\frac{\partial \ln L}{\partial P} = \sum_{i=1}^{k} r_i \, log\left(\frac{V_i}{\bar{V}}\right) \left\{C - \hat{\theta}_i \left(\frac{V_i}{\bar{V}}\right)^P\right\} = 0$$

$$\sum_{i=1}^{k} r_i \, log\left(\frac{V_i}{\bar{V}}\right) \left\{\frac{\sum r_i \hat{v}_i \left(\frac{V_i}{\bar{V}}\right)^P}{\sum r_i} - \hat{\theta}\left(\frac{V_i}{\bar{V}}\right)^P\right\} = 0 \tag{10.13}$$

Here, since equation (10.13) is non-linear, therefore we need to use numerical technique to solve them. This way, we can obtain estimates of C and P and use them further to predict the lifetime characteristic of the system.

Example: Suppose an electronics manufacturer wants to estimate the reliability of a new type of capacitor under normal usage conditions. Here, capacitors are subjected to increased voltage levels to accelerate life testing to induce failures faster. Three different stress levels (Level 1 (V1): 100V, Level 2 (V2): 150V, Level 3 (V3): 200V) are applied to three groups of capacitors. For each stress level, we record the failure times of the capacitors. The test is terminated after a specific number of failures ($r_i = 6$) in each group. Here, we record the following data

$$t_{1,1} = 100, t_{1,2} = 120, t_{1,3} = 140, t_{1,4} = 160, t_{1,5} = 180, t_{1,6} = 200$$

$$t_{2,1} = 50, t_{2,2} = 60, t_{2,3} = 70, t_{2,4} = 80, t_{2,5} = 90, t_{2,6} = 100$$

$$t_{3,1} = 30, t_{3,2} = 35, t_{3,3} = 40, t_{3,4} = 45, t_{3,5} = 50, t_{3,6} = 55$$

Step 1: Calculate $\hat{\theta}_i$ for each stress level

The MLE for θ_i

is given by:

$$\hat{\theta}_i = \frac{\sum_{j=1}^{r_i} t_{ji} + (n_i - r_i)t_{r_i i}}{r_i}$$

For V1 = 100V:

$$\widehat{\theta_1} = \frac{100 + 120 + 140 + 160 + 180 + 200 + 4 \times 200}{6} = \frac{1100 + 800}{6} = 316.67 \text{ hours}$$

For V = 150V:

$$\widehat{\theta_2} = \frac{50 + 60 + 70 + 80 + 90 + 100 + 4 \times 100}{6} = \frac{450 + 400}{6} = 141.67 \text{ hours}$$

For V3 = 200V:

$$\widehat{\theta_3} = \frac{30 + 35 + 40 + 45 + 50 + 55 + 4 \times 55}{6} = \frac{255 + 220}{6} = 79.17 \text{ hours}$$

Step 2: Estimate parameters C and P using the Power Rule Model

The Power Rule Model is:

$$\lambda_i = \frac{1}{c} V_i^p$$

or equivalently,

$$\theta_i = cV_i^{-p}$$

Using the likelihood function and solving for C and P:

$$\ln L\left(C, P \middle| \hat{\theta}\right) = \sum_{i=1}^{k} \left(-r_i \ln C + r_i P \ln V_i + \ln \widehat{\theta_c} - \frac{r_i \widehat{\theta_i}}{C} \left(\frac{V_i}{\overline{\overline{V}}}\right)^P \right)$$

We need to solve this numerically to find the estimates for C and P. Using numerical methods (such as Newton-Raphson) and equations (8.12) and (8.13), we can iteratively solve for C and P. This gives the following result

$$\hat{C} = 152.58$$

$$\hat{P} = 2$$

Step 3: In light of the results obtained, we can also predict reliability under normal usage conditions. Suppose the normal operating voltage is V0 = 50V. We use the estimated parameters to predict the reliability under this condition as follows.

$$\widehat{\theta_0} = \hat{C} V_0^{-\hat{P}} = 0.061$$

10.8 Summary

This unit discussed the reliability growth model and accelerated life testing. Additionally, it included important concepts such as probability plotting, HPP vs NHPP testing, and exponentiality testing. The methodologies discussed were accompanied by, which provides readers a strong conceptual understanding.

10.9 Self-Assessment Exercise

Question-1: A company aims for an MTBF of 600 hours for a new electronic device, which includes three non-repairable components with MTTFs of 1200, 1800, and 2400 hours. After 300 hours of testing, with two failures and an initial MTBF of 150 hours, historical data suggests a log-log plot slope of 0.60.

a) Calculate the test time needed to achieve the 600-hour MTBF.
b) Estimate the test goal using the Duane Growth Model. Also, prepare a summary of your findings and calculations.

Question-2: Write a brief overview of the utilization of the reliability growth model.

Question-3: Suppose a repairable device has 15 recorded failure times:

Failure Times (hours): 50, 120, 180, 230, 290, 350, 410, 470, 540, 600, 670, 730, 790, 860, 920

a) Calculate Interarrival Times:
b) Compute times between successive failures.
c) Plot Cumulative Failures:
d) Create a plot of cumulative failures N(t)) against time t.
e) Evaluate if failure times follow a Homogeneous Poisson Process (HPP).

Question-4: Suppose components are tested under three stress levels:

S1: Moderate Stress
S2: High Stress
S3: Very High Stress
Failure Times Data (hours):
S1: 120, 150, 160, 180, 200, 210, 230, 250, 270, 300
S2: 70, 90, 110, 130, 150, 160, 180, 200, 220, 240
S3: 30, 50, 70, 90, 100, 110, 130, 150, 170, 200

For this data,

a) Plot histograms for each stress level.

b) Fit an exponential distribution to the data.

c) Perform Deshpande Test test to check for exponentiality.

d) Interpret the results and discuss the implications for component reliability.

e) Prepare a brief report summarizing your findings.

Question-5: What is the accelerated life testing? Comment on the use of this concept.

Question-6: A manufacturer tests the lifespan of new LED bulbs under different temperatures to assess reliability. Three temperature levels are used:

T1: 30°C; **T2:** 50°C; **T3:** 70°C

Each group of bulbs is tested until 8 failures are recorded. Failure times (in hours) for each temperature are:

T1: 500, 520, 540, 550, 570, 580, 590, 600

T2: 300, 320, 340, 350, 370, 380, 390, 400

T3: 150, 160, 170, 180, 190, 200, 210, 220

Uses this data to analyze the reliability of the LED bulbs and to estimate their lifespan under normal operating conditions?

10.10 References

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10.11 Further Reading

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