Master in Computer Application

MCA-104

Uttar Pradesh Rajarshi Tandon Open University

Numerical Analysis

COURSE INTRODUCTION

Numerical analysis is a branch of Mathematics that deals with devising efficient methods for obtaining numerical solutions to difficult Mathematical problems. Most of the Mathematical problems that arise in science and engineering and applied engineering mathematics are very hard and sometime impossible to solve exactly. Thus, an approximation to a difficult Mathematical problem is very important to make it more easy to solve. Due to the immense development in the computational technology, numerical approximation has become more popular and a modern tool for scientists and engineers.

The objective of Block-1 is to review the basic concepts of calculus, properties of equations and roots of equation how the initial approximation is taken and convergence is calculated using iterative methods like bisection method, fixed point iterative method, Regula –Falsi method and Newton Raphson Method Here order of convergence is also calculated which clearly indicates the speed by which approximation of the root could be done.

In this block our objective is to review the direct methods for solution of system of linear equations such as Cramer's rule , inverse of matrix, Gauss Elimination Method, Gauss- Jordon Reduction Method, LU decomposition method and Crout's method for calculation of exact solution of system of equations. The iterative methods for solution of system of linear equations such as Jacobi's Iteration Method and Gauss-Seidal Iteration gives the solution to desired accuracy and has vital importance in field of applied engineering.

Interpolation is the method of estimating unknown values with the help of given set of observations. Also interpolation means insertion or filling up intermediate terms of the series. It is the technique of obtaining the value of a function for any intermediate values of the independent variables i.e., of argument when the values of the function corresponding to number of values of argument are given. The study of finite difference calculus has become very important due to its wide variety of application in routine life. There it helps in reducing complex mathematical expressions like trigonometric functions in terms of simple arithmetic operations. When the values of the independent variable occur with unequal spacing, the formula discussed earlier is no longer applicable. In this situation another formula which is based on divided difference is used.

There are number of differential equations which do not possess closed form of finite solutions. Even if they possess closed form solutions, we do not know the method of getting it. In such situations, depending upon the need of the hour, we go in for numerical solutions of differential equations. In researches, especially after the advent of computer, the

numerical solutions of the differential equations have become easy for manipulations. These methods yield solutions either as power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. The methods discussed in this block Picards and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta belong to the latter class. However Runge-Kutta forth order method is iterative and gives solution to desired accuracy.

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BLOCK

1

SOLUTIONS OF NON-LINEAR EQUATIONS IN ONE VARIABLE

UNIT-1

Review of Calculus and Properties of Equations

UNIT-2

Iterative Methods and Order of Convergence criteria

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BLOCK INTRODUCTION

Overview :

Numerical technique is widely used by scientists and engineers to solve their problems. A major advantage for numerical technique is that a numerical answer can be obtained even when a problem has no analytical solution. However, result from numerical analysis is an approximation, in general, which can be made as accurate as desired. The reliability of the numerical result will depend on an error estimate or bound, therefore the analysis of error and the sources of error in numerical methods is also a critically important part of the study of numerical technique.

UNIT-1 REVIEW OF CALCULUS AND PROPERTIES OF EQUATIONS

Structure

1.1   OBJECTIVE

In this unit, our objective is to review the basic concepts of calculus, properties of equations and roots of equation how the initial approximation is taken and convergence is calculated using bisection method.

1.2   INTRODUCTION

Numerical analysis is a branch of Mathematics that deals with devising efficient methods for obtaining numerical solutions to difficult Mathematical problems. Most of the Mathematical problems that arise in science and engineering are very hard and sometime impossible to solve exactly. Thus, an approximation to a difficult Mathematical problem is very important to make it more easy to solve. Due to the immense development in the computational technology, numerical approximation has become more popular and a modern tool for scientists and engineers.

1.3   REVIEW OF CALCULUS

MATHEMATICAL PRELIMINARIES

Theorem 1.1 If $f(x)$ is continuous in $a \leq x \leq b$, and if $f(a)$ and $f(b)$ are of opposite signs, then, $f(\xi) = 0$ for at least one number ξ such that $a < \xi < b$.

Theorem 1.2(*Rolle's theorem*) If $f(x)$ is continuous in $a \leq x \leq b$, $f'(x)$ exists in $a < x < b$ and $f(a) = f(b) = 0$, then, there exists at least one value x, say ξ, such that

$f'(\xi) = 0$, $a < \xi < b$.

Theorem 1.3(*Generalized Rolle's theorem*) Let $f(x)$ be a function which is n times differentiable on [a, b]. If $f(x)$ vanishes at the ,$(n + 1)$ distinct points, $x_0, x_1, \ldots\ldots, x_n$, in (a, b) such that $f^n(\xi) = 0$.

Theorem 1.4(*Intermediate value theorem*)If $f(x)$ be continuous in $[a, b]$ and let k be any number between $f(a)$ and $f(b)$. Then there exists a number ξ in (a, b) such that

$f(\xi) = k$
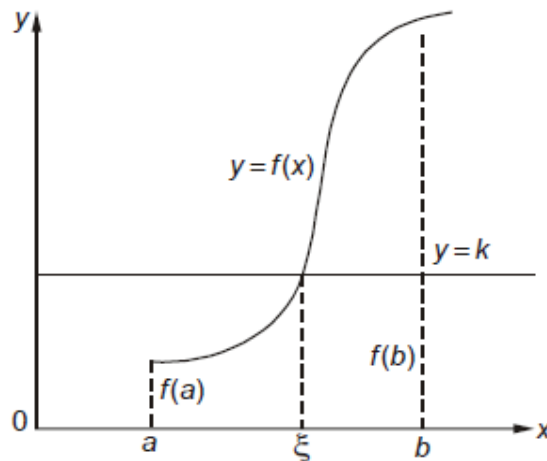
Figure:1 Intermediate value theorem

Theorem 1.5 (*Mean-value theorem for derivatives*) If $f(x)$ be continuous in $[a, b]$ and $f'(x)$ exists in (a, b), then there exists at least at least one value of x say ξ, between a and b such that (It is also known as Lagrange's Mean Value Theorem)

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}, \quad a < \xi < b$$

Setting $b = a + h$, this theorem takes the form

$f(a + h) = f(a) + hf'(a + \theta h)$, $\quad 0 < \theta < 1$.

Theorem 1.6 (*Taylor's series for a function of one variable*) If $f(x)$ is continuous and possesses continuous derivatives of order n in an interval that includes $= a$, then in that interval

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a)$$
$$+ \cdots \frac{(x - a)^{n-1}}{(n - 1)!}f^{(n-1)}(a) + R_n(x)$$

Where $R_n(x)$, the remainder term, can be expressed in the form

$$R_n(x) = \frac{(x-a)^n}{n!}f^n(\xi), \quad a < \xi < b$$

Theorem 1.6 *(Maclaurin's expansion) It states that*

If $f(x)$ is continuous and possesses derivatives of all order and R_n tends to Zero as n $\rightarrow\infty$ then Maclaurin's theorem becomes the Maclaurin's series.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots \frac{x^n}{n!}f^n(0) + \cdots \ldots \ldots$$

Theorem 1.7 *(Taylor's series for a function of two variables) It states*

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2) = f(x_1, x_2) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2$$
$$+ \frac{1}{2}[\frac{\partial^2 f}{\partial x^2_1}(\Delta x_1)^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}(\Delta x_1 \Delta x_2) + \frac{\partial^2 f}{\partial x^2_2}(\Delta x_2)^2] + \ldots \ldots$$

This can easily be generalized.

1.4 ROUND OFF ERROR AND TRUNCATION ERROR

Machine epsilon

The computer has a finite word length so only a fixed number of digits are stored and used during computation. Hence even in storing an exact decimal number in its converted form in the computer memory, an error is introduced. This error is machine dependent and is called machine epsilon.

Error = True value – Approximate value

In any numerical computation, we come across following types of errors:

1. **Inherent errors :** Errors which are already present in the statement of a problem before its solution are called inherent errors. Such errors arise either due to the given data being approximate or due to limitations of mathematical tables, calculators or the digital computer.

    Inherent errors can be minimized by taking better data or by using high precision computing aids. Accuracy refers to the number of significant digits in a value e.g. 53.965 is accurate to 5 significant digits.

Precision refers to the number of decimal position or order of magnitude of the last digit in the value e.g. in 53.965, precision in 10^{-3}.

Example: Which of the following numbers have greatest precision?

4.3201, 4.32, 4.320106.

Solution: In 4.3201, precision is 10^{-4}.

In 4.32, precision is 10^{-2}

In 4.320106, precision is 10^{-6}

Hence the number 4.320106 has the greatest precision.

2. **Rounding errors :** They arise from the process of rounding off the numbers during the computation. It is also called procedural error or numerical error. Such errors are unavoidable in most of the calculations due to limitations of computing aids. These errors can be reduced however by

(i) Changing the calculation procedure so as to avoid subtraction of nearly equal numbers or division by a small number

(ii) Retaining atleast one more significant digit at each step and rounding off at last step.

Rounding off may be executed in two ways:

(a) **Chopping :** In it, extra digits are dropped by truncation of number. Suppose we are using a computer with a fixed word length of four digits then a number like 12.92364 will be stored as 12.92.

(b) **Symmetric round off :** In it, the last retained significant digit is rounded up by unity if the first discarded digit is ≥ 5 otherwise the last retained digit is unchanged.

3. **Truncation errors:**

They are caused by using approximate results or on replacing an infinite process by a finite one.

If we are using a decimal computer having a fixed word length of 4 digits, rounding off of 13.658 gives 13.66 whereas truncation gives 13.65.

Note: Truncation error is the difference between a truncated value and the actual value. A truncated quantity is represented by a numeral with a fixed number of allowed digits, with any excess digits "chopped off" (hence the expression "truncated").

As an example of truncation error, consider the speed of light in a vacuum. The official value is 299,792,458 meters per second. In scientific (power-

of-10) notation, that quantity is expressed as 2.99792458 x 10^8. Truncating it to two decimal places yields 2.99 x 10^8. The truncation error is the difference between the actual value and the truncated value, or 0.00792458 x 10^8. Expressed properly in scientific notation, it is 7.92458 x 10^5.

In computing applications, truncation error is the discrepancy that arises from executing a finite number of steps to approximate an infinite process.

Example : Find the truncation error for e^x at $x = \frac{1}{5}$ if

First three terms are retained in expansion

Solution: Error = True value – Approximate value

$$= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) - (1 + x + \frac{x^2}{2!})$$

$$= (\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots)$$

Put $x = \frac{1}{5}$

$$error = \frac{.008}{6} + \frac{.0016}{24} + \frac{.0032}{120} + \ldots$$

$$= .0013333 + .0000666 + .0000026 + \cdots \ldots = .\mathbf{0014025}$$

Check Your Progress

1.   Round off the following numbers to two decimal places

48.21416 and 2.3742

1.5   SOME PROPERTIES OF EQUATIONS

Algebraic and Transcendental Equation

Introduction

We have seen that expression of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots \ldots + a_{n-1}x + a_n$$

where $a_i's$ are constant $(a_0 \neq 0)$ and n is a positive integer, is called a polynomial in x of degree n, and the equation $f(x) = 0$ is called an algebraic equation of degree n. If $f(x)$ contains some other functions like exponential, trigonometric, logarithmic etc., then $f(x) = 0$ is called a transcendental equation.

For example,

$$x^3 - 3x + 6 = 0 , x^5 - 7x^4 + 3x^2 + 36x - 7 = 0$$

are algebraic equations of third and fifth degree, whereas

$x^2 - 3cosx + 1 = 0$, $xe^x - 2 = 0$, $x \log_{10} x = 1.2$ etc are transcendental equations. In both the cases, if the coefficients are pure numbers, they are called numerical equations. Here we shall describe some numerical methods for the solution of $f(x) = 0$ where $f(x)$ is algebraic or transcendental or both.

Check Your Progress

1. Give two examples each of algebraic and transcendental equation.What is the difference between the two explain with suitable example.

1.6 ITERATION METHODS FOR FINDING THE ROOTS (ZERO'S) OF AN EQUATION

Methods For Finding The Root Of An Equation

Method for finding the root of an equation can be classified into following two parts:

1. Direct methods

2. Iterative methods

1. **Direct methods**

    In some cases, roots can be found by using direct analytical methods. For example, for a quadratic equation $ax^2 + bx + c = 0$ the roots of the equation, obtained by

    $$x_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

    These are called closed form solution. Similar formulae are also available for cubic and biquadratic polynomial equations but we rarely remember them. For higher order polynomial equations and non-polynomial equations, it is difficult and in many cases impossible, to get closed form solutions. Besides this, when numbers are substituted in available closed form solutions, rounding errors reduce their accuracy.

2. **Iterative Methods**

    These methods, also known as *trail* and *error* methods, are based on the idea of successive approximations, *i.e.*, starting with one or more initial approximations to the value of the root, we obtain the sequence of approximations by repeating a fixed sequence of steps over and over again till we get the solution with reasonable accuracy. These methods generally give only one root at a time.

For the human problem solver, these methods are very cumbersome and time consuming , but on other hand, more natural for use on computers, due to the following reasons:

1. These methods can be concisely expressed as computational algorithms.

2. It is possible to formulate algorithms which can handle class of similar problems. For example, algorithms to solve polynomial equations of degree *n* may be written.

3. Rounding errors are negligible as compared to methods based on closed form solutions.

Order (Or Rate) Of Convergence Of Iterative Method

Convergence of an iterative method is judged by the order at which the error between successive approximations to the root decreases.

The order of convergence of an iterative method is said to be kth order convergent if *k* is the largest positive real number such that

$$\lim_{i \to \infty} \left| \frac{e_{i+1}}{e_i^k} \right| \leq A$$

Where *A*, is a non-zero finite number called asymptotic error constant and it depends on derivative of *f(x)* at an approximate root *x*. e_i and e_{i+1} are the errors in successive approximation.

In other words, the error in any step is proportional to the *k*th power of the error in the previous step. Physically, the *k*th order of convergence means that in each iteration, the number of significant digits in each approximation increases *k* times.

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<u>Check Your Progress</u>

1. What does iteration mean and how iterative methods converge after every step.

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1.7   BISECTION METHOD

Bisection (Or Bolzano) Method

This is one of the simplest iterative method and is strongly based on the property of intervals. To find a root using this method, let the function *f(x)* be continuous between *a* and *b*. For definiteness, let *f(a)* be negative and *f(b)* be positive.

Then there is a root of *f(x)* = 0, lying between *a* and *b*.

Let the first approximation be

$$x_1 = \frac{1}{2}(a + b) \ (i.e., \text{ average of the ends of the range}).$$

Now of $f(x_1) = 0$ then x_1 is a root of $f(x) = 0$. Otherwise, the root will lie between a and x_1 or x_1 and b depending upon whether $f(x_1)$ is positive or negative.
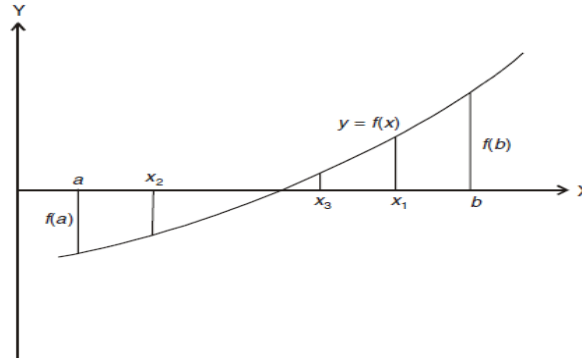
Then, we bisect the interval and continue the process till the root is found to be desired accuracy. In the above figure, $f(x_1)$ is positive; therefore, the root lies in between a and x_1.

The second approximation to the root now is $x_2 = \frac{1}{2}(a + x_1)$.

If $f(x_2)$ is negative as shown in the figure then the root lies in between x_2 and x_1, and the third approximation to the root is $x_3 = \frac{1}{2}(x_2 + x_1)$ and so on.

This method is simple but slowly convergent. It is also called as Bolzano method or Interval halving method.

Procedure For The Bisection Method To Find The Root Of The Equation $f(x) = 0$

Step 1 : Choose two initial guess values (approximation) a and b (where $(a > b)$) such that $f(a).f(b) < 0$

Step 2 : Evaluate the mid point x_1 of a and b given by $x_1 = \frac{1}{2}(a + b)$ and also evaluate $f(x_1)$.

Step 3 : If $f(a).f(x_1) < 0$, then set b $= x_1$ else set $a = x_1$. Then apply the formula of step 2.

Step 4 : Stop evaluation when the difference of two successive values of x_1 obtained from step 2, is numerically less than the prescribed accuracy.

Order of Convergence of Bisection Method

In Bisection Method, the original interval is divided into half interval in each iteration. If we take mid points of successive intervals to be the approximations of the root, one half of the current interval is the upper

bound to the error. Now $f(x_i) = 0$, then x_1 is the root of $f(x)$ In Bisection Method, $e_{i+1} = 0.5 \, e_i$ or $\dfrac{e_{i+1}}{e_i^k} = 0.5$

Here e_i and e_{i+1} are the errors i^{th} and $(i+1)^{th}$ iterations respectively. Comparing the above equation with

$$\lim_{i \to \infty} \left| \frac{e_{i+1}}{e_i^k} \right| \leq A$$

We get $k = 1$ and A = 0.5. Thus the Bisection Method is first order convergent or linearly convergent.

Example 1 Find the root of the equation $x^3 - x - 1 = 0$ lying between 1 and 2 by bisection method.

Solution Let $f(x) = x^3 - x - 1 = 0$

Since $f(1) = 1^3 - 1 - 1 = -1$, which is negative

A nd $f(2) = 2^3 - 2 - 1 = 5$, which is positive

Therefore, $f(1)$ is negative and $f(2)$ is positive, so at least one real root will lie between 1 and 2.

First iteration: Now using Bisection Method, we can take first approximation

$$x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

Then , $f(1.5) = (1.5)^3 - 1.5 - 1$

$$= 3.375 - 1.5 - 1 = 0.875$$

$\therefore$ $f(1.5) > 0$ that is, positive

So root will now lie between 1 and 1.5.

Second iteration: The Second approximation is given by

$$x_2 = \frac{1+1.5}{2} = \frac{2.5}{2} = 1.25$$

Then, $f(1.25) = (1.25)^3 - 1.25 - 1$

$$= 1.953 - 2.25 \ = - 0.875 < 0$$

$\therefore$ $f(1.25) < 0$ that is, negative

Therefore, $f(1.5)$ is positive and $f(1.25)$ is negative, so that root will lie between 1.25 and 1.5.

Third iteration: The third approximation is given by

$$x_3 = \frac{1.25+1.5}{2} = \ 1.375$$

$$x_3 = \ 1.375$$

Then, $\quad\quad\quad\quad f(1.375) = (1.375)^3 - 1.375 - 1$

$$f(1.375) = 0.2246$$

$\therefore \quad\quad\quad\quad f(1.375) > 0$ is, positive

$\therefore$ The required root lies between 1.25 and 1.375

Fourth iteration: The fourth approximation is given by

$$x_4 = \frac{1.25+1.375}{2} = 1.313$$

Then, $\quad\quad\quad\quad f(1.313) = (1.313)^3 - 1.313 - 1$

$$f(1.313) = -0.0494$$

Therefore $f(1.313)$ is, negative and $f(1.375)$ is positive. The root lies between 1.313 and 1.375.

Fifth iteration: The fifth approximation is given by

$$x_5 = \frac{1.313+1.375}{2} = 1.344$$

Then, $\quad\quad\quad\quad f(1.344) = (1.344)^3 - 1.344 - 1$

$$f(1.344) = 0.0837$$

Therefore $f(1.313)$ is, negative and $f(1.344)$ is positive. The root lies between 1.313 and 1.344.

Sixth iteration: The sixth approximation is given by

$$x_6 = \frac{1.313+1.344}{2} = 1.329$$

Then, $\quad\quad\quad\quad f(1.329) = (1.329)^3 - 1.329 - 1$

$$f(1.329) = 0.0183$$

Therefore $f(1.313)$ is, negative and $f(1.329)$ is positive. The root lies between 1.313 and 1.329.

Seventh iteration: The seventh approximation is given by

$$x_7 = \frac{1.313+1.329}{2} = 1.321$$

Then, $\quad\quad\quad\quad f(1.321) = (1.321)^3 - 1.321 - 1$

$$f(1.321) = -0.0158$$

Therefore $f(1.321)$ is, negative and $f(1.329)$ is positive. The root lies between 1.321and 1.329.

From the above iterations, the root of $\quad f(x) = x^3 - x - 1 = 0$ up to three places of decimals is **1.325**, which is desired accuracy.

Example 2 Find the root of the equation $x^3 - x - 4 = 0$ lying between 1 and 2 by bisection method.

Solution Given $f(x) = x^3 - x - 4$

We want to find the root lie between 1 and 2

At $a = 1 \Rightarrow f(a) = (1)^3 - 1 - 4 = -4$ negative

At $b = 2 \Rightarrow f(b) = (2)^3 - 2 - 4 = 2$ positive

This implies that root lies between 1 and 2

First iteration: Here, $a = 1$ and $b = 2$, $x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5$

Now, $f(a) = -4$, $f(b) = 2$. Then, $f(x_1) = (1.5)^3 - 1.5 - 4 = -2.125$

Since $f(1.5)$ is negative and $f(2)$ is positive.

So root will now lie between 1.5 and 2.

Second iteration: Here, $x_1 = 1.5$ and $b = 2$, $x_2 = \frac{1.5+2}{2} = 1.75$

Now, $f(x_1) = -2.125$, $f(b) = 2$.

Then, $f(x_2) = (1.75)^3 - 1.75 - 4 = -0.39062$

Since $f(1.75)$ is negative and $f(2)$ is positive therefore the root lies between 1.75 and 2.

Third iteration: Here, $x_2 = 1.75$ and $= 2$, $x_3 = \frac{1.75+2}{2} = 1.875$

Now, $f(x_2) = -0.39062$, $f(b) = 2$.

Then, $f(x_3) = (1.875)^3 - 1.875 - 4 = 0.7169$

Since $f(1.75)$ is negative and $f(1.875)$ is positive therefore the root lies between 1.75 and 1.875.

Fourth iteration: Here, $x_2 = 1.75$ and $x_3 = 1.875$, $x_4 = \frac{1.75+1.875}{2} = 1.8125$

Now, $f(x_2) = -0.39062$, $f(x_3) = 0.71679$

Then, $f(x_4) = (1.8125)^3 - 1.8125 - 4 = 0.14184$

Since $f(1.75)$ is negative and $f(1.8125)$ is positive therefore the root lies between 1.75 and 1.8125.

Fifth iteration: Here, $x_2 = 1.75$ and $x_4 = 1.8125$, $x_5 = \frac{1.75+1.8125}{2} = 1.78125$

Now, $f(x_2) = -0.39062$, $f(x_4) = 0.14184$

Then, $f(x_5) = (1.78125)^3 - 1.78125 - 4 = -0.12960$

Since $f(1.78125)$ is negative and $f(1.8125)$ is positive therefore the root lies between 1.75 and 1.8125.

Repeating the process, the successive approximations are

$x_6 = 1.79687$, $x_7 = 1.78906$, $x_8 = 1.79296$, $x_9 = 1.79491$,

$x_{10} = 1.79589$, $x_{11} = 1.79638$, $x_{12} = 1.79613$.

From the above discussion, the value of the root to three decimal places is **1.796**.

Example 3 Using the Bisection Method, find the real root of the equation

$$f(x) = 3x - \sqrt{1 + \sin x} = 0$$

Solution The given equation

$f(x) = 3x - \sqrt{1 + \sin x} = 0$ is a transcendental equation.

$$\text{Given } f(x) = 3x - \sqrt{1 + \sin x} = 0 \qquad\qquad \text{--- (1)}$$

Then $\qquad\qquad f(0) = 0 - \sqrt{1 + \sin 0} = \text{-1}$

And $\qquad\qquad f(1) = 3 - \sqrt{1 + \sin 1} = 3 - \sqrt{1.8414}$

$$= 3 - 1.3570 = 1.643 > 0$$

Thus $f(0)$ is negative and $f(1)$ is positive, therefore, a root lies between 0 and 1.

First approximation: The first approximation of the root is given by

$$x_1 = \frac{0+1}{2} = 0.5$$

Now, $\qquad\qquad f(0.5) = 3(0.5) - \sqrt{1 + \sin(0.5)}$

$$= 1.5 - \sqrt{1.4794} = 1.5 - 1.2163 = 0.2837 > 0$$

Thus, $f(0.5)$ is positive, while $f(0)$ is negative, therefore, a root lies between 0 and 0.5.

Second approximation: The second approximation of the root is given by

$$x_2 = \frac{0+0.5}{2} = 0.25$$

Again, $\qquad\qquad f(0.25) = 3(0.25) - \sqrt{1 + \sin(0.25)}$

$$= 0.75 - \sqrt{1.2474} = 0.75 - 1.1169 = \text{- } 0.3669 < 0$$

Thus, $f(0.25)$ is negative, while $f(0.5)$ is positive, therefore, a root lies between 0.25 and 0.5.

Third approximation: The third approximation of the root is given by

$$x_3 = \frac{0.25 + 0.5}{2} = 0.375$$

Again, $\qquad f(0.375) = 3(0.375) - \sqrt{1 + \sin(0.375)}$

$$= 1.125 - \sqrt{1.3663} = 1.125 - 1.1689 = -0.0439 < 0$$

Thus, $f(0.375)$ is negative, while $f(0.5)$ is positive, therefore, a root lies between 0.375and 0.5.

Fourth approximation: The fourth approximation of the root is given by

$$x_4 = \frac{0.375 + 0.5}{2} = 0.4375$$

Again, $\qquad f(0.4375) = 3(0.4375) - \sqrt{1 + \sin(0.4375)}$

$$= 1.3125 - \sqrt{1.4237} = 1.3125 - 1.1932 = 0.1193 > 0$$

Thus, $f(0.4375)$ is positive, while $f(0.375)$ is negative, therefore, a root lies between 0.375and 0.4375.

Fifth approximation: The fifth approximation of the root is given by

$$x_5 = \frac{0.375 + 0.4375}{2} = 0.4063$$

Again, $\qquad f(0.4063) = 3(0.4063) - \sqrt{1 + \sin(0.4063)}$

$$= 1.2189 - \sqrt{1.3952} = 1.2189 - 1.1812 = 0.0377 > 0$$

Thus, $f(0.4063)$ is positive, while $f(0.375)$ is negative, therefore, a root lies between 0.375and 0.4063.

Sixth approximation: The sixth approximation of the root is given by

$$x_6 = \frac{0.375 + 0.4063}{2} = 0.3907$$

Again, $\qquad f(0.3907) = 3(0.3907) - \sqrt{1 + \sin(0.3907)}$

$$= 1.1721 - \sqrt{1.3808} = 1.1721 - 1.1751 = -0.003 < 0$$

Thus, $f(0.3907)$ is negative, while $f(0.4063)$ is positive, therefore, a root lies between 0.3907 and 0.4063.

Seventh approximation: The seventh approximation of the root is given by

$$x_7 = \frac{0.3907 + 0.4063}{2} = 0.3985$$

From the last two observations, i.e. $x_6 = 0.3907$ and $x_7 = 0.3985$ the approximate value of the root up to two places of decimal is given by 0.39. Hence the root is **0.39** approximate

1.8 SUMMARY

In fact, there is no need of a deeper knowledge of numerical methods and their analysis in most of the cases in order to use some standard softwares as an end user. However, there are at least three reasons to gain a basic understanding of the theoretical background of numerical methods.

1. Learning different numerical methods and their analysis will make a person more familiar with the technique of developing new numerical methods. This is important when the available methods are not enough or not efficient for a specific problem to be solved.

2. In many circumstances, one has more methods for a given problem. Hence, choosing an appropriate method is important for producing an accurate result in lesser time.

3. With a sound background, one can use methods properly (especially when a method has its own limitations and/or disadvantages in some specific cases) and, most importantly, one can understand what is going wrong when results are not as expected.

1.9 EXERCISE

Q. 1.   If X= 2.536, find the absolute error and relative error when

   (i)   X is rounded off

   (ii) X is truncated to two decimal digits

Q. 2.   What do you mean by truncation error? Explain with examples.

Q. 3.   Find the real root of $e^x = 3x$ by Bisection Method.

   [Ans 1.5121375]

Q. 4.   Find a root of $x^3 - x - 11 = 0$ Bisection Method correct to three decimal places which lies between 2 and 3.

   [Ans 2.374]

Q. 5.   Compute the root of log x = cos x correct to 2 decimal places using Bisection Method

   [Ans 1.5121375]

UNIT-2 ITERATIVE METHODS AND ORDER OF CONVERGENCE CRITERIA

Structure

2.1 OBJECTIVE

In this unit, our objective is to review the basic concepts of basic iterative methods like fixed point iterative method, Regula –Falsi method and Newton Raphson Method and how their order of convergence is calculated.

2.2 INTRODUCTION

Majority of the Mathematical problems that arise in science and engineering are very hard and sometimes impossible to solve exactly manually. Thus, by making an initial approximation and using the iterative methods like fixed point iterative method, Regula –Falsi method and Newton Raphson Method one can calculate the root exactly to the accurate number of places required using the computational tools and techniques.

2.3 FIXED POINT ITERATION METHOD

ITERATION METHOD (METHOD OF SUCCESSIVE APPROXIMATION

This method is also known as the direct substitution method or method of fixed iterations.

To find the root of the equation $f(x) = 0$ by successive approximations, we rewrite the give equation in the form

$$x = g(x) \qquad \text{--- (1)}$$

Now, first we assume the approximate value of root (let x_0) , then substitute it in $g(x)$ to have a first approximation x_1 given by

$$x_1 = g(x_0) \qquad\qquad \text{--- (2)}$$

Similarly, the second approximation x_2 is given by

$$x_2 = g(x_1) \qquad\qquad \text{--- (3)}$$

In general, $\qquad\qquad x_{i+1} = g(x_i) \qquad\qquad \text{--- (4)}$

Procedure For Iteration Method To Find The Root of The Equation $f(x) = 0$

Step 1 : Take an initial approximation as x_0.

Step 2 : Find the next (first) approximation x_1 by usin $\quad x_1 = g(x_0)$

Step 3 : Follow the above procedure to find the successive approximations , x_{i+1} by using $\quad x_{i+1} = g(x_i)$, $i = 0, 1, 2, 3 \dots$

Step 4 : Stop the evaluation where relative error $\leq \quad \varepsilon$, where ε is the prescribed accuracy.

Note 1: The iteration method $x = g(x)$ is convergent if $|g^1(x)| < 1$

Rate of convergence of Iteration Method

Let $f(x) = 0$ be the equation which is being expressed as $x = g(x)$. The iterative formula for solving the equation is

$$x_{i+1} = g(x_i)$$

If a is the root of the equation $x = g(x)$ lying in the interval $]a, b[$, $\alpha = g(\alpha)$.

The iterative formula may also be written as

$$x_{i+1} = g(x + \overline{x_i - \alpha})$$

Then by mean value theorem

$$x_{i+1} = g(\alpha) + (x_i - \alpha)g^{'}(c_i) \text{ where } \alpha < c_i < b$$

But $\qquad\qquad g(\alpha) = \alpha$

$\Rightarrow \qquad\qquad x_{i+1} = \alpha + (x_i - \alpha)g^{'}(c_i)$

$\Rightarrow \qquad\qquad x_{i+1} - \alpha = + (x_i - \alpha)g^{'}(c_i) \qquad\qquad \text{-----(1)}$

Now, if e_{i+1} , e_i are the error for the approximation x_{i+1} and x_i

Therefore, $\;e_{i+1} = x_{i+1} - \alpha$, $\;e_i = x_i - \alpha$

Using this in (1), we get

$$e_{i+1} = e_i g^{'}(c_i)$$

Here $g(x)$ is a continuous function, therefore, it is bounded

∴ $\left|g'(c_i)\right| < k$, where $k \in \,]a, b[$ is a constant

∴ $e_{i+1} \le e_i k$

Or $\dfrac{e_{i+1}}{e_i} \le k$

Hence, by definition, the rate of convergence of iteration method is 1. In other words, iteration method converges linearly.

Example 1 Find the root of the equation $x = 0.21 \sin(0.5 + x)$ by iteration method starting with x =0.12.

Solution: Here $x = 0.21 \sin(0.5 + x)$

$\Rightarrow$ $f(x) = 0.21 \sin(0.5 + x)$ --- (1)

Here we observe that $|f(x)| < 1$

$\Rightarrow$ Method of iteration can be applied.

Now, first approximation of x is given by

$$x^{(1)} = 0.21 \sin(0.5 + 0.12) = 0.21 \sin(0.62)$$

$$= 0.21 \, (0.58104) = 0.1220$$

The Second approximation of x is given by

$$x^{(2)} = 0.21 \sin(0.5 + 0.122) = 0.21 \sin(0.622)$$

$$= 0.21 \, (0.58299) = 0.12243$$

The third approximation of x is given by

$$x^{(3)} = 0.21 \sin(0.5 + 0.1224) = 0.21 \sin(0.6224)$$

$$= 0.21 \, (0.58299) = 0.12243$$

The fourth approximation of x is given by

$$x^{(4)} = 0.21 \sin(0.5 + 0.12243) = 0.21 \sin(0.62243)$$

$$= 0.21 \, (0.58301) = 0.12243$$

Here we observe that $x^{(3)} = x^{(4)}$

Hence, the required root is given by $x = \mathbf{0.12243}$

Example 2 Find the root of the equation $f(x) = x^3 + x^2 - 1 = 0$ by using iteration method.

Solution: ∵ $x^3 + x^2 - 1 = 0$

$\Rightarrow x^2(1 + x) = 1 \Rightarrow x^2 = \dfrac{1}{1+x}$

∴ $x = \dfrac{1}{\sqrt{1+x}} = \emptyset(x)$

Here, $f(0) = -1$ and $f(1) = 1$ so a root lies between 0 and 1.

Now, $x = \frac{1}{\sqrt{1+x}}$ so that,

$$\emptyset(x) = \frac{1}{\sqrt{1+x}}$$

$$\therefore \qquad \emptyset'(x) = \frac{1}{2(1+x)^{3/2}}$$

We have , $|\emptyset'(x)| < 1$ for $x < 1$

Hence iterative method can be applied

Take, $\qquad x_0 = 0.5$, we get

$$x_1 = \emptyset(x_0) = \frac{1}{\sqrt{1.5}} = 0.81649$$

$$x_2 = \emptyset(x_1) = \frac{1}{\sqrt{1.81649}} = 0.74196$$

-

-

-

-

$$x_8 = 0.75487$$

Example 3 Find the cube root of 15 correct to four significant figures by iterative method.

Solution: Let $\qquad x = (15)^{1/3}$ therefore $x^3 - 15 = 0$

$$\because \quad 2^3 < 15 < 3^3$$

Real root of the equation lies in (2, 3). The equation may be written as

$$x = \frac{15 + 20\,x - x^3}{20} = \emptyset(x)$$

Now, $\qquad \emptyset'(x) = 1 - \frac{3x^3}{20}$ therefore , $\qquad |\emptyset'(x)| < 1$

Iterative formula is $x_{i+1} = \frac{15 + 20\,x_i - x_i^3}{20}$ ------------(1)

Put $\qquad i = 0, x_0 = 2.5$, we get $x_1 = 2.47$

Put $\qquad i = 1, \ in \ (1)$, we get $x_2 = 2.466$ (where $x_1 = 2.47$)

Similarly, $\qquad x_3 = 2.4661$

Therefore $\sqrt[3]{20}$ correct to 3 decimal places **is 2.466**.

2.4 CHORD METHODS FOR FINDING ROOTS- REGULA FALSI METHOD

REGULA FALSI METHOD (CHORD METHODS FOR FINDING ROOTS)

This method is essentially same as the bisection method except that instead of bisecting the interval.

In this method, we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Since the graph of $y = f(x)$ crosses the X-axis between these two points, a root must lie in between these points.

Consequently, $f(x_0)\ f(x_1) < 0$. Equation of the chord joining points $\{x_0, (x_0)\ \}$ and $\{x_1, f(x_1)\ \}$ is
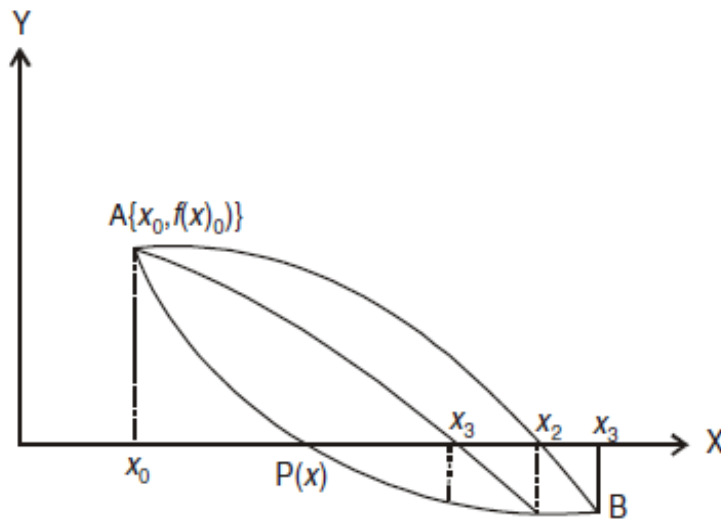
$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

The method consists in replacing the curve AB by means of the chord and taking the point of intersection of the chord with X-axis as an approximation to the root.

So the abscissa of the point where chord cuts y = 0 is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}\ f(x_0)$$

The value of x_2 can also be put in the following form:

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

In general, the $(i+1)th$ approximation to the root is given by

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Procedure for the False Position Method to Find the Root of the Equation $f(x) = 0$

Step 1: Choose two initial guess values (approximation) x_0 and x_1 (where $x_1 > x_0$) such that $f(x_0) . f(x_1) < 0$

Step 2 : Find the next approximation x_2 using the formula

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

and also evaluate $f(x_2)$.

Step 3: If $f(x_2) . f(x_1) < 0$, then go to the next step. If not, rename x_0 and x_1 and then go to next step.

Step 4 : Evaluate successive approximations using the formula

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$, where i =2, 3, 4, ……..

But before applying the formula for x_{i+1}, ensure whether If $f(x_{i-1}) . f(x_i) < 0$ if not, rename x_{i-2} and x_{i-1} and proceed.

Step 5 : Stop the evaluation when $|x_i - x_{i-1}| < \epsilon$, where ϵ is the prescribed accuracy.

Order (or Rate) of Convergence of REGULA FALSI METHOD

The general iterative formula for False Position Method is given by

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$ -------------- (1)

where x_{i-1}, x_i and x_{i+1} are successive approximations to be required root of $f(x) = 0$

The formula given in (1), can also be written as :

$$x_{i+1} = x_i - \frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} f(x_i)$$ -------------- (2)

Let α be the actual (true) root of $f(x) = 0$, i.e. $f(\alpha) = 0$. If e_{i-1} , e_0 and e_{i+1} are the successive errors in $(i-1)th$, $i\,th$ and $(i+1)th$ iterations respectively, then

$$e_{i-1} = x_{i-1} - \alpha \, , e_i = x_i - \alpha \, , e_{i+1} = x_{i+1} - \alpha$$

Or $\quad x_{i-1} = \alpha + e_{i-1}, \quad x_i = \alpha + e_i, \quad x_{i+1} = \alpha + e_{i+1},$

Using these in (2), we obtain

$$\alpha + e_{i+1} = \alpha + e_i - \frac{(e_i - e_{i-1})}{f(\alpha + e_i) - f(\alpha + e_{i-1})} f(\alpha + e_i) \qquad \text{------- (3)}$$

Expanding $\quad f(\alpha + e_i)$ and $f(\alpha + e_{i-1})$ in Taylor's series around α, we have

$$e_{i+1} = e_i - \frac{(e_i - e_{i-1})[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha) + \cdots]}{[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha) + \cdots] - [f(\alpha) + e_{i-1} f'(\alpha) + \frac{e_{i-1}^2}{2} f''(\alpha) + \cdots]}$$

i.e. $e_{i+1} = e_i - \dfrac{(e_i - e_{i-1})[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha)]}{[(e_i - e_{i-1}) f'(\alpha) + (\frac{e_i^2 - e_{i-1}^2}{2}) f''(\alpha)]}$,

[on ignoring the higher order terms]

$$e_{i+1} = e_i - \frac{[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha)]}{[f'(\alpha) + (\frac{e_i + e_{i-1}}{2}) f''(\alpha)]} \qquad [\text{ since } f(\alpha) = 0]$$

. i.e. $\quad e_{i+1} = e_i - \dfrac{[e_i + \frac{e_i^2}{2} \frac{f''(\alpha)}{f'(\alpha)}]}{[1 + (\frac{e_i + e_{i-1}}{2}) \frac{f''(\alpha)}{f'(\alpha)}]}$

[on dividing the numerator and denominator by $f'(\alpha)$]

i.e. $\quad e_{i+1} = e_i - [e_i + \frac{e_i^2}{2} \frac{f''(\alpha)}{f'(\alpha)}][1 + (\frac{e_i + e_{i-1}}{2}) \frac{f''(\alpha)}{f'(\alpha)}]^{-1}$

i.e. $\quad e_{i+1} = e_i - [e_i + \frac{e_i^2}{2} \frac{f''(\alpha)}{f'(\alpha)}][1 - (\frac{e_i + e_{i-1}}{2}) \frac{f''(\alpha)}{f'(\alpha)}]$

i.e. $\quad e_{i+1} = e_i - [\frac{e_i(e_i + e_{i-1})}{2} \frac{f''(\alpha)}{f'(\alpha)} + \frac{e_i^2}{2} \frac{f''(\alpha)}{f'(\alpha)} - \frac{e_i^2(e_i + e_{i-1})}{4} \{\frac{f''(\alpha)}{f'(\alpha)}\}^2]$

If e_{i-1} and e_i are very small, then ignoring higher terms we get

$$e_{i+1} = e_i e_{i-1} \frac{f''(\alpha)}{2 f'(\alpha)} \qquad \text{------- (4)}$$

Which can be written as

$$e_{i+1} = e_i e_{i-1} M \text{ , where M} = \frac{f''(\alpha)}{2 f'(\alpha)} \text{ and would be a constant} \qquad \text{--- (5)}$$

In order to find the order of convergence, it is necessary to find a formula of the type

$$e_{i+1} = A e_i{}^k \text{ with a appropriate value of k.} \qquad \text{----- (6)}$$

With the help of (6), we can write

$$e_{i+1} = A e_i{}^k \text{ or } \quad e_{i-1} = (\frac{e_i}{A})^{1/k}$$

Now substituting the value of e_{i+1} and e_{i-1} in (5), we get

$$Ae_i{}^k = e_i \left(\frac{e_i}{A}\right)^{1/k} \quad \text{M}$$

Or
$$e_i{}^k = M\, A^{-\left(1+\frac{1}{k}\right)} . e_i{}^{\left(1+\frac{1}{k}\right)} \qquad \text{----- (7)}$$

Comparing the powers of e_i on both sides of (7), we get

$$k = 1 + 1/k$$

Or
$$k^2 - k - 1 = 0 \qquad \text{------- (8)}$$

From (8), taking only the positive root, we get $k = 1.618$

By putting the value of k in (6), we have

$$e_{i+1} = Ae_i{}^{1.618} \quad \text{or} \quad \frac{e_{i+1}}{e_i{}^{1.618}} = A$$

Comparing this with $\lim_{i \to \infty}\left(\frac{e_{i+1}}{e_i{}^k}\right) \le A$, we see that order (or rate) of convergence of false position method is 1.618.

Example 1 Find the real root of the equation $f(x) = x^3 - 2x - 5 = 0$ by the method of false position up to three places of decimal.

Solution: Given that $\quad f(x) = x^3 - 2x - 5 = 0$

So that $\quad f(2) = (2)^3 - 2(2) - 5 = -1$

and $\quad f(3) = (3)^3 - 2(3) - 5 = 16$

Therefore, a root lies between 2 and 3.

First approximation: Therefore taking, $x_0 = 2$, $x_1 = 3$, $f(x_0) = -1$, $f(x_1) = 16$, then by Regula- Falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2 - \frac{3-2}{16+1}(-1) = 2 + \frac{1}{17} = 2.0588$$

Now, $\quad f(x_2) = f(2.0588)$

$$= (2.0588)^3 - 2(2.0588) - 5 = -0.3911$$

Therefore, root lies between 2.0588 and 3.

Second approximation: Therefore taking, $x_0 = 2.0588$, $x_1 = 3$, $f(x_0) = -0.3911$, $f(x_1) = 16$, then by Regula- Falsi method, we get

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0588 - \frac{3-2.0588}{16+0.3911}(-0.3911)$$

$$= 2.0588 + 0.0225 = 2.0813$$

Now, $f(x_3) = f(2.0813)$

$$= (2.0813)^3 - 2(2.0813) - 5 = -0.1468$$

Therefore, root lies between 2.0813 and 3.

Third approximation: Therefore taking, $x_0 = 2.0813$, $x_1 = 3$,

$f(x_0) = -0.1468$, $f(x_1) = 16$, then by Regula- Falsi method, we get

$$x_4 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0813 - \frac{3 - 2.0813}{16 + 0.1468} (-0.1468)$$

$$= 2.0813 + 0.0084 = 2.0897$$

Now, $f(x_4) = f(2.0897)$

$$= (2.0897)^3 - 2(2.0897) - 5 = -0.054$$

Therefore, root lies between 2.0897 and 3.

Fourth approximation: Therefore taking, $x_0 = 2.0897$, $x_1 = 3$,

$f(x_0) = -0.054$, $f(x_1) = 16$, then by Regula- Falsi method, we get

$$x_5 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0897 - \frac{3 - 2.0897}{16 + 0.054} (-0.054)$$

$$= 2.0588 + 0.0031 = 2.0928$$

Now, $f(x_5) = f(2.0928)$

$$= (2.0928)^3 - 2(2.0928) - 5 = -0.0195$$

Therefore, root lies between 2.0928 and 3.

Fifth approximation: Therefore taking, $x_0 = 2.0928$, $x_1 = 3$,

$f(x_0) = -0.0195$, $f(x_1) = 16$, then by Regula- Falsi method, we get

$$x_6 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0928 - \frac{3 - 2.0928}{16 + 0.0195} (-0.0195)$$

$$= 2.0928 + 0.0011 = 2.0939$$

Now, $f(x_6) = f(2.0939)$

$$= (2.0939)^3 - 2(2.0939) - 5 = -0.0074$$

Therefore, root lies between 2.0939 and 3.

Sixth approximation: Therefore taking, $x_0 = 2.0939$, $x_1 = 3$,

$f(x_0) = -0.0074$, $f(x_1) = 16$, then by Regula- Falsi method, we get

$$x_7 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0939 - \frac{3 - 2.0939}{16 + 0.0074} (-0.0074)$$

$$= 2.0939 + 0.00042 = 2.0943$$

Now, $f(x_7) = f(2.0943)$

$$= (2.0943)^3 - 2(2.0943) - 5 = -0.0028$$

Therefore, root lies between 2.0943 and 3.

Seventh approximation: Taking, $x_0 = 2.0943$, $x_1 = 3$,

$f(x_0) = -0.0028, f(x_1) = 16$, then by Regula- Falsi method, we get

$$x_8 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0943 - \frac{3 - 2.0943}{16 + 0.0028} (-0.0028)$$

$$= 2.0943 + 0.00016 = 2.0945$$

Therefore, the root is **2.094** correct to three decimal places.

Example 2 Using the method of False Position, find the root of equation $f(x) = x^6 - x^4 - x^3 - 1 = 0$ up to four decimal places.

Solution: Let $\qquad\qquad f(x) = x^6 - x^4 - x^3 - 1$

$$f(1.4) = (1.4)^6 - (1.4)^4 - (1.4)^3 - 1 = -0.056$$

$$f(1.41) = (1.41)^6 - (1.41)^4 - (1.41)^3 - 1 = 0.102$$

Hence the root lies between 1.4 and 1.41.

Using the method of False Position,

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.4 - \frac{1.41 - 1.4}{0.102 + 0.056} (-0.056)$$

$$= 1.4 + \frac{0.01}{0.158} (0.056) = 1.4035$$

Now, $f(1.4035) = (1.4035)^6 - (1.4035)^4 - (1.4035)^3 - 1$

$$f(x_2) = -0.0016(-ve)$$

Hence the root lies between 1.4035 and 1.41.

Using the method of False Position,

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

$$= 1.4035 - \frac{1.41 - 1.4035}{0.102 + 0.016} (-0.0016)$$

$$= 1.4035 \ + \ \frac{0.0065}{0.1036} \ (0.0016) = 1.4036$$

Now, $\quad f(1.4036) = (1.4036)^6 - \ (1.4036)^4 - (1.4036)^3 - 1$

$$f(x_3) = \ -0.00003(-ve)$$

Hence the root lies between 1.4036 and 1.41.

Using the method of False Position,

$$x_4 = x_3 - \frac{x_1 - x_3}{f(x_1) - f(x_3)} \ f(x_3)$$

$$= 1.4036 - \frac{1.41 - 1.4036}{0.102 + 0.00003} \ (-0.00003)$$

$$= 1.4036 \ + \ \frac{0.0064}{0.10203} \ (0.00003) = 1.4036$$

Since, x_3 and x_4 are approximately the same upto four places of decimal, hence the required root of the given equation is **1.4036.**

Check Your Progress

1. Find the real root of the equation $x^3 - 2x - 5 = 0$ by the method of False Position correct to three decimal places.

[Ans 2.094]

2.5 NEWTON RAPHSON METHOD, ORDER OF CONVERGENCE.

NEWTON-RAPHSON METHOD (or NEWTON'S METHOD)

This method can be derived from Taylor's series as follows:

Let $f(x) = 0$ be the equation for which we are assuming x_0 be the initial approximation and h be a small corrections to x_0, so that

$$f(x_0 + h) = 0$$

Expanding it by Taylor's series, we get

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ \dots .. = \ 0$$

Since h is small, we can neglect second and higher degree terms in h and therefore, we get

$$f(x_0) + hf'(x_0) = \ 0$$

From which we have,

$$h = -\frac{f(x_0)}{f'(x_0)}$$

Hence, if x_0 be the initial approximation, then next (or first) approximation x_1 is given by

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The next and second approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, 3, \dots\dots$

This formula is well known as Newton-Raphson formula.

The iterative procedure terminates when the relative error for two successive approximations becomes less than or equal to the prescribed tolerance

Procedure For Newton Raphson Method To Find The Root of The Equation $f(x) = 0$

Step 1: Take the trail solution (initial approximation) as x_0. Find $f(x_0)$ and $f'(x_0)$.

Step 2: Find next (first) approximation x_1 by using the formula $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Step 3: Follow the above procedure to find the successive approximations x_{n+1} using the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, where n =1, 2, 3, …….

Step 4: Stop the process when $|x_{n+1} - x_n| < \varepsilon, where\ \epsilon$ is prescribed accuracy.

Order (or Rate) of Convergence of Newton-Raphson Method

Let α be the actual root of equation $f(x) = 0$ i.e. $f(\alpha) = 0$. Let x_n and x_{n+1} be two successive approximations to the actual root α. If e_n and e_{n+1} are the corresponding errors we have, $x_n = \alpha + e_n$ and $x_{n+1} = \alpha + e_{n+1}$. By Newton-Raphson formula,

$$e_{n+1} = e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_{n+1})}$$

$$e_{n+1} = e_n - \frac{f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2}f''(\alpha) + \cdots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2}f'''(\alpha) + \cdots} \quad \text{(By Taylor's Series)}$$

$$e_{n+1} = e_n - \frac{e_n f'(\alpha) + \frac{e_n^2}{2}f''(\alpha) + \cdots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2}f'''(\alpha) + \cdots} \quad \text{(since . } f(\alpha) = 0 \text{)}$$

$$e_{n+1} = \frac{e_n^2 f''(\alpha)}{2[f'(\alpha) + e_n f''(\alpha)]} \quad \text{(On neglecting high powers of } e_n)$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)\{1+e_n \frac{f''(\alpha)}{f'(\alpha)}\}}$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \{1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\}^{-1}$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \{1 - e_n \frac{f''(\alpha)}{f'(\alpha)} + \cdots .\}$$

$$\frac{e_{n+1}}{e_n^2} \approx \frac{f''(\alpha)}{2f'(\alpha)} \quad \text{(Neglecting terms containing powers of } e_n)$$

Hence by definition, the order of convergence of Newton-Raphson method is 2 i.e., Newton-Raphson method is **quadratic convergent.**

This also shows that subsequent error at each step is proportional to the square of the previous error and as such the convergence is quadratic.

Example 1 Find the real root of the equation $x^2 - 5x + 2 = 0$ between 4 and 5 by Newton-Raphson method.

Solution: Let that $f(x) = x^2 - 5x + 2$ ………….. (1)

   Now, $f(4) = 4^2 - 5 \times 4 + 2 = -2$

   and $f(5) = 5^2 - 5 \times 5 + 2 = 2$

   Therefore, the root lies between 4 and 5

   From (1), we get $f'(x) = 2x - 5$ ………….. (2)

   Now, Newton-Raphson's method becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 5x_n + 2}{2x_n - 5}$$

$$x_{n+1} = \frac{x_n^2 - 2}{2x_n - 5} \quad n = 0,1,2, \dots \dots ..$$

   Let us take $x_0=4$ to obtain the approximation to the root by putting $n = 0,1,2, \dots \dots ..$ into (3), we get

First approximation:

$$x_1 = \frac{x_0^2 - 2}{2x_0 - 5} = \frac{4^2 - 2}{2(4) - 5} = \frac{14}{3} = 4.667$$

Second approximation:

   The root is given by

$$x_2 = \frac{x_1^2 - 2}{2x_1 - 5} = \frac{(4.667)^2 - 2}{2(4.667) - 5} = \frac{19.7781}{4.3334} = 4.5641$$

Third approximation:

The root is given by

$$x_2 = \frac{x_1{}^2 - 2}{2x_1 - 5} = \frac{(4.667)^2 - 2}{2(4.667) - 5} = \frac{19.7781}{4.3334} = 4.5641$$

Fourth approximation:

The root is given by

$$x_4 = \frac{x_3{}^2 - 2}{2x_3 - 5} = \frac{(4.5641)^2 - 2}{2(4.5641) - 5} = \frac{18.8082}{4.1232} = 4.5616$$

Since $x_3 = x_4$, hence the root of the equation is **4.5616** correct to four decimal places.

Example 2 Find the real root of the equation $3x = \cos x + 1$ by Newton-Raphson method.

Solution: Let that $f(x) = 3x - \cos x - 1 = 0$ ………….. (1)

So $f(x) = -2$

$$f(1) = 3 - \cos 1 - 1 = 1.4597$$

So the root lies between 0 and 1

Let us take $x_0 = 0.6$

From (1) $f'(x) = 3 + \sin x$ ………….. (2)

Therefore the Newton's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Or $x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}$ ……………(3)

First approximation:

Putting $n = 0$, in (3) we get first approximation

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0}$$

$$= \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)}$$

$$= \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)}$$

$$= \frac{(0.6)(0.5646) + (0.8253) + 1}{3 + (0.5646)}$$

$$= \frac{2.16406}{3.5646}$$

Or

$$x_1 = 0.6071$$

Second approximation:

Putting $n = 1$, in (3) we get second approximation

$$x_2 = \frac{x_1 Sin\, x_1 + \cos x_1 + 1}{3 + Sin\, x_1}$$

$$= \frac{(0.6071)Sin(0.6071) + \cos(0.6071) + 1}{3 + Sin\,(0.6071)}$$

$$= \frac{(0.6071)(0.5705) + (0.8213) + 1}{3 + (0.5705)}$$

$$= \frac{2.1677}{3.5705}$$

Or

$$x_2 = 0.6071$$

Since $x_1 = x_2$ Therefore the root as 0.6071 correct to four decimal places.

+---+
| **Check Your Progress** |
| |
| 1. Find the cube root of 10   [Ans 2.15466] |
+---+

2.6   SUMMARY

Numerical analysis include three parts. The first part of the subject is about the development of a method to a problem. The second part deals with the analysis of the method, which includes the error analysis and the efficiency analysis. Error analysis gives us the understanding of how accurate the result will be if we use the method and the efficiency analysis tells us how fast we can compute the result. The third part of the subject is the development of an efficient algorithm to implement the method as a computer code. A complete knowledge of the subject includes familiarity in all these three parts. Here in this unit we have learnt those mathematical iterative methods which can be used to find the root of transcendental equation upto desired no of accuracy.

2.7   EXERCISE

Q.1. Find by iterative method, the real root of the equation $3x - \log_{10} x = 6$ correct to four significant figures.

[Ans 0.5671477]

Q.2. Find by iteration method $\sqrt{30}$

[Ans 5.477225575]

Q.3. Find the postive root of $xe^x = 2$ by method of False Postion.

[Ans 0.852605]

Q.4. Find the real root of the equation $x = \tan x$ using False Position method

[Ans 4.4934]

Q.5. Find the real root of the equation $x = e^{-x}$ using the Newton-Raphson method.

[Ans 0.5671]

Q.6. Use Newton-Raphson method to obtain a root, correct to three decimal places of following equation $Sinx = \frac{x}{2}$

[Ans 1.896]

Master in Computer Application

MCA-104

**Uttar Pradesh Rajarshi Tandon
Open University**

Numerical Analysis

BLOCK

2

SOLUTION OF SYSTEM OF LINEAR EQUATIONS

UNIT-3

Direct Methods for Solution of System of Linear Equations

UNIT-4

Iterative Methods for Solution of System of Linear Equations

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BLOCK INTRODUCTION

Overview :

Numerical technique is widely used by scientists and engineers to solve their problems. A major advantage for numerical technique is that a numerical answer can be obtained even when a problem has no analytical solution. Engineers are often confronted with the problem of finding a solution for a set of simultaneous equations involving multiple variables. Traditional mathematical methods are not amenable to computerization & hence too cumbersome to use in case of large number of variables. Numerical methods are programmable, can deliver solutions rapidly with reasonable accuracy & hence provide a viable alternative for handling such problems.

UNIT-3 DIRECT METHODS FOR SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Structure

3.1 OBJECTIVE

In this unit, our objective is to review the direct methods for solution of system of linear equations such as Cramer's rule , inverse of matrix, Gauss Elimination Method, Gauss- Jordon Reduction Method, LU decomposition method and Crout's method for calculation of exact solution of system of equations

3.2 INTRODUCTION

Numerical methods deal with devising efficient methods for obtaining numerical solutions to difficult system of linear simultaneous equations. Most of the Mathematical problems that arise in science and engineering are very hard and sometime impossible to solve exactly. Here in this unit direct and iterative methods such as Elimination Method, Gauss- Jordon Reduction Method etc are discussed to find out the approximate root of simultaneous linear equation.

3.3 DIRECT METHODS- PRELIMINARIE

Direct Methods

When two equations, each in two variables, are given, they can be solved in following two ways (i.e. these are known as direct methods)

1. Elimination by cancellation
2. Elimination by substitution

Elimination by cancellation

Example 1: $2x + 3y = 19$ and $5x + 4y = 19$ then find the values of x and y.

Solution

In this method, the two equations are reduced to a single variable equation by eliminating one of the variables.

Step 1: Here, let us eliminate the y term, and in order to eliminate the y term, we have to multiply the first equation with the coefficient of y in the second equation and the second equation with the coefficient of y in the first equation so that the coefficients of y terms in both the equations become equal.

$$(2x + 3y = 19\,)4 \Rightarrow 8x + 12y = 76 \qquad\qquad \ldots\ldots\ldots\ldots(1)$$
$$(5x + 4y = 37\,)3 \Rightarrow 15x + 12y = 111 \qquad\qquad \ldots\ldots\ldots\ldots(2)$$

Step 2: Subtract Eq. (2) from Eq. (1),

$$(15x + 12y) - (\,8x + 12y) = 111 - 76$$
$$\Rightarrow 7x = 35$$
$$\Rightarrow x = 5$$

Step 3: Substitute the value of x in Eq. (1) or Eq. (2) to find the value of y. Substituting the value of x in the first equation, we get

$$2(5) + 3y = 19$$
$$\Rightarrow 3y = 19 - 10 \Rightarrow 3y = 9$$
$$\Rightarrow y = 3$$

$\therefore$ The solution of the given pair of equation is $x = 5\,;y = 3$

Elimination by Substitution

Example 2: $4x - 3y = 32$ and $x + y = 1$ then find the values of x and y.

Solution

In this method, the two equations are reduced to a single variable equation by eliminating one of the variables equation by substituting the value of one variable, obtained from one equation, in the other equation.

Step 1: Using the second equation, find x in terms of y, i.e.,

$$x + y = 1$$

$$\Rightarrow y = 1 - x \qquad\qquad \text{……………..} (1)$$

Step 2: Substitute the value of y in the first equation to find the value of

$$\therefore 4x - 3(1 - x) = 32$$

Step 3: Simplify the equation in terms of x and find the value of x.

$$4x - 3 + 3x = 32$$

$$\Rightarrow 7x = 32 + 3$$

$$\Rightarrow x = 5$$

Step 4: Simplify the value of x in Eq. (1), we have

$$y = 1 - x = 1 - 5$$

$$\Rightarrow y = -4$$

$$\therefore \text{ The solution of the given pair of equation is } x = 5 \, ; y = -4$$

Nature of Solutions

When we try to solve a pair of equations we could arrive at three possible results. They are, having

1. a unique solution.

2. an infinite number of solutions.

3. no solution.

Let the pair of equations be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where a_1 and a_2 are the coefficients of x; b_1 and b_2 are the coefficients of y; while c_1 and c_2 are the known constant quantities.

1. **A pair of equations having a unique solution:**

   If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the pair of equations will have a unique solution.

   We have already solved such equations.

2. **A pair of equations having a infinite solution:**

   If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the pair of equations be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, will have infinite number of solutions.

Note In fact this means that there are no two equations as such and one of the two equations is simply obtained by multiplying the other with a constant. These equations are known as dependent equations.

Example:
$$3x + 4y = 8$$

$$9x + 12y = 24$$

For these two equations, $a_1 = 3, a_2 = 9, b_1 = 4, b_2 = 12, c_1 = -8, c_2 = -24$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Since $\frac{3}{9} = \frac{4}{12} = \frac{-8}{-24}$

The above pair of equations will have infinite solutions.

3. **A pair of equations having no solution at all:**

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the pair of equations be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, will have no solutions.

Notes

1. In other words, the two equations will contradict each other or be inconsistent with each other.

2. A pair of equations is said to be consistent if it has a solution (finite or infinite).

Example: $5x + 6y = 30$

$10x + 12y = 40$

For these two equations, $a_1 = 5, a_2 = 10, b_1 = 6, b_2 = 12, c_1 = -30, c_2 = -40$

Here,

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Since $\frac{5}{10} = \frac{6}{12} \neq \frac{-30}{-40}$

Hence, the pair of equations have no solution at all.

3.4 CRAMER'S RULE AND METHOD OF SOLUTION USING INVERSE OF MATRIX

Method of determinants -Cramer's rule

Consider the equations $\quad a_1x + b_1y + c_1z = d_1$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3 \qquad \text{------- (1)}$$

If the determinant of the coefficient be $\quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then $\quad x\Delta = \begin{vmatrix} xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \\ xa_3 & b_3 & c_3 \end{vmatrix} \qquad [opertae\ c_1 \to c_1 + yc_2 + zc_3]$

$$= \begin{vmatrix} a_1x + b_1y + c_1 & b_1 & c_1 \\ a_2x + b_2y + c_2 & b_2 & c_2 \\ a_3x + b_3y + c_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad [\text{by (1)}]$$

Thus $\quad x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \ provided\ \Delta \neq 0 \qquad \text{------- (2)}$

Similarly, $\quad y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \text{------- (3)}$

$$z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \text{------- (4)}$$

Equation (2), (3) and (4) giving the values x, y and z constitute the Cramer's rule, which reduces the solution of the linear equations (1) to a problem in evaluation of determinants.

Example : Solve the equations

$$3x + y + 2z = 3,$$

$$2x - 3y - z = -3,$$

$$x + 2y + z = 4 \text{ by}$$

Method of determinants -Cramer's rule.

Solution:

Here $\quad \Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} \qquad = 3(-3 + 2) - 2(1 - 4) + (-1 + 6) = 8$

[Expanding by c_1]

$$\therefore \quad x = \frac{1}{\Delta} \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} \quad \text{[Expand by } c_1\text{]}$$

$$= \frac{1}{8}[3(-3 + 2) + 3(1 - 4) + 4(-1 + 6)] = 1$$

Similarly $\quad y = \frac{1}{\Delta} \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 2 \quad$ and $\quad z = \frac{1}{\Delta} \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -1$

Hence $x = 1, y = 2, z = -1$

Matrix inversion method

Consider the equations $\quad a_1 x + b_1 y + c_1 = d_1$

$$a_2 x + b_2 y + c_2 = d_2$$

$$a_3 x + b_3 y + c_3 = d_3 \qquad\text{------- (1)}$$

If $\quad A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then the equation (1) are equivalent to the matrix equation

$$AX = D \qquad\text{---------(2)}$$

Where A is the *coefficient matrix.*

Multiplying both sides of (2) by the reciprocal matrix A^{-1}, we get

$$A^{-1}AX = A^{-1}D \qquad \text{or } IX = A^{-1}D \qquad [\because\ A^{-1}A = I]$$

Or $\qquad\qquad X = A^{-1}D$

i.e. $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad\text{---------- (3)}$

where A_1, B_2 etc. are the cofactors of a_1, b_1 etc. in the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad (\Delta \neq 0)$$

Hence equating the values of x, y, z to the corresponding elements in the product on the right side (3) we get the desired solution.

Note:

The cofactor of any element in a determinant is obtained by deleting the row and column which intersect in that element with the proper sign. The sign of an element in the ith row and jth column is $(-1)^{i+j}$. The cofactor of an element is usually denoted by the corresponding capital letter.

For instance, in $\quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

the cofactor of b_3 i.e. $\boldsymbol{B_3} = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$, and

$\boldsymbol{C_2} = (-1)^{2+3} \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$

Example : Solve the equations

$$3x + y + 2z = 3,$$

$$2x - 3y - z = -3,$$

$$x + 2y + z = 4$$

by using Matrix inversion method

Solution:

Here $\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (say)

$= 3(-3+2) - 2(1-4) + (-1+6) = 8$

Therefore $\quad A_1 = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = \begin{vmatrix} -3 & -1 \\ 2 & 1 \end{vmatrix} = (-3 \times 1) - (-1 \times$

$2) = -1$

Therefore $A_1 = -1, \ A_2 = 3, \ A_3 = 5, \ B_1 = -3, \ B_2 = 1 \ B_3 = 7,$

$C_1 = 7, \ C_2 = -5, \ C_3 = -11$

Also $\Delta = a_1 A_1 + a_2 A_2 + a_3 A_3 = 8$

$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

$= \frac{1}{8} \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

$= \frac{1}{8} \begin{bmatrix} -3 - 9 + 20 \\ -9 - 3 + 28 \\ 21 + 15 - 44 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Hence $\quad x = 1, y = 2, z = -1$

Check Your Progress

1. Solve the equations x+y+ z=4, x-y+ z=0 2x+y+ z=5 by Method of determinants -Cramer's rule.

    [Answer x= 1,y = 2,z= 1]

2. Solve the equations x+y+ z=3, x+2y+3z=4, x+4y+9z=6 by using Matrix inversion method.

    [Answer x= 2, y = 1,z= 0]

3.5 GAUSS ELIMINATION METHOD

Simultaneous linear equations occur in various engineering problems. The student knows that a given system of linear equations can be solved by Cramer's rule or by Matrix method. But these methods

become tedious for large systems. However, there exist other numerical methods of solution which are well-suited for computing machines. There are some direct and iterative methods of solution.

Gauss Elimination Method

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. The method is quite general and is well-adapted for computer operations. Here we shall explain it by considering a system of three equations for the sake of clarity.

Consider the equations $\quad a_1 x + b_1 y + c_1 z = d_1$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3 \qquad \text{------- (1)}$$

Step I *To eliminate x from second and third equation*

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting (a_2/a_1) times the first equation from the second equation. Similarly we eliminate x from the third equation by eliminating (a_3/a_1) times the first equation from the third equation. We thus, get the new system

$$a_1 x + b_1 y + c_1 z = d_1$$

$$b'_2 y + c'_2 z = d'_2$$

$$b'_3 y + c'_3 z = d'_3 \qquad \text{------- (2)}$$

Here the first equation is called the pivotal equation and a_1 is called the first pivot.

Step II *To eliminate y from third equation in (2)*

Assuming $b'_2 \neq 0$, we eliminate y from the third equation of (2) by subtracting (b'_3/b'_2) times the second equation from the third equation.

We thus, get the new system

$$a_1 x + b_1 y + c_1 z = d_1$$

$$b'_2 y + c'_2 z = d'_2$$

$$c''_3 z = d''_3 \qquad \text{------- (3)}$$

Here the second equation is called the pivotal equation and b'_2 is called the new pivot.

Step III *To evaluate the unknowns*

The values of x, y, z are found from the reduced system (3) by back substitution.

Example 1 : Apply Gauss elimination method to solve the equations

$$x + 4y - z = -5, \quad x + y - 6z = -12, \quad 3x - y - z = 4 .$$

Solution   We have

$$x + 4y - z = -5 \qquad \text{……. (i)}$$
$$x + y - 6z = -12 \qquad \text{……….(ii)}$$
$$3x - y - z = 4 \qquad \text{-------- (iii)}$$

Step I Operate (ii) –(i) and (iii) – 3(i) to eliminate x:

$$-3y - 5z = -7 \qquad \text{-------- (iv)}$$
$$-13y + 2z = 19 \qquad \text{-------- (v)}$$

Step II Operate (v) –13/3 and (iv) to eliminate y:

$$\frac{71}{3}z = \frac{148}{3} \qquad \text{-------- (vi)}$$

Step III By back-substitution, we get

From (vi) $z = \frac{148}{71} = 2.0845$

From (iv) $y = \frac{7}{3} - \frac{5}{3}\left(\frac{148}{71}\right) = -\frac{81}{71} = -1.1408$

From (i) $x = -5 - 4\left(\frac{81}{71}\right) + \frac{148}{71} = \frac{117}{71} = 1.6479$

Hence $x = 1.6479, \ y = -1.1408, z = 2.0845$

Example 2 : Apply Gauss elimination method to solve the equations

$$2x + 4y + 6z = 22, \quad 3x + 8y + 5z = 27, \qquad -x + y + 2z = 2 .$$

Solution   We have

$$2x + 4y + 6z = 22 \qquad \text{……. (i)}$$
$$3x + 8y + 5z = 27 \qquad \text{……….(ii)}$$
$$-x + y + 2z = 2 \qquad \text{-------- (iii)}$$

Step I Multiply equation (i) by 1/2

$$x + 2y + 3z = 11 \qquad \text{------ (iv)}$$
$$3x + 8y + 5z = 27 \qquad \text{----(v)}$$
$$-x + y + 2z = 2$$

Step II Now multiply -3 in (iv) and add (v) to eliminate x

$$x + 2y + 3z = 11 \qquad \text{------ (iv)}$$
$$2y - 4z = -6 \qquad \text{----(v)}$$
$$-x + y + 2z = 2 \qquad \text{--- (vi)}$$

Step III Now from equation (iv) and add (vi) we get

$$x + 2y + 3z = 11 \qquad \text{------ (iv)}$$

$$2y - 4z = -6 \qquad \text{--- -(v)}$$

$$3y + 5z = 13 \qquad \text{-----(vii)}$$

Step IIV Now eliminate y equation (v) and add (vi) we get

$$x + 2y + 3z = 11 \qquad \text{------ (iv)}$$

$$y - 2z = -3 \qquad \text{--- -(viii)}$$

$$11z = 22 \qquad \text{-----(ix)}$$

Then by back substitution we get

$$z = 2 \text{ from equation (ix)}$$

$$y = 1 \text{ from equation (viii)}$$

$$x = 3 \text{ from equation (iv)}$$

Check Your Progress

1. Example: Apply Gauss elimination method to solve the equations   2x+y+z=10,   3x+2y+3z=18, x+4y+9z=16 .

    [Answer x=7,   y=-9 , z=5]

3.6 GAUSS- JORDON REDUCTION METHOD

This is a modification of the Gauss elimination method. In this method, elimination of unknown is performed not in the equations below but in the equations above also, **ultimately reducing the system to a diagonal matrix form i.e.**, each equation involving only one unknown. From these equations the unknowns x , y, z can be obtained readily.

Thus in this method, the labour of back-substitution for finding the unknowns is saved at the cost of additional calculations.

Example : Apply Gauss- Jordon Reduction Method to solve the equations
$$x + y + z = 9,$$

$$2x - 3y + 4z = 13,$$

$$3x + 4y + 5z = 40 .$$

Solution   We have

$$x + y + z = 9 \qquad \text{………. (i)}$$

$$2x - 3y + 4z = 13 \qquad \text{………. (ii)}$$

$$3x + 4y + 5z = 40 \qquad \text{...…….. (iii)}$$

Step I Operate (ii) –2(i) and (iii) – 3(i) to eliminate x from (ii) and (iii).

$$x + y + z = 9 \qquad \text{-------- (iv)}$$

$$-5y + 2z = -5 \qquad \text{-------- (v)}$$

$$y + 2z = 13 \qquad \text{-------- (vi)}$$

Step II Operate (iv) +1/5 (v) and (vi) +1/5(v) to eliminate y from (iv) and (vi):

$$x + 7/5z = 8 \qquad \text{-------- (vii)}$$

$$-5y + 2z = -5 \qquad \text{-------- (viii)}$$

$$\frac{12}{5}z = 12 \qquad \text{-------- (ix)}$$

Step III Operate (vii) -7/12 (ix) and (viii) -5/6(ix) to eliminate z from (vii) and (viii):

$$x = 1 \qquad \text{-------- (vii)}$$

$$-5y = -15 \qquad \text{-------- (viii)}$$

$$\frac{12}{5}z = 12 \qquad \text{-------- (ix)}$$

Hence $x = 1$, $y = 3$, $z = 5$

Check Your Progress

1. Apply Gauss- Jordon Reduction to solve the equations
    2x + 5y + 7z=52,   2x + y – z =0,   x + y + z = 9 .

    [Answer x=1,   y=-3 , z=5]

3.7 LU DECOMPOSITION METHOD

LU decomposition method is also known as matrix factorization or Crout's reduction method.

This method is based on the fact that every matrix A can be expressed as the product of lower triangular matrix and an upper triangular matrix, provided all the principal minors of A are non-singlar i.e. if $A = [a_{ij}]$, then

$$a_{11} \neq 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{vmatrix} \neq 0, \text{etc.}$$

Also such a factorization if it exists, is unique.

Now consider the equations

$$a_{11}x_1 + a_{12}x_1 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_1 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

which can be written as $\quad AX = B$ ------- (1)

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let $\quad\quad\quad\quad\quad\quad A = LU$ ------- (2)

Where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Here L is known as the Lower triangular matrix and l_{21}, l_{31}, l_{32} are the coefficients. Similarly U is known as the Upper triangular matrix and $u_{11}, u_{12}, u_{13}, etc$ are the coefficients.

Then (1) becomes $\quad\quad\quad LUX = B$ ------- (3)

Writing $\quad\quad\quad UX = V$ ------- (4)

(3) becomes $LV = B$ which is equivalent to the equations

$$x_1 = b_1; \ l_{21}v_1 + v_2 = b_2; \ l_{31}v_1 + l_{31}v_2 + v_3 = b_3;$$

Solving these for v_1, v_2, v_3 we know V. Then, (4) becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = v_1; \ u_{22}x_2 + u_{23}x_3 = v_2; \ u_{33}x_3 = v_3$$

From which $x_3, x_2, \ and \ x_1$ can be found by back-substitution.

To compute the matrices L and U, we write (2) as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and equating corresponding elements from both sides, we obtain

(i) $u_{11} = a_{11}, \quad u_{12} = a_{12}, \quad u_{13} = a_{13}$

(ii) $l_{21}u_{11} = a_{21} \quad\quad$ or $l_{21} = \dfrac{a_{21}}{a_{11}}$

          $l_{31}u_{11} = a_{31} \quad\quad$ or $l_{31} = \dfrac{a_{31}}{a_{11}}$

(iii) $l_{21}u_{12} + u_{22} = a_{22} \quad$ or $\quad u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}}a_{12}$

$$l_{21}u_{13} + u_{23} = a_{23} \quad \text{or} \quad u_{23} = a_{23} - \frac{a_{21}}{a_{11}}a_{13}$$

(iv) $\quad l_{31}u_{12} + l_{32}u_{22} = a_{32} \quad$ or $\quad l_{32} = \frac{1}{u_{22}}\left[a_{32} - \frac{a_{31}}{a_{11}}a_{12}\right]$

(v) $\quad l_{31}u_{13} + l_{33}u_{23} + u_{33} = a_{33} \quad$ which gives u_{33}.

Thus we compute the elements of L and U in the following set order:

(i)	First row of U,	(ii)	First column of L,
(iii)	Second row of U,	(iv)	Second column of L,
(iv)	Third row of U,		

Example : Apply LU decomposition (factorization) method to solve the equations
$$3x + 2y + 7z = 4,$$

$$2x + 3y + z = 5,$$

$$3x + 4y + z = 7.$$

Solution: Let

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

$(i.e. A)$,

So that

(i) $\quad u_{11} = 3, \quad u_{12} = 2, \quad u_{13} = 7$

(ii) $\quad l_{21}u_{11} = 2 \quad\quad$ or $l_{21} = \frac{2}{3}$

$\quad\quad l_{31}u_{11} = 3 \quad\quad$ or $l_{31} = 1$

(iii) $\quad l_{21}u_{12} + u_{22} = 3 \quad$ or $\quad u_{22} = 5/3$

$\quad\quad l_{21}u_{13} + u_{23} = 1 \quad$ or $\quad u_{23} = -11/3$

(iv) $\quad l_{31}u_{12} + l_{32}u_{22} = 4 \quad$ or $\quad l_{32} = 6/5$

(v) $\quad l_{31}u_{13} + l_{33}u_{23} + u_{33} = 1 \quad$ which gives $u_{33} = -8/5$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

Writing $\quad UX = V$, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

Solving this system, we have $v_1 = 4$

$\frac{2}{3}v_1 + v_2 = 5$ or $v_2 = \frac{7}{3}$

$v_1 + \frac{6}{5}v_2 + v_3 = 7$ or $v_3 = \frac{1}{5}$

Hence the original system becomes

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

i.e. $3x + 2y + 7z = 4$, $\frac{5}{3}y - \frac{11}{3}z = \frac{7}{3}$, $-\frac{8}{5}z = \frac{1}{5}$

By back-substitution, we have $z = -\frac{1}{8}, y = \frac{9}{8}$ and $x = \frac{7}{8}$

Check Your Progress

1. Apply LU decomposition (factorization) method to solve the equations

    $10x + y + z = 12$, $2x + 10y + z = 13$, $2x + 2y + 10z = 14$.

    [Answer x=y= z=1]

3.8   CROUT'S METHOD

This method is based on the fact that every square matrix A can be expressed as the product of a lower triangular matrix and an upper triangular matrix, provided all the principle minors of A are non-singular. Also, such a factorisation, if exists, is unique.

This method is also called triangularisation or factorisation method. Here, we factorise the given matrix as $= LU$, where L is a lower triangular matrix with unit diagonal elements and U is an upper triangular matrix. Then,

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$$

Now consider the equations

$$a_{11}x_1 + a_{12}x_1 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_1 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \qquad \text{------- (1)}$$

which can be written as $\qquad AX = B$

Let $\qquad\qquad\qquad\qquad\qquad A = LU \qquad\qquad\qquad$ ------- (2)

where $\quad L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$ ------- (3)

Here, L is a lower triangular matrix and U is an upper triangular matrix with diagonal elements equal to unity.

$$A = LU \Rightarrow A^{-1} = U^{-1}L^{-1} \qquad\qquad\qquad \text{------- (4)}$$

Now $A = LU \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Or

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23 +} + l_{33} \end{bmatrix}$

Equating the corresponding elements, we obtain

$l_{11} = a_{11} \qquad\qquad l_{21} = a_{21} \qquad\qquad l_{31} = a_{31} \qquad\qquad$ ------- (4)

$l_{11}u_{12} = a_{12} \qquad\qquad l_{11}u_{13} = a_{13} \qquad\qquad$ ------- (5)

$l_{21}u_{12} + l_{22} = a_{22} \qquad\qquad l_{31}u_{12} + l_{32} = a_{32} \qquad$ ------- (6)

$l_{21}u_{13} + l_{22}u_{23} = a_{23} \qquad\qquad\qquad$ ------- (7)

and $\quad l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33} \qquad\qquad$ ------- (8)

from (5) we find $\qquad u_{12} = \dfrac{a_{12}}{l_{11}} = \dfrac{a_{12}}{a_{11}}$

from (6) we obtain $l_{21} = a_{23} - l_{21}u_{12} \qquad\qquad$ ------- (9)

$\qquad\qquad\qquad l_{32} = a_{32} - l_{31}u_{12} \qquad\qquad$ ------- (10)

from (7) gives $\qquad u_{23} = (a_{23} - l_{21}u_{23})/l_{22} \qquad$ ------- (11)

from (8) gives $\qquad l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} \qquad$ ------- (12)

Thus, we have determined all the elements of L and U

From equations (2) and (3) we have

$$LU = B \qquad\qquad\qquad \text{------- (13)}$$

Let $\qquad\qquad\qquad UX = V$

where $\qquad\qquad\qquad V = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$

from equation (13) we have $LV = b$, which on forward substitution yields V .

From $UX = V$, we find x (by backward substitution).

Example : Solve the following set of equations by Crout's method:

$$2x + y + 4z = 12$$

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

Solution: We have

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; B = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

$$AX = B$$

Let $\quad A = LU$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23 +} + l_{33} \end{bmatrix}$$

$$l_{11}u_{12} = 1 \Rightarrow u_{12} = \frac{1}{2}$$

$$l_{11}u_{13} = 4 \Rightarrow u_{13} = \frac{4}{2} = 2$$

$$l_{22} + l_{21}u_{12} = -3 \Rightarrow l_{22} = -3 - 8\left(\frac{1}{2}\right) = -7$$

$$l_{32} + l_{21}u_{12} = -3 \Rightarrow l_{32} = 11 - 4\left(\frac{1}{2}\right) = 9$$

$$l_{21}u_{13} + l_{22}u_{23} = 2 \Rightarrow u_{23} = \frac{2 - 8 \times 2}{-7} = 2$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = -1 \Rightarrow l_{33} = -1 - 4 \times 2 - 9 \times 2 = -27$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 8 & -7 & 0 \\ 4 & 9 & -27 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & 1/2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$LV = B$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 8 & -7 & 0 \\ 4 & 9 & -27 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{matrix} 12 \\ 20 \\ 33 \end{matrix}$$

$$2v_1 = 12 \Rightarrow v_1 = 6$$

$$8v_1 - 7v_2 = 20 \Rightarrow v_2 = \frac{-20 + 8 \times 6}{7} = 4$$

$$4v_1 + 9v_2 - 27v_3 = 33 \Rightarrow v_3 = \frac{-33 + 4 \times 6 + 9 \times 4}{27} = 1$$

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}; \qquad Ux = V$$

$$\begin{bmatrix} 1 & 1/2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

$$x + \frac{2}{2}y + 2z = 6$$

$$y + 2z = 4$$

$$z = 1$$

$$y = 4 - 2 \times 1$$

$$\Rightarrow y = 2$$

$$x = 6 - \frac{1}{2} \times 2 - 2 \times 1 \Rightarrow x = 3$$

Check Your Progress

1. Solve the following set of equations by Crout's method:

   x + 2y + 2z = 14,   2x + 3y + 4z = 20, 3x + 4y + z = 14.

   [Answer x=1, y=2 , z=3]

3.9   SUMMARY

Methods for the solution of algebraic linear simultaneous equations fall naturally into two classes. First, there is the direct approach, and the other is iterative methods. The best known method of this class is that known classically as the Gaussian algorithm, or more recently as the method of 'pivotal condensation'. Here the matrix is reduced to a triangular form by successive elimination of variables, and the solution is finally obtained by a process known as back-substitution. Alternatively, by

extensions of the elimination process, the matrix can be reduced to a diagonal form, and the back-substitution avoided.

3.10 EXERCISE

Q.1. Example Solve the equations

x+3y+ 6z=2, 3x-y+ 4z =9, x-4y+ 2z=7 by Method of determinants -Cramer's rule.

[Answer   x= 2,y = -1,z= 1/2]

Q.2. Solve the equations $2x - 3y + 4z = -4, \ x + z = 0 \quad -y + 4z = 2$ by using Matrix inversion method.

[Answer $x = 1/7, y = 10/7, z = 1/7$]

Q.3. Apply Gauss elimination method to solve the equations

$2x + 2y + z = 12, \quad 3x + 2y + 2z = 8, \ 5x + 10y - 8z = 10$.

[Ans $x = -51/4, \ y = 115/8, z = 35/4$]

Q.4. Solve

$10x - 7y + 3z + 5u = 6, \ -6x + 8y - z - 4u = 5,$

$3x + y + 4z + 11u = 2, \quad 5x - 9y - 2z + 4u = 7$ \qquad by Gauss elimination method.

[Ans: $u = 1, \ z = -7, y = 4, and \ x = 5$]

Q.5. Apply Gauss- Jordon Reduction to solve the equations

$2x - 3y + z = -1, \quad x + 4y + 5z = 25, \ 3x - 4y + z = 2$.

[Ans: $x = 8.7, \ y = 5.7, z = -1.3$]

Q.6. Solve

Solve $10x - 7y + 3z + 5u = 6, \ -6x + 8y - z - 4u = 5,$

$3x + y + 4z + 11u = 2, \quad 5x - 9y - 2z + 4u = 7$ \qquad by Gauss-Jordon Reduction.

[Ans $u = 1, \ z = -7, y = 4, and \ x = 5$]

Q.7. Apply LU decomposition (factorization)   method to solve the equations

$x + 2y + 2z = 14, \quad 2x + 3y + 4z = 20, \ 3x + 4y + z = 14.$

[Ans $x = 1, \ y = 2, z = 3$]

Q.8. Solve the following set of equations by Crout's method:

$$2x_1 + x_2 + x_3 = 7$$
$$x_1 + 2x_2 + x_3 = 8$$
$$x_1 + x_2 + 2x_3 = 9$$

[Ans $x = 1, \ y = 2, z = 3$]

UNIT-4 ITERATIVE METHODS FOR SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Structure

4.1   Objective

4.2   Introduction

4.3   General Iteration Method and Jacobi's Iteration Method

4.4   Gauss-Seidal Iteration Method

4.5   Summary

4.6   Exercise

4.1   OBJECTIVE

In this unit, our objective is to review the iterative methods for solution of system of linear equations such as Jacobi's Iteration Method and Gauss-Seidal Iteration Method for calculation of exact solution of system of equations upto desired accuracy.

4.2   INTRODUCTION

Iterative methods for solution of system of linear   equations such as Jacobi's Iteration Method and Gauss-Seidal Iteration Method for calculation of exact solution of system of equations upto desired accuracy. There is little doubt that indirect methods at their best are quicker and less laborious than direct methods when performed on desk computing machines.

4.3   GENERAL   ITERATION   METHOD   AND   JACOBI'S ITERATION METHOD

ITERATIVE METHODS OF SOLUTION

The preceding methods of solving simultaneous linear equations are *known as direct methods* as they yield exact solutions. On the other hand, an iterative method is that in which we start from an approximation to the true solution and obtain better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy.

Simple iteration methods can be devised for systems in which the coefficients of the leading diagonal are large compared to others. We now explain two such methods:

Jacobi's iteration method

Consider the equations

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3 \qquad \text{------- (1)}$$

If a_1, b_2, c_3 are large compared to other coefficients, then solving these x, y, z respectively, the system can be written in the form

For example eq'n (1) can be written as

$a_1 x + b_1 y + c_1 z = d_1$ { As a_1 is large compared to other coefficients}

$\therefore \qquad\qquad a_1 x = d_1 - b_1 y - c_1 z$

Hence $\qquad x = \dfrac{d_1}{a_1} - \dfrac{b_1}{a_1} y - \dfrac{c_1}{a_1} z$

$\therefore$ We get $k_1 = \dfrac{d_1}{a_1}, l_1 = \dfrac{b_1}{a_1}, \ m_1 = \dfrac{c_1}{a_1}$

Similarly we get $\quad k_2 = \dfrac{d_2}{b_2}, l_2 = \dfrac{a_2}{b_2}, \ m_2 = \dfrac{c_2}{b_2}$ and $k_3 = \dfrac{d_3}{c_3}, l_3 = \dfrac{a_3}{c_3}$, $m_3 = \dfrac{b_3}{c_3}$

$$x = k_1 - l_1 y - m_1 z$$

$$y = k_2 - l_2 x - m_2 z$$

$$z = k_3 - l_3 x - m_3 y \qquad \text{------- (2)}$$

Let us start with the initial approximations x_0, y_0, z_0 (each $= 0$) for the values of x, y, z. Substituting these on the right, we get the first approximations

$$x_1 = k_1, y_1 = k_2, z_1 = k_3$$

Substituting these on the right-hand sides of (2), the second approximations are given by

$$x_2 = k_1 - l_1 y_1 - m_1 z_1$$

$$y_2 = k_2 - l_2 x_1 - m_2 z_1$$

$$z_2 = k_3 - l_3 x_1 - m_3 y_1$$

This process is repeated till the difference between two consecutive approximations is negligible.

Example 1 : Solve by Jacobi's iteration method, the equations

$$10x + y - z = 11.19$$

$$x + 10y + z = 28.08$$

$$-x + y + 10z = 35.61$$

Correct to two decimal places.

Solution: Rewriting the following equations as

$$x_1 = \frac{1}{10}(11.19 - y_0 + z_0),$$

$$y_1 = \frac{1}{10}(28.08 - x_0 - z_0),$$

$$z_1 = \frac{1}{10}(35.61 + x_0 - y_0)$$

We start from an approximation ($x_0, y_0 , z_0 = 0$)

First iteration $\quad x_1 = \frac{11.19}{10} = 1.119, \quad y_1 = \frac{28.08}{10} = 2.808, \quad z_1 = \frac{35.61}{10} = 3.561,$

Second iteration $\quad x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19$

$$y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.24$$

$$z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$$

Third iteration $\quad x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22$

$$y_3 = \frac{1}{10}(28.08 - x_2 - z_2) = 2.35$$

$$z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$$

Fourth iteration $\quad x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23$

$$y_4 = \frac{1}{10}(28.08 - x_3 - z_3) = 2.34$$

$$z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$$

Fifth iteration $\quad x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = 1.23$

$$y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = 2.34$$

$$z_5 = \frac{1}{10}(35.61 + x_4 - y_4) = 3.45$$

Hence the solution

$$x = 1.23, \ y = 2.34 , \ z = 3.45$$

Example 2 : Solve by Jacobi's iteration method, the equations

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

Solution: We write the given equations in the form

$$x = \frac{1}{20}(17 - y + 2z),$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

We start from an approximation ($x_0, y_0, z_0 = 0$)

Substituting these on the right sides of the equations, we get

First iteration $x_1 = \frac{17}{20} = 0.85, \quad y_1 = -\frac{18}{20} = -0.9, \quad z_1 = \frac{25}{20} = 1.25,$

Putting these on the right sides of the equations, we get

Second iteration $x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.1515$$

Third iteration $x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0134$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = -0.9954$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.0032$$

Fourth iteration $x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0009$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.0018$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9993$$

Fifth iteration $x_5 = \frac{1}{20}(17 - y_4 + 2z_4) = 1.0000$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.0002$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.9996$$

Sixth iteration $x_6 = \frac{1}{20}(17 - y_5 + 2z_5) = 1.0000$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = -1.0000$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 1.0000$$

The values of the 5th and 6th iterations being practically the same, we can stop

Hence the solution

$$x = 1, \ \ y = -1 , z = 1$$

<hr>

Check Your Progress

1.   Solve by Jacobi's iteration method, the equations

2x+y+6z=9

8x+3y+2z=13

x+5y+z=7

[Answer x = y = z = 1]

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4.4   GAUSS-SEIDAL ITERATION METHOD

Gauss-Seidel iteration method

This is a modification of the Jacobi's iteration method. As before, we start with the initial approximations x_0, y_0 , z_0 (each = 0) for x, y, z respectively.

Consider the equations

$$a_1 x + b_1 y + \ c_1 z = d_1$$
$$a_2 x + b_2 y + \ c_2 z = d_2$$
$$a_3 x + b_3 y + \ c_3 z = d_3 \qquad \text{------- (1)}$$

If a_1, b_2 , c_3 are large compared to other coefficients, then solving these x, y, z respectively, the system can be written in the form

For example eq'n (1) can be written as

$a_1 x + b_1 y + \ c_1 z = d_1$ *{ As a_1 is large compared to other coefficients}*

∴ $a_1 x = d_1 - b_1 y - \ c_1 z$

Hence $x = \frac{d_1}{a_1} - \frac{b_1}{a_1} y - \frac{c_1}{a_1} z$

∴ We get $k_1 = \frac{d_1}{a_1}, l_1 = \frac{b_1}{a_1}, \ m_1 = \frac{c_1}{a_1}$

Similarly we get $\quad k_2 = \dfrac{d_2}{b_2}, l_2 = \dfrac{a_2}{b_2}$, $\quad m_2 = \dfrac{c_2}{b_2}\quad$ and $\quad k_3 = \dfrac{d_3}{c_3}, l_3 = \dfrac{a_3}{c_3}$, $m_3 = \dfrac{b_3}{c_3}$

$$x = k_1 - l_1 y - m_1 z$$

$$y = k_2 - l_2 x - m_2 z$$

$$z = k_3 - l_3 x - m_3 y \qquad\qquad \text{------- (2)}$$

Let us start with the initial approximations x_0, y_0, z_0 (each $= 0$) for the values of x, y, z . Substituting $y = y_0 , z = z_0$ in equation (2)

$$x_1 = k_1$$

Then putting $\quad x = x_1 , z = z_0$ in the equation (2)

$$y_1 = k_2 - l_2 x_1 - m_2 z_0$$

Then putting $\quad x = x_1 , y = y_1$ in the equation (2)

$$z_1 = k_3 - l_3 x_1 - m_3 y_1$$

and so on i.e. as soon as new approximation for an unknown is found, it is immediately used in the next step.

The process of iteration is continued till convergence to the desired degree of accuracy is obtained.

Note 1- Since the most recent approximation of the unknowns are used while proceeding to the next step, the convergence in the Gauss-Seidel method is faster than in Jacobi's method.

Note 2- Gauss-Seidel method converges if in each equation, the absolute value of the largest coefficient is greater than the sum of the absolute values of the remaining coefficients.

Example 1 : Apply Gauss-Seidel method to solve the equations

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

Solution: We write the given equations in the form

$$x = \frac{1}{20}(17 - y + 2z),$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 2x + 3y) \qquad \text{............ (1)}$$

We start from an approximation ($x_0, y_0, z_0 = 0$)

Substituting $y = y_0$, $z = z_0$ in the right side of the first of equation (1) we get

First iteration $x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$

Then putting $x = x_1$, $z = z_0$ in the equation

$$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$$

Then putting $x = x_1$, $y = y_1$ in the equation

$$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$$

Second iteration $x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$

$$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$$

$$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$$

Third iteratioz $x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1.0000$$

Hence values in the 2nd and 3rd iterations being practically the same, we can stop.

Hence the solution is

$$x = 1, \quad y = 1, z = 1$$

Example 2 : Solve the equations:

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

by Gauss-Seidel iteration method.

Solution: Rewriting the given equations as

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \quad \text{...….......…..(1)}$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \quad \text{……….……..(2)}$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \quad \text{……….……..(3)}$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \quad \text{…...........…..(4)}$$

First iteration

Putting $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, in (1), we get $x_1 = 0.3$

Putting $x_1 = 0.3$, $x_3 = 0$, $x_4 = 0$, in (2), we get $x_2 = 1.56$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_4 = 0$, in (3), we get $x_3 = 2.886$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_{43} = 2.886$, in (4), we get $x_4 = -0.1368$

Second iteration

Putting $x_2 = 1.56$, $x_3 = 2.886$, $x_4 = -0.1368$, in (1), we get

$x_1 = 0.8869$

Putting $x_1 = 0.8869$, $x_3 = 2.886$, $x_4 = -0.1368$ in (2), we get

$x_2 = 1.9523$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_4 = -0.1368$, in (3), we get

$x_3 = 2.9566$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_3 = 2.9566$ in (4), we get

$x_4 = -0.0248$

Third iteration

Putting $x_2 = 1.9523$, $x_3 = 2.9566$, $x_4 = -0.0248$, in (1), we get

$x_1 = 0.9836$

Putting $x_1 = 0.9836$, $x_3 = 2.9566$, $x_4 = -0.0248$ in (2), we get

$x_2 = 1.9899$

Putting $x_1 = 0.9836$, $x_2 = 1.9899$, $x_4 = -0.0248$, in (3), we get

$x_3 = 2.9924$

Putting $x_1 = 0.9836$, $x_2 = 1.9899$, $x_3 = 2.9924$ in (4), we get

$x_4 = -0.0042$

Fourth iteration

Proceeding as above

$x_1 = 0.9968$, $x_2 = 1.9982$, $x_3 = 2.9987$, $x_4 = -0.0008$

Fifth iteration

Proceeding as above

$x_1 = 0.9994$, $x_2 = 1.9997$, $x_3 = 2.9997$, $x_4 = -0.0001$

Sixth iteration

Proceeding as above

$x_1 = 0.9999, \ x_2 = 1.9999, \ x_3 = 2.9999, \ x_4 = -0.0001$

Hence the solution is $x_1 = 1, \ x_2 = 1, \ x_3 = 1, \ x_4 = 0$.

Check Your Progress

1. Solve Solve the following equation by Gauss-Seidel method

    2x+y+6z=9

    8x+3y+2z=13

    x+5y+z=7

    [Answer   x = y = z = 1]

4.9   SUMMARY

Methods for the solution of algebraic linear simultaneous equations fall naturally into two classes. First, there is the direct approach, and the other is iterative methods. By the direct method the solution can always be obtained.(provided it exists), and the only limit on accuracy is the number of figures to which the computer is prepared or compelled to work. Indirect methods, on the other hand, may never converge to a solution, or may converge so slowly as to be impracticable. Again, if an indirect method converges reasonably quickly, the number of figures retained in the computation can be increased gradually as the solution improves, reducing to a minimum the labour involved. Finally, mistakes in the indirect process can merely delay the convergence to an accurate solution, while mistakes in direct methods may completely invalidate the solution obtained, so that careful checks have to be applied throughout.

4.10 EXERCISE

Q.1. Solve by Jacobi's iteration method, the equations

$$5x - y + z = 10$$

$$2x + 4y = 12$$

$$x + y + 5z = -1$$

Start with the solution (2, 3, 0).

[Ans $x = 2.556, \ y = 1.722, z = -1.055$]

Q.2. Solve by Jacobi's iteration method, the equations

$$27x + 6y - z = 85$$

$$x + y + 54z = 110$$

$$6x + 15y + 2z = 72$$

[Ans $x = 2.426$, $y = 3.573$, $z = 1.926$]

Q.3. Solve the following equation by Gauss-Seidel method

$$27x + 6y - z = 85$$

$$x + y + 54z = 110$$

$$6x + 15y + 2z = 72$$

[Ans: $x = 2.426$, $y = 3.573$, $z = 1.926$]

Q.4. Solve the following equation by Gauss-Seidel method

$$28x + 4y - z = 32$$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

[Ans: $x = 0.998$, $y = 1.723$, $z = 2.024$]

Master in Computer Application

MCA-104

Uttar Pradesh Rajarshi Tandon Open University

Numerical Analysis

BLOCK

3

INTERPOLATION

UNIT-5

Interpolation with Equal Intervals

UNIT-6

Interpolation with Unequal Intervals

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BLOCK INTRODUCTION

Overview :

Numerical technique is widely used by scientists and engineers to solve their problems. A major advantage for numerical technique is that a numerical answer can be obtained even when a problem has no analytical solution. However, result from numerical analysis is an approximation, in general, which can be made as accurate as desired. The reliability of the numerical result will depend on an error estimate or bound, therefore the analysis of error and the sources of error in numerical methods is also a critically important part of the study of numerical technique.

UNIT-5 INTERPOLATION WITH EQUAL INTERVAL

Structure

5.1   OBJECTIVE

Finite Differences- The calculus of finite differences deals with the changes that take place in the value of the dependent variable due to finite changes in the independent variable from this we study the relations that exist between the values, which can be assumed by function, whenever the independent variable changes by finite jumps whether equal or unequal.

5.2   INTRODUCTION

The study of finite difference calculus has become very important due to its wide variety of application in routine life. It has been originated by Sir Issac Newton. It has been of great use for Mathematicians as well as Computer Scientists for solution of the Scientific, business and engineering problems. There it helps in reducing complex mathematical

expressions like trigonometric functions in terms of simple arithmetic operations.

Let $y = f(x)$ be an explicit function. Suppose we are given that following values of $y = f(x)$ for asset of values of x:

$$x: \quad x_0 \quad x_1 \quad x_2 \quad x_3.......x_n$$

$$y: \quad y_0 \quad y_1 \quad y_2 \quad y_3.......y_n$$

Thus the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called **interpolation**.

5.3   FINITE DIFFERENCES

The calculus of finite differences deals with the changes in the values of the function (dependent variable) due to changes in the independent variable.

Suppose the function $y = f(x)$ has the values $y_0, y_1, y_2, y_3.......y_n$ for the values of $x = x_0, x_1, x_2, x_3 ... . x_n$ where $x_1, x_2, x_3 ... . x_n$ are equally spaced and given as $x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h ... . x_n = x_0 + nh$.

Generally there are following three types of differences:

(a)   Forward Differences   (b)   Backward Differences

(c)   Central Differences

5.3.1   FORWARD DIFFERENCES

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, ... y_n - y_{n-1}$ are called the first differences of the function y and if denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \Delta y_3......$ then are said to be first forward differences.

$$\square\text{We Have} \quad \Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

$$........$$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

The symbol Δ is called the forward difference operator. The second differences are

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta y_1 - \Delta y_0 = y_2 - 2y_1 + y_0$$

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta y_2 - \Delta y_1 = y_3 - 2y_2 + y_1$$

$$........$$

$$\Delta^2 y_{n-1} = \Delta(\Delta y_{n-1}) = \Delta y_n - \Delta y_{n-1}$$

Δ^2 is called second difference operator.

The third forward differences may be calculated as

$$\Delta^3 y_0 = \Delta^2(\Delta y_0) = \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

In general, $\Delta^{m+1} y_n = \Delta^m y_{n+1} - \Delta^m y_n$.

Table : Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

The above table is called a diagonal difference table. The first term in the table is y_0. It is called the leading term.

Properties of Δ

1. If $f(x)$ is a constant then $\Delta f(x) = 0$.

2. Δ is distributive, i.e. $\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$.

3. If c is a constant then $\Delta[c\, f(x)] = c\Delta f(x)$.

4. If m and n are positive integers then $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$.

5. $\Delta[f_1(x) + f_2(x) + f_3(x) \dots + f_n(x)] = \Delta f_1(x) + \Delta f_2(x) + \Delta f_3(x) \dots + \Delta f_n(x)$

6. $\Delta[f(x)g(x)] = f(x)\Delta g(x) + g(x)\Delta f(x)$

7. $\Delta\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x).g(x+h)}$

Example 1. Find (a) Δe^{ax} (b) $\Delta^2 e^x$ (c) $\Delta \sin x$ (d) $\Delta \log x$
(e) $\Delta \tan^{-1} x$

Solution. (a) $\Delta e^{ax} = e^{a(x+h)} - e^{ax}$

$$= e^{ax}(e^{ah} - 1) \qquad \text{Ans.}$$

(b) $\Delta^2 e^x = \Delta[\Delta e^x] = \Delta[e^{x+h} - e^x$

$$= \Delta[e^x(e^h - 1)] = (e^h - 1)) \; \Delta e^x$$

$$= (e^h - 1)) \; [e^{x+h} - e^x]$$

$$= (e^h - 1)^2 e^x \qquad \text{Ans.}$$

(c) $\Delta sinx = \sin(x + h) - sinx$

$$= 2cos\left(\frac{x+h+x}{2}\right) sin\left(\frac{x+h-x}{2}\right)$$

$$= 2cos\left(x + \frac{h}{2}\right) sin\frac{h}{2} \qquad \text{Ans.}$$

(d) $\Delta logx = \log(x + h) - logx$

$$= log\left(\frac{x+h}{x}\right)$$

$$= log\left(1 + \frac{h}{x}\right) \qquad \text{Ans.}$$

(e) $\Delta tan^{-1}x = tan^{-1}(x + h) - tan^{-1}x$

$$= tan^{-1}\left[\frac{(x+h)-x}{1+(x+h)x}\right]$$

$$= tan^{-1}\left[\frac{h}{1+hx+x^2}\right] \qquad \text{Ans.}$$

Example 2. Construct a difference table for $y = f(x) = x^3 + 2x + 1$ for $x = 1,2,3,4,5$.

Solution. Using the functional form we get,

$$f(1) = 4, \; f(2) = 13, \; f(3) = 34, \; f(4) = 73, \; f(5) = 136$$

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	4				
		9			
2	13		12		
		21		6	
3	34		18		0
		39		6	
4	73		24		
		63			
5	136				

*Theorem. "The nth difference of a polynomial of nth degree are constant when the value of independent variable are at equal intervals."

*Note. The converse of the above theorem is true, i.e., "if the nth difference of a tabulated function and the values of the independent variable are equally spaced then the function is a polynomial of degree n."

Example 3. By constructing a difference table and taking the second order difference as constant find the fourth term of the series 8, 12, 19, ….

Solution. Let k be the fourth term of the series. The difference table is

x	$y = f(x)$	Δy	$\Delta^2 y$
1	8		
		4	
2	12		3
		7	
3	19		k-26
		k-19	
4	k		

The second differences are constant.

$$k - 26 = 3$$

$\Longrightarrow$
$$k = 29$$

The fifth term of the series is 29.

Example 4. Evaluate

(a) $\Delta\left[\dfrac{5x+12}{x^2+5x+6}\right]$ (b) $\Delta^n\left[\dfrac{1}{x}\right]$

Solution. (a) $\Delta\left[\dfrac{5x+12}{x^2+5x+6}\right] = \Delta\left[\dfrac{2(x+3)+3(x+2)}{(x+2)(x+3)}\right]$

$$= \Delta\left[\dfrac{2}{x+2} + \dfrac{3}{x+3}\right]$$

$$= \Delta\left[\dfrac{2}{x+2}\right] + \Delta\left[\dfrac{3}{x+3}\right]$$

$$= \left[\frac{2}{x+1+2} - \frac{2}{x+2}\right] + \left[\frac{3}{x+1+3} - \frac{3}{x+3}\right]$$

$$= -\frac{2}{(x+2)(x+3)} - \frac{3}{(x+3)(x+4)}$$

(b) $\qquad \Delta^n \left[\frac{1}{x}\right]$

Now $\qquad \Delta \left[\frac{1}{x}\right] = \frac{1}{x+1} - \frac{1}{x} = \frac{-1}{x(x+1)}$

Similarly $\qquad \Delta^2 \left[\frac{1}{x}\right] = \frac{(-1)^2}{x(x+1(x+2)}$ and so on proceeding as above, we get

$$\cdots\cdots\cdots$$

$$\Delta^n \left[\frac{1}{x}\right] = \frac{(-1)^n}{x(x+1(x+2)\ldots\ldots\ldots(x+n)}$$

5.3.2 BACKWARD DIFFERENCES

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots\ldots\ldots y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \nabla y_3, \nabla y_4 \ldots\ldots \nabla y_n$ respectively, are called the first backward differences wher ∇ is the backward difference operator. Similarly, we may define higher order backward differences as

$$\nabla y_1 = y_1 - y_0, \qquad \nabla y_2 = y_2 - y_1, \quad \ldots\ldots..\nabla y_n = y_n - y_{n-1},$$

$$\nabla^2 y_n = \nabla(\nabla y_n) = \nabla y_n - \nabla y_{n-1}$$

∇^2 is called second backward difference operator.

The third backward differences may be calculated as

$$\nabla^3 y_n = \nabla^2 y_n - \nabla^2 y_{n-1}$$

Table : Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
		$\nabla y_1 = y_1 - y_0$			

x_1	y_1				
			$\nabla^2 y_2$ $= \nabla y_2 - \nabla y_1$		
		∇y_2 $= y_2 - y_1$		$\nabla^3 y_3$ $= \nabla^2 y_3 - \nabla^2 y_2$	
x_2	y_2		$\nabla^2 y_3$ $= \nabla y_3 - \nabla y_2$		$\nabla^4 y_4$ $= \nabla^3 y_4 - \nabla^3 y_3$
		∇y_3 $= y_3 - y_2$		$\nabla^3 y_4$ $= \nabla^2 y_4 - \nabla^2 y_3$	
x_3	y_3		$\nabla^2 y_4$ $= \nabla y_4 - \nabla y_3$		
		∇y_4 $= y_4 - y_3$			
x_4	y_4				

5.3.3 CENTRAL DIFFERENCES

The central difference operator δ is defined as

$$y_1 - y_0 = \delta y_{\frac{1}{2}}, y_2 - y_1 = \delta y_{\frac{3}{2}}, \ldots, y_n - y_{n-1} = \delta y_{(2n-1)/2}$$

Similarly, we may define higher order backward differences as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1,$$

$$\delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2, \ldots\ldots\text{.etc}$$

In general, $\delta^{m+1} y_{(2n-1)/2} = \delta^m y_n = \delta^m y_{n-1}$

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f(x - \frac{h}{2})$$

Or $\delta y_x = y_{x+h/2} - y_{x-h/2}$

Table: Central Difference Table

x	y	δ	δ^2	δ^3	δ^4
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

From this table we see that the central differences on the same horizontal line have the same suffix. The difference of odd order are known only for half values of the suffix and those of even order for only integral values of the suffix:

THE SHIFT OPERATOR "E"

The $y = f(x)$ be function of x and $x + h, x + 2h, x + 3h, \ldots . etc$ be the corresponding values of x, then the operator E is defined as:

$$Ef(x) = f(x + h),$$

Similarly $E^n f(x) = Ef(x + (n-1)h) = f(x + nh)$

$$E^{-n} f(x) = Ef(x - (n-1)h) = f(x - nh)$$

E is called shift operator. It is also called displacement operator.

$E^2 f(x)$ means the operator $E[Ef(x)]$

i.e., $E^2 f(x) = Ef(x + h) = f(x + 2h)$

Similarly, $E^n f(x) = Ef(x + nh)$

and $E^{-n} f(x) = Ef(x - nh)$

THE OPERATOR "D"

Dy or $Df(x)$ denotes the differential coefficient of y with respect to x where $D \equiv \frac{d}{dx}$. We have $Dy = \frac{dy}{dx}$. The nth derivative of y with respect to x is denoted by $D^n y = \frac{d^n y}{dx^n}$.

MEAN OPERATOR "μ"

In addition to the operator Δ, ∇, E and δ, we define the operator (averaging operator) μ as follows:

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right) \right]$$

Or

$$\mu y = \frac{1}{2}\left[y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}} \right].$$

5.4   RELATION BETWEEN OPERATORS

$(i) E = 1 + \Delta$ $\qquad (ii) E.\Delta = \Delta.E$ $\qquad (iii) E = e^{hD}$

$(iv) \Delta. \nabla = \Delta - \nabla$ $\qquad (v) \Delta - \nabla = \delta^2$ $\qquad (vi) E^{1/2} = \mu + \frac{1}{2}\delta$

$(vii)\ \Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$ $\qquad\qquad (viii)\ \mu.\delta = \frac{1}{2}\Delta.E^{-1} + \frac{1}{2}\Delta$

Proof: $(i) E = 1 + \Delta$

From the definition of Δ, we know that

$$\Delta f(x) = f(x + h) - f(x)$$

Where h is the interval of differencing. Using the operator E, we can write

$$\Delta f(x) = f(x + h) - f(x)$$
$$= Ef(x) - f(x)$$
$$= (E - 1)f(x)$$

Or $\qquad\qquad \Delta = E - 1$

Or $\qquad\qquad E = 1 + \Delta$ $\qquad$ Hence Proved

$(ii) \qquad E.\Delta = \Delta.E$

$$E.\Delta f(x) = E[f(x + h) - f(x)]$$
$$= f(x + 2h) - f(x + h)$$
$$= \Delta.f(x + h)$$
$$= \Delta.Ef(x)$$
$$\qquad\qquad E.\Delta = \Delta.E$$

(iii) $E = e^{hD}$

We have $Ef(x) = f(x + h)$

Expanding by Taylor's series, we have

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots \ldots + \frac{h^n}{n!}f^n(x) + \cdots \ldots$$

$$= f(x) + h.Df(x) + \frac{h^2}{2!}D^2f(x) + \cdots \ldots + \frac{h^n}{n!}D^nf(x) + \cdots$$

$$= \left(1 + hD + \frac{h^2 D^2}{2!} + \cdots \ldots + \frac{h^n D^n}{n!} + \cdots \ldots\right)f(x)$$

$$Ef(x) = e^{hD}.f(x)$$

Hence, $E = e^{hD}$ …………(1)   Hence Proved

Corollary: Taking log of equation (1) we get

$$hD = logE = \log(1 + \Delta)$$

$$= \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots \ldots$$

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right]$$

(*iv*) $\Delta.\nabla = \Delta - \nabla$

We have $\Delta.\nabla f(x) = (E + 1)(1 - E^{-1})f(x)$

$$= (E - 1)[f(x) - E^{-1}f(x)]$$

$$= (E - 1)[f(x) - f(x - h)]$$

$$= [f(x + h) - f(x) - f(x) + f(x - h)]$$

$$= [f(x + h) - f(x)] - [f(x) - f(x - h)]$$

$$= \Delta f(x) - \nabla f(x)$$

$$= (\Delta - \nabla)f(x)$$

Or $\Delta.\nabla = \Delta - \nabla$

(*v*) $\Delta - \nabla = \delta^2$

L.H.S $\Delta - \nabla = \delta.E^{\frac{1}{2}} - \delta.E^{-1/2}$

$$= \delta(E^{\frac{1}{2}} - E^{-\frac{1}{2}})$$

$$= \delta.\delta = \delta^2$$

(*vi*) $E^{1/2} = \mu + \frac{1}{2}\delta$

R.H.S $\mu + \frac{1}{2}\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)$

$$= \frac{1}{2}\left(2E^{\frac{1}{2}}\right) = E^{1/2}$$

(vii) $\Delta = \dfrac{1}{2}\delta^2 + \delta\sqrt{1 + \dfrac{\delta^2}{4}}$

R.H.S $\qquad = \dfrac{1}{2}\delta^2 + \delta\sqrt{1 + \dfrac{\delta^2}{4}}$

$$= \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2 + \delta\sqrt{1 + \frac{\left(E^{1/2} - E^{-1/2}\right)^2}{4}}$$

$$= \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 + \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{\frac{\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)^2}{4}}$$

$$= \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 + \frac{\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)}{2}$$

$$= \frac{1}{2}(E + E^{-1} - 2) + \frac{(E - E^{-1})}{2}$$

$$= \frac{1}{2}(2E - 2) = E - 1$$

$$= \Delta \quad = \text{L.H.S.}$$

(vii) $\mu.\delta = \dfrac{1}{2}\Delta.E^{-1} + \dfrac{1}{2}\Delta$

R.H.S $\qquad = \mu.\delta = \dfrac{1}{2}\left(E^{1/2} + E^{-1/2}\right).\left(E^{1/2} - E^{-1/2}\right)$

$$= \frac{1}{2}(E - E^{-1})$$

$$= \frac{1}{2}(E - 1 + 1 - E^{-1})$$

$$= \frac{1}{2}\Delta + \frac{1}{2}(1 - E^{-1})$$

$$= \frac{1}{2}\Delta + \frac{1}{2}\frac{E-1}{E}$$

$$= \frac{1}{2}\Delta + \frac{1}{2}\Delta E^{-1}$$

Example 1   Prove that $\Delta \log f(x) = \log\left[1 + \dfrac{\Delta f(x)}{f(x)}\right]$.

Solution.   Let h be the interval of differencing

$$f(x + h) = E\, f(x) = (\Delta + 1)f(x) = \Delta f(x) + f(x)$$

$$\frac{f(x+h)}{f(x)} = \frac{\Delta f(x)}{f(x)} + 1$$

$$\log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

$\Rightarrow \qquad \log\left[1 + \dfrac{\Delta f(x)}{f(x)}\right] = \log f(x + h) - \log f(x)$

Example 2 Evaluate $\left(\frac{\Delta^2}{E}\right) x^3$

Solution. Let h be the interval of differencing

$$\left(\frac{\Delta^2}{E}\right) x^3 = (\Delta^2 E^{-1}) x^3 = (E-1)^2 E^{-1} x^3$$

$$= (E^2 - 2E + 1) E^{-1} x^3$$

$$= (E - 2 + E^{-1}) x^3 = (Ex^3 - 2x^3 +$$

$E^{-1} x^3)$

$$= (x + h)^3 - 2x^3 + (x - h)^3$$

$$= 6xh$$

Example 3 Given $u_0 = 1, u_1 = 11, u_2 = 21, u_3 = 28 \ and \ u_4 = 29, find \ \Delta^4 u_0.$

Solution. $\Delta^4 u_0 = (E-1)^4 u_0$

$$= (E^4 - {}^4C_1 E^3 + {}^4C_2 E^2 - {}^4C_3 E + 1) u_0$$

$$= E^4 u_0 - {}^4C_1 E^3 u_0 + {}^4C_2 E^2 u_0 - {}^4C_3 E u_0 + u_0$$

$$= u_4 - 4u_3 + 6u_2 - 4u_1 + u_0$$

$$= 29 - 112 + 126 - 44 + 1$$

$$= 0$$

Example 4 Find the first term of the series whose second and subsequent terms are $8, 3, 0, -1, 0, \dots \dots \dots$

Solution. Given $f(2) = 8, \ f(3) = 3, \ f(4) = 0, \ f(5) = -1,$ $f(6) = 0$ we are to find $f(1).$

We construct the difference table with the given values.

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	8				
		-5			
3	3		2		
		-3		0	
4	0		2		0
		-1		0	

5	-1		2		
		1			
6	0				

We have $\Delta^3 f(x) = \Delta^4 f(x) = \cdots = 0$

Using the displacement operator

$$E^{-1}f(x) = f(x - h)$$
$$f(1) = E^{-1}f(2)$$
$$= (1 + \Delta)^{-1}f(2)$$
$$= (1 - \Delta + \Delta^2 - \Delta^3 + \cdots \ldots \ldots)f(2)$$
$$= f(2) - \Delta f(2) + \Delta^2 f(2) - \Delta^3 f(2) + \cdots.$$
$$= 8 - (-5) + 2$$
$$f(1) = 15$$

Check Your Progress

Prove with the usual notations, that

(i) $hD = \log(1 + \Delta) = -\log(1 - \nabla) = \sinh^{-1}(\mu\delta)$

(ii) $(E^{1/2} + E^{-1/2})(1 + \Delta)^{1/2} = 2 + \Delta$

This technique is based on the principle that for a set of to (n+1) number of observations, a polynomial of degree n may be fitted and hence it's nth forward difference will be a constant and (n+1)th forward difference will be zero.

Example 1.   Following are the population of a district

Year (x)	1881	1891	1901	1911	1921	1931
Population (y)in lakh	363	391	421	?	467	501

Find the population of the year 1911.

Solution.   We have

$$y_0 = 363, y_1 = 391, y_2 = 421, y_3 =?, y_4 = 467, y_5 = 501$$

Five value are given. Hence we may assume that y is a polynomial in x of degree 4.

$$\Delta^5 y_0 = 0 \qquad \Rightarrow (E-1)^5 y_0 = 0$$

$$\Rightarrow \qquad (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_0 = 0$$

$$\Rightarrow \qquad y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$

$$\Rightarrow \quad 501 - 5 \times 461 + 10y_3 - 10 \times 421 + 5 \times 391 - 363 = 0$$

$$\Rightarrow \qquad\qquad\qquad 10y_3 - 4452 = 0$$

$$\Rightarrow \qquad\qquad\qquad\qquad\qquad y_3 = 445.2$$

The population of the district in the year 1911 is 445.2 lakh.

Example 2. Interpolate the missing entries

x	0	1	2	3	4	5
$Y = f(x)$	0	--	8	15	--	35

Solution. We have

$$y_0 = 0, y_1 =?, y_2 = 8, y_3 = 15, y_4?, y_5 = 35$$

Four value are given. Hence we may assume that y is a polynomial in x of degree 3.

$$\Delta^4 y_0 = 0 \qquad \Rightarrow (E-1)^4 y_0 = 0$$

$$\Rightarrow \qquad (E^4 - 4E^3 + 6E^2 - 4E + 1)y_0 = 0$$

$$\Rightarrow \qquad y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$\Rightarrow \qquad y_4 - 4 \times 15 + 6 \times 8 - 4y_1 - 0 = 0$$

$$\Rightarrow \qquad\qquad y_4 - 4y_1 = 12 \qquad\qquad \dots\dots..(1)$$

Also $\qquad\qquad \Delta^5 y_0 = 0 \qquad \Rightarrow (E-1)^5 y_0 = 0$

$$\Rightarrow \qquad (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_0 = 0$$

$$\Rightarrow \qquad y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$

$$\Rightarrow \qquad 35 - 5y_4 + 10 \times 15 - 10 \times 8 + 5y_1 - 0 = 0$$

$$\Rightarrow \qquad\qquad y_4 - y_1 = 21 \qquad\qquad \dots\dots..(2)$$

Solving (1) and (2), we get, $\qquad\qquad y_1 = 3, \ y_4 = 24$

Check Your Progress

1. Find the missing entry in the following table

x	0	1	2	3	4
$Y = f(x)$	1	3	9	--	81

5.5   MISSING TERM TECHNIQUE

The technique is based on the principle that for a set of to (n+1) number of observations, a polynomial of degree n may be fitted and hence it's nth forward difference will be a contant and (n+1)th forward difference will be zero.

Example 1. Following are the population of a district.

Year (x)	1881	1891	1901	1911	1921	1931
Population (y) (in lakh)	363	391	421	?	467	501

Find the population of the year 1911.

Solution. We have

5.6   INTERPOLATION   WITH   EQUAL INTERVALS

The study of interpolation is based on the assumption that there are no sudden jumps in the values of the dependent variable for the period under consideration. It is also assumed that the rate of change of figures from one period to another is uniform.

Thus Let $y = f(x)$ be a function i.e. y for corresponding value of x:

$$x \quad : \quad x_0 x_1 x_2 \quad \dots\dots\dots\dots\dots x_n$$

$$y \quad : \quad y_0 y_1 y_2 \quad \dots\dots\dots\dots\dots y_n$$

When $x = x_0$ then $y_0 = f(x_0)$.

Then the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called interpolation. Thus, interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called extrapolation.

The study of interpolation is based on the calculus of finite differences. First we define two important interpolation formulae by means of forward and backward differences of a function. These formulae are employed in engineering and scientific investigations.

5.7   NEWTON   GREGORY   FORMULA   FOR FORWARD DIFFERENCES

Given the set of (n+1) values $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots\dots\dots\dots. (x_n, y_n)$ of x and y, it is required to

find $y_n(x)$, a polynomial of nth degree, so that y and $y_n(x)$ concide at tabulated points.

Let the values of x be the equidistant so that,

$$x_i = x_0 + ih, \qquad i = 0,1,2,3, \ldots \ldots \ldots \ldots . n.$$

Since $y_n(x)$ is a polynomial of degrre n, let us assume that

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$
$$+ \cdots .. + a_n(x - x_0)(x - x_1) \ldots . (x$$
$$- x_{n-1}) \qquad \ldots \ldots \ldots \ldots \ldots (1)$$

Using condition that that y and $y_n(x)$ concide at tabulated points, we obtain on putting

$$x = x_0; y_n(x_0) = y_0, \quad x = x_1; y_n(x_1) = y_1, \quad x = x_2; y_n(x_2) = y_2$$

And so on in equation (1), we get

$$a_0 = y_0, \qquad a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}, \qquad a_2 = \frac{\Delta^2 y_0}{h^2 2!}, \qquad a_3$$
$$= \frac{\Delta^3 y_0}{h^3 3!}, \qquad a_n = \frac{\Delta^n y_0}{h^n n!}$$

Hence equation (1) gives

$$y_n(x)$$
$$= y_0 + \frac{(x - x_0)}{h} \Delta y_0 + \frac{(x - x_0)(x - x_1)}{h^2 2!} \Delta^2 y_0$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_2)}{h^3 3!} \Delta^3 y_0$$
$$+ \cdots .. + \frac{(x - x_0)(x - x_1) \ldots . (x - x_{n-1})}{h^n n!} \Delta^n y_0 \qquad \ldots \ldots \ldots \ldots . (2)$$

On putting $x = x_0 + uh$, we have, from (2)

$$y_n(x)$$
$$= y_0 + u\Delta y_0 + \frac{u(u - 1)}{2!} \Delta^2 y_0 + \frac{u(u - 1)(u - 2)}{3!} \Delta^3 y_0$$
$$+ \cdots .. + \frac{u(u - 1)(u - 2) \ldots (u - n + 1)}{n!} \Delta^n y_0 \qquad \ldots \ldots \ldots \ldots . (3)$$

Where $\qquad u = \frac{x - x_0}{h}, \qquad u \leq 1$

Equation (3) is called Newton Gregory forward interpolation formula.

NOTE[*]: This formula is particularly useful for interpolationg the value of f(x) near the beginning of the set of given values.

Example1. Evaluate $y = e^{2x}$ for $x = 0.05$ using the following table

x	0.00	0.10	0.20	0.30	0.40

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$Y = e^{2x}$ 1.000   1.2214   1.4918   1.8221   2.255

Solution.   The forward difference table is

x	$y = e^{2x}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.00	**1.000**				
		0.2214			
0.10	1.2214		**0.0490**		
		0.2704		**0.0109**	
0.20	1.4918		0.0599		**0.0023**
		0.3303		0.0132	
0.30	1.8221		0.0731		
		0.4034			
0.40	2.255				

We have $x_0 = 0.00, \quad x = 0.05, \quad h = 0.1$

$\therefore$ $$u = \frac{x - x_0}{h} = \frac{0.05 - 0.00}{0.1} = \frac{0.05}{0.1} = 0.5$$

Using Newton's forward Interpolation formula, we get

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

We have,

$$y = 1.000 + 0.5 \times 0.2214 + \frac{0.5(0.5-1)}{2} \times 0.0490$$
$$+ \frac{0.5(0.5-1)(0.5-2)}{6} \times 0.0109$$
$$+ \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24} \times 0.0023$$
$$= 1.000 + 0.1107 - 0.006125 + 0.000681 - 0.000090$$

$$= 1.105166$$

$$y \approx \mathbf{1.052}$$

Example 2. Find the value of $sin30^0 15'30''$ from the following table

Angle x^0	c	31^0	32^0	33^0	34^0
$f(x)$ $= sinx^0$	0.5000	0.5150	0.5299	0.5446	0.5592

Solution. The forward difference table is

x	$f(x)$ $= sinx^0$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
30^0	**0.5000**				
		0.0150			
31^0	0.5150		$\mathbf{-0.0490}$		
		0.0149		$\mathbf{-0.0001}$	
32^0	0.5299		-0.0002		**0.0002**
		0.0147		0.0001	
33^0	0.5446		-0.0001		
		0.0146			
34^0	0.5592				

Given $\qquad x = 30^0 15'30''$, $\qquad x_0 = 30^0$ $\qquad\qquad h = .1$

$\therefore \qquad\qquad u = \dfrac{x - x_0}{h} = \dfrac{30^0 15'30'' - 30^0}{1} = 15'30''$

Using Newton's forward Interpolation formula, we get

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0)$$
$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0)$$

$$+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(x_0)$$

We have,

$f(30^0 15' 30'')$

$$= 0.5000 + 0.2583 \times 0.0150$$

$$+\frac{00.2583(0.2583-1)}{2}(-.0001)$$

$$+\frac{0.2583(0.2583-1)(0.2583-2)}{6}(-.0001)$$

$$+\frac{0.2583(0.2583-1)(0.2583-2)(0.2583-3)}{24}(.0002)$$

$$= .5000 + .0039 + .0000$$

$$y = .5039$$

Example3. Find the number of students whose weight is between60 and 70 from the data given below:

Weight in KG: (x)	$0-40$	$40-60$	$60-80$	$80-100$	$100-120$
No of students: (y)	250	120	100	70	50

Solution.   The forward difference table is

x (Weight)	y (No.of students)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	**250**				
		120			
Below 60	370		-20		
		100		-10	
Below 80	470		-30		20
		70		10	
Below 100	540		-20		

		50			
Below 120	590				

Let us calculate the number of students whose weight is les than 70 kg.

We have, $\qquad x_0 = 40, \qquad x = 70, \qquad h = 20$

$\therefore \qquad\qquad u = \dfrac{x - x_0}{h} = \dfrac{70 - 40}{20} = 1.5$

Using Newton's forward Interpolation formula, we get

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

We have,

$$y = 250 + 1.5 \times 1204 + \frac{1.5(1.5-1)}{2}(-20)$$
$$+ \frac{1.5(1.5-1)(1.5-2)}{6}(-10)$$
$$+ \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{24}(20)$$
$$= 250 + 180 - 7.5 + 0.625 + 0.46875$$
$$= 423.59$$
$$y \approx 424$$

Hence the number of students whose weight is between 60 and 70 is

$$= y(70) - y(60) = 424 - 370 = \mathbf{54}$$

Example4. Find the cubic polynomial which takes the following values, Hence find $f(4)$

x	0	1	2	3
$f(x)$	1	2	1	10

Solution. The forward difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**1**			
		1		

1	2		−2	
		−1		**12**
2	1		10	
		9		
3	10			

We take $\qquad x_0 = 0, \qquad u = \dfrac{x - x_0}{h} = \dfrac{x - 0}{1} = x$

∴ $\qquad$ Using Newton's forward Interpolation formula, we get

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0)$$
$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \cdots .$$

$\Rightarrow \quad f(x) = f(x_0) + x\Delta f(x_0) + \frac{x(x-1)}{2!}\Delta^2 f(x_0)$
$$+ \frac{x(x-1)(x-2)}{3!}\Delta^3 f(x_0)$$

$\Rightarrow \quad f(x) = 1 + x \times 1 + \dfrac{x(x-1)}{2}(-2) + \dfrac{x(x-1)(x-2)}{6}(12)$

$\Rightarrow \quad f(x) = 1 + x - x^2 + x + 2x(x^2 - 3x + 2)$

$\Rightarrow \quad f(x) = 1 + 2x - x^2 + 2x^3 - 6x^2 + 4x$

$\Rightarrow \quad f(x) = 2x^3 - 7x^2 + 6x + 1$

Which is the required polynomial.

Hence $f(4) = 2(4)^3 - 7(4)^2 + 6(4) + 1$

$\qquad f(4) = 41$

Check Your Progress

1. The following table gives the population of a town during the last six censuses.

Estimate the increase in the population during the period from 1946 to 1948:

Year	1911	1921	1931	1941	1951	1961
Population (in thousand)	12	15	20	27	39	52

5.8 NEWTON GREGORY FORMULA FOR BACKWARD DIFFERENCES

In this case we assume the polynomial as

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1})$$

$$+ \cdots .. + a_n(x - x_n)(x - x_{n-1}) \ldots (x - x_1) \qquad \ldots \ldots \ldots \ldots (1)$$

Using condition that that y and $y_n(x)$ concide at tabulated points, we obtain on putting

$$x = x_n; y_n(x_n) = y_n, \qquad x = x_{n-1}; y_n(x_{n-1}) = y_{n-1}, \quad x = x_{n-2}; y_n(x_{n-2}) = y_{n-2}$$

And so on in equation (1), we get

$$a_0 = y_n, \qquad a_1 = \frac{y_{n-1} - y_n}{x_{n-1} - x_n} = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

$$= \frac{\nabla y_n}{h}, \qquad a_2 = \frac{\nabla^2 y_n}{h^2 2!},$$

$$a_3 = \frac{\nabla^3 y_n}{h^3 3!}, \qquad a_n = \frac{\Delta^n y_n}{h^n n!}$$

Hence equation (1) gives

$$y_n(x) = y_n + \frac{(x - x_n)}{h} \nabla y_n + \frac{(x - x_n)(x - x_{n-1})}{h^2 2!} \nabla^2 y_n$$

$$+ \cdots .. + \frac{(x - x_n)(x - x_{n-1}) \ldots (x - x_1)}{h^n n!} \nabla^n y_n \qquad \ldots \ldots \ldots (2)$$

On putting $(x = x_n + uh)$ or $(u = \frac{x - x_n}{h})$, we have, from (2)

$$y_n(x) = y_n + u \nabla y_n + \frac{u(u + 1)}{2!} \nabla^2 y_n + \frac{u(u + 1)(u + 2)}{3!} \nabla^3 y_n$$

$$+ \cdots .. + \frac{u(u + 1)(u + 2) \ldots (u + n - 1)}{n!} \nabla^n y_n \qquad \ldots \ldots \ldots (3)$$

Equation (3) is called Newton Gregory forward interpolation formula.

NOTE[*]: This formula is particularly useful for interpolating the value of f(x) near the end of the set of given values.

Example 1. The following data gives the melting point of an alloy of lead and zinc, where t is the temperature in degrees, c and p is the percentage of lead in the alloy:

p	40	50	60	70	80	90
t	184	204	226	250	276	304

Find the melting point of the alloy containing 84% lead.

Solution. Since 84 lies near the end of the table, therefore we will use Newton Gregory backward interpolation formula.The backward difference table is

p	t	∇t	$\nabla^2 t$	$\nabla^3 t$	$\nabla^4 t$	$\nabla^5 t$
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		**0**
70	250		2		**0**	
		26		**0**		
80	276		**2**			
		28				
90	**304**					

We have, $\qquad x_n = 90, \qquad x = 84, \qquad h = 10$

$\therefore \qquad\qquad u = \dfrac{x - x_n}{h} = \dfrac{84 - 90}{10} = -0.6$

Using Newton's backward Interpolation formula, we have,,

$t(p)$

$$= t_n + u\nabla t_n + \frac{u(u+1)}{2!}\nabla^2 t_n$$
$$+ \frac{u(u+1)(u+2)}{3!}\nabla^3 t_n$$
$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 t_n + \cdots \ldots \ldots$$

$$t(84) = 304 - 0.6 \times 28 + \frac{(-0.6)(-0.6+1)}{2} \quad (2)$$

$$= 304 - 16.8 - 0.24$$

$$= 286.96$$

$$\text{melting point} \approx 287$$

Example 2. The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

$x = height$	100	150	200	250	300	350	400
$y = distance$	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the value of y when x = 410 ft.

Solution. Since 410 lies near the end of the table, therefore we will use Newton Gregory backward interpolation formula.The backward difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
100	10.63						
		2.40					
150	13.03		-0.39				
		2.01		0.15			
200	15.04		0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		**0.02**
		1.61		0.03		**0.04**	
300	18.42		-0.13		**-0.01**		
		1.48		**0.02**			
350	19.90		**-0.11**				
		1.37					
400	**21.27**						

We have, $\qquad x_n = 400, \qquad x = 410, \qquad h = 50$

$\therefore \qquad\qquad\qquad u = \dfrac{x - x_n}{h} = \dfrac{410 - 400}{50} = 0.2$

Using Newton's backward Interpolation formula, we have,,

$$y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n + \cdots \ldots \ldots$$

$y(410)$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11)$$

$$+ \frac{(0.2)(1.2)(2.2)}{3!}(0.02)$$

$$+ \frac{(0.2)(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$+ \frac{(0.2)(1.2)(2.2)(3.2)(4.2)}{5!}(0.04)$$

$$+ \frac{(0.2)(1.2)(2.2)(3.2)(4.2)(5.2)}{6!}(0.02)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007 + .0023 + .0010$$

$$= 21.5352 \text{ nauutical miles.}$$

Example 3. In the table below, the values are consecutive terms of a series of which 23.2 is the 6th term. Find the first and tenth terms of the series:

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Find the value of y when x = 410 ft.

Solution. The difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
3	**4.8**				
		3.6			
4	8.4		**2.5**		
		6.1		**0.5**	
5	14.5		3.0		**0**
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.53	

7	36.2		4.0		**0**
		16.6		**0.05**	
8	52.8		**4.5**		
		21.1			
9	**73.9**				

To find the first term ,use Newton Gregory forward interpolation formula with

$$x_0 = 3, \qquad x = 1, \qquad h = 1 \qquad and \qquad u = \frac{x - x_0}{h} = -2$$

$$y(1)$$

$$= 4.8 + (-2)(3.6) + \frac{(-2)(-3)}{2!}(2.5)$$
$$+ \frac{(-2)(-3)(-4)}{3!}(0.5)$$

$$= 3.1$$

To find the tenth term ,use Newton Gregory backward interpolation formula with

$$x_n = 9, \qquad x = 10, \qquad h = 1 \qquad and \qquad u = \frac{x - x_n}{h} = 1$$

$$y(10)$$

$$= 73.9 + (1)(21.1) + \frac{(1)(2)}{2!}(4.5)$$
$$+ \frac{(1)(2)(3)}{3!}(0.5)$$

$$= 100$$

Check Your Progress

1.   Calculate the value of f(7.5) for the table

x	1	2	3	4	5	6	7	8
f(x)	1	8	27	64	125	216	343	512

Answer: 421.87

5.9   SUMMARY

        Interpolation is the method of estimating unknown values with the help of given set of observations. Also interpolation means insertion or

filling up intermediate terms of the series. It is the technique of obtaining the value of a function for any intermediate values of the independent variables i.e., of argument when the values of the function corresponding to number of values of argument are given. The study of finite difference calculus has become very important due to its wide variety of application in routine life.There it helps in reducing complex mathematical expressions like trigonometric functions in terms of simple arithmetic operations.

5.10 EXERCISE

1. Show that $\nabla = 1 - e^{-hD}$

2. Prove the following operator relations

    (a) $\nabla = 1 - (1 + \Delta)^{-1}$

    (b) $\delta = \Delta(1 + \Delta)^{-\frac{1}{2}}$

    (c) $\delta = 2sin\left(\frac{hD}{2}\right)$

    (d) $\Delta^3 y_2 = \nabla^3 y_5$

3. Show that $y_4 = y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1$

4. Obtain the function whose first difference is

    (a) e^x (b) $2x^3 + 5x^2 - 6x + 13$

5. If $f(x) = sinx$, then show that $\Delta^2 f(x) = -\left(2sin\frac{1}{2}\right)^2 Ef(x)$

6. Estimate the missing term in the following table:

x	0	1	2	3	4
y=f(x)	1	3	9	?	81

    [Ans 31]

7. Estimate the missing term in the following table:

x	2.0	2.1	2.2	2.3	2.4	2.5	2.6
y=f(x)	0.135	--	0.111	0.100	--	0.082	0.024

    [Ans 123.090]

8. Given $u_0 = 580, u_1 = 556, u_2 = 520$ and$u_4 = 385$ findu_3.

    [Ans 465.25]

9. Estimate the production of cotton in the year 1935 from the data given below:

Year	1931	1932	1933	1934	1935	1936	1937
Production (in million)	17.1	13.0	14.0	9.6	--	12.4	18.2

[Ans 6.609]

10. In an examination the number of candidates who secured marks

Marks	$0 - 40$	$40 - 60$	$60 - 80$	$80 - 100$	$100 - 120$
No of candidates	250	120	100	70	50

between certain limits were as follows:

Estimate the number of candidates getting marks less than 70.

[Ans 196]

UNIT-6 INTERPOLATION WITH UNEQUAL INTERVALS

Structure

6.1   OBJECTIVE

In todays era interpolation is considered to be an important technique of approximation, forecasting and trend analysis. Interpolation is a statistical method using which we may estimate the value of an unknown dependent variable (function) corresponding to any intermediate value of independent variable (argument) for a given set of tabulated data. As the tabulated datapoints may be at equal intervals or unequal intervals so in this unit we will cover both the concept.

6.2   INTRODUCTION

Interpolation means insertion or filling up intermediate terms of the series. It is the technique of obtaining the value of a function for any intermediate values of the independent variables When the values of the independent variable occur with unequal spacing, the formula discussed earlier is no longer applicable. In this situation another formula which is based on divided difference is used. Before presenting the formula let us first discuss divided differences.

6.3   LAGRANGE'S INTERPOLATION FORMULA

When the values of independent variable are not equally spaced and the differences of dependent variable are not small, ultimately, we will use Lagrange interpolation formula.

Let $y = f(x)$ be a function such that $f(x)$ takes the values $y_0, y_1, y_2, \ldots \ldots . y_n$, corresponding to $x = x_0, x_1, x_2, \ldots \ldots, x_n$ i.e.,

$$y_i = f(x_i), \qquad i = 0,1,2,3,\dots\dots n$$

Now, there are (n+1) paired values $(x_i, y_i), i = 0,1,2,\dots\dots n$ and hence $f(x)$ can be represented by a polynomial function of degree n in x.

We can assume $f(x)$ as follows:

$$f(x) = A_0(x - x_1)(x - x_2)\dots\dots(x - x_n)$$
$$+ A_1(x - x_0)(x - x_2)\dots\dots(x - x_n)$$
$$+ A_2(x - x_0)(x - x_1)(x - x_3)\dots\dots(x - x_n)\dots\dots$$
$$+ A_i(x - x_0)(x - x_1)\dots(x - x_{i-1})(x - x_{i+1})\dots\dots(x - x_n)$$
$$+ \cdots\dots + A_n(x - x_0)(x - x_1)(x - x_3)\dots\dots(x - x_{n-1})\dots\dots\dots. (1)$$

This is true for all values of x.

Substituting in (1), $x = x_0, y = y_0$, we get

$$y_0 = A_0(x_0 - x_1)(x_0 - x_2)\dots\dots(x_0 - x_n)$$

$$\therefore A_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\dots\dots(x_0 - x_n)}$$

Similarly, substituting $x = x_1, y = y_1$, we get

$$A_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)\dots\dots(x_1 - x_n)}$$

$$A_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\dots\dots(x_2 - x_n)}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$A_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\dots\dots(x_n - x_{n-1})}$$

Substituting these values in equation (1), we get

$$y = f(x) = \frac{(x - x_1)(x - x_2)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)}y_0$$
$$+ \frac{(x - x_0)(x - x_2)\dots\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)\dots(x_1 - x_n)}y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)\dots\dots\dots\dots\dots\dots(x - x_n)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\dots\dots\dots\dots\dots\dots(x_2 - x_n)}y_2 + \cdots\dots$$

$$+ \frac{(x - x_0)(x - x_1)\dots(x - x_{i-1})(x - x_{i+1})\dots(x - x_n)}{(x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)}y_i + \cdots$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)\dots\dots\dots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\dots\dots(x_n - x_{n-1})}y_n \dots\dots\dots\dots\dots. (2)$$

Equation (2) is called Lagrange's interpolation formula for unequal intervals.

Corrollary. Dividing equation (2) by $(x - x_0)(x - x_1)(x - x_2)...(x - x_n)$, we get

$$\frac{f(x)}{(x - x_0)(x - x_1)\,...\,....\,(x - x_n)}$$
$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\,...\,(x_0 - x_n)}\frac{1}{(x - x_0)} +$$

$$\frac{y_1}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)\,...\,(x_1 - x_n)}\frac{1}{(x - x_1)} + ...$$

$$+ \frac{y_n}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\,...\,....\,(x_n - x_{n-1})}\frac{1}{(x - x_n)}$$

Example1. For the table below, Evaluate f(9) using Lagrange's Interpolation formula:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Solution. The Lagrange's interpolation formula is

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)}f(x_0)$$
$$+ \frac{(x - x_0)(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)}f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)}f(x_2)$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)}f(x_3)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}f(x_4)$$

$$f(9) = \frac{(9 - 7)(9 - 11)(9 - 13)(9 - 17)}{(5 - 7)(5 - 11)(5 - 13)(5 - 17)}(150)$$
$$+ \frac{(9 - 5)(9 - 11)(9 - 13)(9 - 17)}{(7 - 5)(7 - 11)(7 - 13)(7 - 17)}(392)$$

$$+ \frac{(9 - 5)(9 - 7)(9 - 13)(9 - 17)}{(11 - 5)(11 - 7)(11 - 13)(11 - 17)}(1452)$$
$$+ \frac{(9 - 5)(9 - 7)(9 - 11)(9 - 17)}{(13 - 5)(13 - 7)(13 - 11)(13 - 17)}(2366)$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}(5202)$$

$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5}$$

$$= 810$$

Example 2. For the given $log_{10}654 = 2.8156$, $log_{10}658 = 2.8182$, $log_{10}659 = 2.8189$,

$log_{10}661 = 2.8202$ Find $log_{10}656$.

Solution. Here $f(x) = log_{10}656$, $x_0 = 654$, $x_1 = 658$, $x_2 = 659$, $x_3 = 661$

Thus by using Lagrange's interpolation formula, we get

$$log_{10}656 = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times 2.8156$$

$$+ \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2.8182$$

$$+ \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times 2.8189$$

$$+ \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2.8202$$

$$= 0.6033 + 7.045 - 5.6378 + .8057$$

$$= 2.8170$$

Example 3. Find the form of the function given by:

x	3	2	1	-1
$f(x)$	3	12	15	-21

Solution. Here $x_0 = 3$, $x_1 = 2$, $x_2 = 1$, $x_3 = -1$

Thus by using Lagrange's interpolation formula, we get

$$f(x) = \frac{(x-2)(x-1)(x+1)}{(3-2)(3-1)(3+1)} \times 3 + \frac{(x-3)(x-1)(x+1)}{(2-3)(2-1)(2+1)} \times 12$$

$$+ \frac{(x-3)(x-2)(x+1)}{(1-3)(1-2)(1+1)} \times 15 + \frac{(x-3)(x-2)(x-1)}{(-1-3)(-1-2)(-1-1)} \times (-21)$$

On solving we get,

$$f(x) = x^3 - 9x^2 + 17x + 6$$

Example 4.   Using Lagrange's formula, find the value of

$$y_x \text{ if } y_1 = 4, \ y_3 = 120, \ y_4 = 340, y_5 = 2544$$

Solution.   Here $x_0 = 1, \ x_1 = 3, \ x_2 = 4, \ x_3 = 5$

$$f(x_0) = 4, \ f(x_1) = 120, f(x_2) = 340, f(x_3) = 2544,$$

Now using Lagrange's interpolation formula, we get

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)} f(x_0)$$
$$+ \frac{(x - x_0)(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} f(x_2)$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} f(x_3)$$

$$f(x) = \frac{(x - 3)(x - 4)(x - 5)}{(1 - 3)(1 - 4)(1 - 5)} \times 4 + \frac{(x - 1)(x - 4)(x - 5)}{(3 - 1)(3 - 4)(3 - 5)}$$
$$\times 120$$

$$+ \frac{(x - 1)(x - 3)(x - 5)}{(4 - 1)(4 - 3)(4 - 5)} \times 340 + \frac{(x - 1)(x - 3)(x - 4)}{(5 - 1)(5 - 3)(5 - 4)}$$
$$\times (2544)$$

$$y_x = f(x) = -\frac{1}{6}(x - 3)(x - 4)(x - 5) + \ 30 \ (x - 1)(x - 4)(x - 5) +$$

$$- \frac{340}{3}(x - 1)(x - 3)(x - 5) \quad + \quad 318 \ (x - 1)(x -$$
$$3)(x - 4)$$

Example 5   Values of $f(x)$ are given at a, b, c. Show that the maximum is obtained by

$$x = \frac{f(a)(b^2 - c^2) + f(b)(c^2 - a^2) + f(c)(a^2 - b^2)}{[f(a)(b - c) + f(b)(c - a) + f(c)(a - b)]}$$

Solution   Here $x_0 = a, \ x_1 = b, \ x_2 = c,$

$$f(x_0) = a, \ f(x_1) = b, f(x_2) = c,$$

By applying Lagrange's formula, we have

$$f(x) = \frac{(x - b)(x - c)}{(a - b)(a - c)} f(a) + \frac{(x - b)(x - c)}{(b - a)(b - c)} f(b)$$
$$+ \frac{(x - b)(x - c)}{(c - a)(c - b)} f(c)$$

$$f(x) = \frac{x^2 - (b+c)x + bc}{(a-b)(a-c)} f(a) + \frac{x^2 - (a+c)x + ac}{(b-a)(b-c)} f(b)$$
$$+ \frac{x^2 - (a+b)x + ab}{(c-a)(c-b)} f(c)$$

For maximum, we have

$$f'(x) = 0$$

i.e.

$$f'(x) = \frac{2x - (b+c)f(a)}{(a-b)(a-c)} + \frac{2x - (a+c)f(b)}{(b-a)(b-c)} + \frac{2x - (a+b)f(c)}{(c-a)(c-b)}$$

$= \{2x - (b+c)\}(b-c)f(a) + \{2x - (a+c)\}(c-a)f(b) + \{2x - (a+b)\}(a-b)f(c) = 0$

$= 2x[(b-c)f(a) + (c-a)f(b) + (a-b)f(c)] =$

$$(b^2 - c^2)f(a) + (c^2 - a^2)f(b) + (a^2 - b^2)f(c)$$
$$(\because 2 \neq 0)$$

$\therefore \qquad x = \frac{f(a)(b^2-c^2)+f(b)(c^2-a^2)+f(c)(a^2-b^2)}{[f(a)(b-c)+f(b)(c-a)+f(c)(a-b)]}$ **Proved**

Check Your Progress

1. Using Lagrange's interpolation formula, find $y(8)$ from following table:

X	:	5	6	9	11
Y	:	12	13	14	16

[Ans- 14.6666667]

6.4 DIVIDED DIFFERENCE AND PROPERTIES OF DIVIDED DIFFERENCE

Divided Differences

The Lagrange formula has the drawback that if another interpolation value were inserted, then the ineterpolation coefficients are required to be recalculated. The labour of recomputing the interpolation formula which employs what are called 'divided differences'. Before deriving this formula, we shall first define these differences.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots$ be given points, then the first divided difference for the arguments, x_0, x_1 is defined by the relation

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly $[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3] = \frac{y_3 - y_2}{x_3 - x_2}$ etc

The *second divided difference* for x_0, x_1, x_2 is defined as

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$ and so on.

Properties of Divided Differences

1. Divided differences are symmetric with respect to the arguments i.e., independent of the order of arguments.

   i.e. $[x_0, x_1] = [x_1, x_0]$

   Also $[x_0, x_1, x_2] = [x_2, x_0, x_1]$ or $[x_1, x_2, x_0]$

2. The nth divided differences of a polynomial of degree n provided $A_0 \neq 0$ and arguments be equally spaced so that

   $$x_1 - x_0 = x_2 - x_1 = \ldots\ldots\ldots\ldots\ldots\ldots = x_n - x_{n-1} = h$$

   The first divided difference

   $$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

   The *second divided difference* for x_0, x_1, x_2

   $$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} = \frac{1}{2!h^2} = \Delta^2 y_0$$

   Therefore $[x_0, x_1, x_2 \ldots\ldots x_{n-1}, x_n] = \frac{1}{n!h^n} = \Delta^n y_0$

   Since, function is a nth degree polynomial

   Therefore, $\Delta^n y_0 = $ constant

   $\therefore$ nth divided will also be constant.

3. The nth divided difference can be expressed as the quotient of two determinants of order (n +1). i.e.,

   $$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1}{x_1 - x_0} - \frac{y_0}{x_1 - x_0}$$

   $$\Rightarrow \quad [x_0, x_1] = \begin{vmatrix} y_1 & y_0 \\ 1 & 1 \end{vmatrix} \div \begin{vmatrix} x_1 & x_0 \\ 1 & 1 \end{vmatrix}$$

   Similarly, $\Rightarrow [x_0, x_1, x_2] = \begin{vmatrix} y_0 & y_1 & y_2 \\ x_0 & x_1 & x_2 \\ 1 & 1 & 1 \end{vmatrix} \div \begin{vmatrix} x_0^2 & x_1^2 & x_2^2 \\ x_0 & x_1 & x_2 \\ 1 & 1 & 1 \end{vmatrix}$ ...... and so on.

4. The nth divided difference can be expressed as the product of multiple integral.

Relation Between Divided Differences and Ordinary Differences

Let the arguments $x_0, x_1, x_2, \ldots \ldots x_n$, be equally spaced such that

$$x_1 - x_0 = x_2 - x_1 = \cdots \ldots \ldots \ldots = x_n - x_{n-1} = h$$

$$\therefore \quad x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$

$$\ldots \ldots \ldots \ldots \ldots \ldots$$

$$x_n = x_0 + nh$$

Now, find divided difference for arguments (x_0, x_1) be given by

$$_{x_1}^{\Delta}f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{\Delta f(x_0)}{h} \qquad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Similarly, second divided difference be given as

$$_{x_1 x_2}^{\Delta^2}f(x_0) = \frac{1}{x_2 - x_0}[f(x_1, x_2) - f(x_0, x_1)]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]$$

$$= \frac{1}{2h}\left[\frac{f(x_0 + 2h) - f(x_0 + h)}{h} - \frac{f(x_0 + h) - f(x_0)}{h}\right]$$

$$= \frac{1}{2h^2}[f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)] = \frac{\Delta^2 f(x_0)}{2!h^2} \text{------} (2)$$

Similarly we get

$$_{x_1 x_2 x_3}^{\Delta^3}f(x_0) = \frac{\Delta^3 f(x_0)}{3! \, h^3}$$

In general

$$_{x_1 \ldots x_n}^{\Delta^n}f(x_0) = \frac{\Delta^n f(x_0)}{n!h^n}$$ which shows the relation between divided difference and ordinary difference.

6.5 NEWTON'S DIVIDED DIFFERENCE INTERPOLATION FORMULA

Let $y_0, y_1, y_2, \ldots \ldots y_n$ be the values $y = f(x)$ corresponding to the arguments $x_0, x_1, x_2, \ldots \ldots x_n$, then from the definition of divided differences, we have

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

So that, $\qquad y = y_0 + (x - x_0)[x_0, x_1] \qquad \ldots\ldots\ldots\ldots\ldots (1)$

Again, $\qquad [x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$

Which gives, $[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$ . ... (2)

From (1) and (2),

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \,\text{.........} \, (3)$$

Also $[x, x_0, x_1, x_2] = \dfrac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$

Which gives

$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2] \quad \text{.................... (4)}$$

From (3) and (4)

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$y = f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2] + \cdots \cdots$$
$$\cdots \cdots .. + (x - x_0)(x - x_1)(x - x_2) \dots . (x -$$
$$x_n)[x, x_0, x_1, x_2 \dots \dots x_n]$$

This is called Newton's general interpolation formula with divided differences, the last term being the remainder term after (n + 1) terms.

Newton's divided difference formula can also be wriiten as

$$y = f(x) = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0$$
$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + \cdots \cdots$$
$$\cdots \cdots .. + (x - x_0)(x - x_1)(x - x_2) \dots . (x - x_n)\Delta^n y_0$$

Example 1. Apply Newton's divided differnce formula to find the value of $f(8)$ if

$$f(1) = 3, \ f(3) = 31, f(6) = 31, f(10) = 1011, f(11) = 1343,$$

Solution

We construct the difference table with the given values.

x	$f(x)$	Δ	Δ^2	Δ^3
1	3			
		28/2 =14		
3	31		50/5 =10	

		192/3 =64		9/9=1
6	223		133/7 =19	
		788/4 =197		9/9=1
10	1011		135/5 =27	
		332/1=332		
11	1343			

On, applying Newton's divided diifernce formula, we have

$$y = f(x) = y_0 + (x - x_0)\Delta y_0 \ + \ (x - x_0)(x - x_1)\Delta^2 y_0$$

$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 \ + \cdots \ldots$$

$$\ldots \ldots .. + (x - x_0)(x - x_1)(x - x_2) \ldots . (x - x_n)\Delta^n y_0$$

$$y = f(x) = 3 + (x - 1) \times 14 \ + \ (x - 1)(x - 3) \times 10$$

$$+ (x - 1)(x - 3)(x - 6) \times 1 + \cdots \ldots$$

For $f(8)$, we put $x = 8$ in above equation, we get

$$f(8) = 3 + 7 \times 14 \ + 7 \times 5 \times 14 \ + 7 \times 5 \times 2$$

$$f(8) = 3 + 98 \ + 350 \ + 70$$

$$f(8) = 521 \text{ Ans}$$

Example 2. Find the function u_x in powers of $x - 1$ given that

$$u_0 = 8, \ u_1 = 11, \ \ u_4 = 68, \ \ \ u_5 = 123$$

Solution: Here

$$x_0 = x_0, \ x_1 = 1, \ \ x_2 = 4, \ x_3 = 5,$$

$$y_0 = 8, \ y_1 = 11, \ \ y_2 = 68, \ y_3 = 123,$$

The divided difference table is given by

x	$f(x)$	Δ	Δ^2	Δ^3
0	8			
		3		
1	11		4	

		19		1
4	68		9	
		55		
5	123			

On, applying Newton's divided diifernce formula, we have

$$y = f(x) = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0$$
$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + \cdots \cdots$$
$$\cdots \cdots \cdots + (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)\Delta^n y_0$$
$$y_x = 8 + (x - 0) \times 3 + (x - 0)(x - 1) \times 4$$
$$+ (x - 0)(x - 1)(x - 4) \times 1 + \cdots \cdots$$

$$y_x = 8 + 3x + 4x^2 - 4x + x(x^2 - 5x + 4)$$
$$y_x = x^3 - x^2 + 3x + 8$$

Example3 . Find the Newton's divided difference interpolation polynomial for:

x   :   0.5   1.5   3.0   5.0   6.5   8.0

f(x) :   1.625   5.875   31.0   131.0   282.125   521.0

Solution : The divided difference table is given as, if here

x   :   0.5   1.5   3.0   5.0   6.5   8.0

f(x) :   1.625   5.875   31.0   131.0   282.125   521.0

The table for divided difference is as :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.5	1.625				
		4.25			
1.5	5.875		5		
		16.75		1	
3.0	31.0		9.5		0

		50		1	
5.0	131.0		14.5		0
		100.75		1	
6.5	282.125		19.5		
		159.25			
8.0	521.0				

By using Newton's divided difference formula, we have

$$y_x = 1.625 + (x - 0.5) \times 4.25 + (x - 0.5)(x - 1.5) \times 5 + (x - 0.5)(x - 1.5)(x - 3) \times 1$$

$$y_x = 1.625 + 4.25x - 2.125 + 5x^2 - 10x + 3.75 + (x - 0.5)(x^2 - 4.5x + 4.5)$$

$$\therefore \ y_x = x^3 + 11x - 10x + 1$$

$$y_x = x^3 + x + 1 \ \text{Ans}$$

Example 4. Find the third divided difference with arguments 2, 4, 9, 10 of the function

$$f(x) = x^3 - 2x$$

Solution :

We construct the difference table with the given values.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
		$\dfrac{56 - 4}{4 - 2} = 26$		
4	56		$\dfrac{131 - 26}{9 - 2} = 15$	
		$\dfrac{711 - 56}{9 - 4} = 131$		9/9=1

9	711		$\dfrac{269-131}{10-4}$ $=23$	
		$\dfrac{980-711}{10-9}$ $=269$		$\dfrac{23-15}{10-2}$ $=1$
10	980			

Hence the third divided difference is 1.

Example 5. The mode of a certain frequency curve $y = f(x)$ is very near to $x = 9$ and the values of frequency density $f(x)$ for $x = 8.9, 9.0 \ and \ 9.3$ are respectively equal to 0.30, 0.35 and 0.25. Calculate the approximate value of mode.

Solution : Divide difference table for given frequency density is as

x	$100f(x)$	$100\Delta f(x)$	$100\Delta^2 f(x)$
8.9	30		
		$\dfrac{50}{9}$	
9.0	35		$-\dfrac{3500}{36}$
		$-\dfrac{100}{3}$	
9.3	25		

On, applying Newton's divided diifernce formula, we have

$$y = f(x) = y_0 + (x - x_0)\Delta y_0 \ + \ (x - x_0)(x - x_1)\Delta^2 y_0$$
$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + \cdots \ldots$$
$$\ldots \ldots \ldots + (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_n)\Delta^n y_0$$

$$100f(x) = 30 + (x - 8.9) \times \frac{50}{9} + (x - 8.9)(x - 9) \times \left(-\frac{3500}{36}\right)$$
$$= -97.222x^2 + 1745.833x - 1759.7217$$
$$\therefore \quad f(x) = -97.222x^2 + 1745.833x - 1759.7217$$
$$f'(x) = -1.9444x - 17.45833$$

Putting $f'(x) = 0$, we get

$$x = -\frac{17.45833}{1.9444} = 8.9788$$

Also, $f''(x) = -1.9444$ i.e. (- ve)

$\therefore \; f(x) \; is \; maximum \; at \; \boldsymbol{x = 8.9788}$

Hence, mode is 8.9788.

Example 6. Using Newton's divided difference formula, calculate the value of $f(6)$ from the following data:

x   :   1   2   7   8

(x)   :   1   5   5   4

Solution:

The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		4		
2	5		$-\dfrac{2}{3}$	
		0		$\dfrac{1}{14}$
7	5		$-\dfrac{1}{6}$	
		-1		
8	4			

Applying Newton's divided difference formula,

$$f(x) = 1 + (x-1) \times 4 + (x-1)(x-2) \times \left(-\frac{2}{3}\right)$$
$$+ (x-1)(x-2)(x-7) \times (\frac{1}{14})$$

$$f(x) = 1 + 20 + (5)(4) \times \left(-\frac{2}{3}\right) + (5)(4)(-1)(\frac{1}{14})$$

$$= 6.2381$$

Check Your Progress

1. By means of Newton's divided differnce formula

   Find the value of $f(8)$ if and $f(15)$ from the following table:

x	:	4	5	7	10	11	13
$f(x)$	:	48	100	294	900	1210	202 [Ans- 448, 3150]

6.6 SUMMARY

When the values of the independent variable occur with unequal spacing, the formula discussed earlier is no longer applicable. In this situation another formula which is based on divided difference is used. Before presenting the formula let us first discuss divided differences. One can use, Lagrange's Interpolation and Divided difference formula when there is an uneven spread in the domain set.

6.7 EXERCISE

1. Use Lagrange's interpolation formula, to fit a polynomial for the data

x	:	-1	0	2	3
u_x	:	-8	3	1	12

   Hence or otherwise find the value of u_1 .

   $$[\text{Ans- } u_x = 2x^3 - 6x^2 + 3x + 3, \ u_1 = 2]$$

2. Compute the value of $f(x)$ for $x = 2.5$ from the following data:

x	:	1	2	3	4
$f(x)$	:	1	8	27	64

Using Lagrange's interpolation method.   [Ans- 15.625]

3. If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$ and $y(6) = 132$, find the four pint Lagrange's interpolation polynomial which takes the same values as the function y at the given points.

$$[\text{Ans-} \ x^3 - 3x^2 + 5x - 6]$$

4. The percentage of Criminals for different age group are given below:

Age less than	:	25	30	40	50
Percentage of Criminals	:	52	67	84	94

Apply Lagrange's formula to find the percentage of criminals under 35 years of age.   [Ans- 77]

5. Using Newton's divided differnce formula, find a polynomial function satisfying the following data:

x	:	-4	-1	0	2	5
$f(x)$	:	1245	3	5	9	1335

Hence find $f(3)$

$$[\text{Ans-} \ f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5 \ , \ f(3) = 89]$$

6. For the value of $log_{10}33$ from the data given below:

$$log_{10}2 = 0.3010, \ log_{10}3 = 0.4771, \ log_{10}7 = 0.8451,$$

$$[\text{Ans-} \ log_{10}33 = 1.5184]$$

7. Find $f'(10)$ from the given data:

x	:	3	5	11	27	34
$f(x)$	:	-13	23	899	17315	35606

$$[\text{Ans-} \ f'(10) = 233]$$

8. Evaluate $f(9)$ using Newton's divided difference formula

x	:	5	7	11	13	17
$f(x)$	:	150	392	1452	2366	5202

$$[\text{Ans-} \ f(9) = 810]$$

Master in Computer Application

MCA-104

Uttar Pradesh Rajarshi Tandon Open University

Numerical Analysis

BLOCK

4

NUMERICAL DIFFERENTIATION, INTEGRATION AND SOLUTIONS OF DIFFERENTIATION EQUATIONS

UNIT-7

Numerical Differentiation and Integration

UNIT-8

Numerical Solution of Ordinary Differential Equations

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BLOCK INTRODUCTION

Overview :

Numerical technique is widely used by scientists and engineers to solve their problems. A major advantage for numerical technique is that a numerical answer can be obtained even when a problem has no analytical solution. However, result from numerical analysis is an approximation, in general, which can be made as accurate as desired. The reliability of the numerical result will depend on an error estimate or bound, therefore the analysis of error and the sources of error in numerical methods is also a critically important part of the study of numerical technique.

UNIT-7 NUMERICAL DIFFERENTIATION AND INTEGRATION

Structure

7.1   OBJECTIVE

The general method for deriving the numerical differentiation formulae is to differentiate the interpolating polynomial. Hence, corresponding to each off the formulae we derive a formula for derivative. We have taken Newton's forward difference and backward difference formulae for derivation etc.

7.2 INTRODUCTION

The differentiation are integration are loosely linked processes which are actually inversely related. For example, if the given function y(t) represents an objects position as a function of time, its differentiation provides its velocity. On the other hand, if we are provided with velocity v(t) as a function of time, its integration denotes its position. There are so many methods available to find the derivative and definite integration of a function. But when we have a complicated function or a function given in tabular form, then we use numerical methods. Here we shall discuss with the problem of numerical differentiation and integration.

7.3   NUMERICAL DIFFERENTIATION

The method of obtaining the derivatives of a function using a numerical technique is known as numerical differentiation. There are essentially two situations where numerical differentiation is required.

They are:

1. The function values are known but the function is unknown, such functions are called tabulated function.

2. The function to be differentiated is complicated and, therefore, it is difficult to differentiate.

The choice of the formula is the same as discussed for interpolation if the derivative at a point near the beginning of a set of values given by a table is required then we use Newton forward formula, and if the same is required at a point near the end of the set of given tabular values, then we use Newton's backward interpolation formula. The central difference formula (Bessel's and Stirling's) used to calculate value for points near the middle of the set of given tabular values. If the values of x are not equally spaced, we use Newton's divided difference interpolation formula or Lagrange's interpolation formula to get the required value of the derivative.

Derivation Using Newton's Forward Interpolation Formula

Newton's forward interpolation is given by

$$y_n(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \ldots\ldots\ldots\ldots (1)$$

Where $u = \frac{x-x_0}{h}$ Here h is called the interval of difference.

Differentiating equation (1) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \cdots \ldots \qquad \ldots\ldots\ldots (2)$$

Now $\because$ $\quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{h} \cdot \frac{dy}{du} \qquad (\because \frac{du}{dx} = \frac{1}{h})$

Therefore

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots \ldots\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

As $x = x_0$, u =0 , therefore, putting u =0 in (3) , we get

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4!}\Delta^4 y_0 + \cdots \ldots\right]$$

Differentiating equation (3) again w.r.t. $'x'$ we get

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx} = \frac{1}{h} \times \frac{d}{du}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 - \cdots \ldots\right] \qquad \ldots\ldots\ldots (4)$$

Putting $u = 0$ in (4), we get

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots \ldots\right] \qquad \ldots\ldots\ldots (5)$$

Otherwise $E = 1 + \Delta = e^{hD}$

Taking log of equation we get

$$hD = logE = \log(1 + \Delta)$$

$$= \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots \ldots \ldots$$

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots \right]$$

and

$$D^2 = \frac{1}{h^2}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots \right]^2$$

$$= \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \ldots \right]$$

Hence Dy_0 i.e.

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4!}\Delta^4 y_0 + \cdots \ldots\right]$$

Derivatives Using Newton's Backward Interpolation Formula

Newton's backward interpolation formula is given by

$$y_n(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 \ldots \qquad \ldots (1)$$

Where $u = \frac{x - x_n}{h}$

Diffferentiating both sides of equation (1) with respect to x we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2 \mp 6u + 2}{3!}\nabla^3 y_n + \cdots \ldots \ldots \qquad \ldots (2)\right.$$

As $x = x_n$, u $=0$, therefore, putting u $=0$ in (2) , we get

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots \ldots \ldots \right] \qquad \ldots (3)$$

Again differentiating both sides of (2) w.r.t. x, we get

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6u + 6}{6}\nabla^3 y_n + \frac{12u^2 + 36u + 22}{24}\nabla^4 y_n + \cdots \ldots\right]$$

As $x = x_n$, u $=0$, therefore, putting u $=0$ in (3) , we get

$$\left[\frac{d^2 y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots \ldots\right] \qquad \ldots (4)$$

Similarly, we know

$$E^{-1} = 1 - \nabla = e^{-hD}$$

Taking log of equation we get

$$-hD = \log(1 - \nabla)$$

$$= -\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots \ldots \ldots\right]$$

$$D = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots \ldots \ldots\right]$$

and
$$D^2 = \frac{1}{h^2}\left[\nabla^2 + \frac{1}{2}\nabla^3 + \frac{11}{12}\nabla^4 \ldots \ldots\right]^2$$

Example 1 Find $\frac{dy}{dx}$ at $x = 0.1$ from the following table:

x	0.1	0.2	0.3	0.4
y	0.9975	0.9900	0.9776	0.9604

Solution Difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	0.9975			
		-0.0075		
0.2	0.9900		-0.0049	
		-0.0124		0.0001
0.3	0.9976		-0.0048	
		-0.0172		
0.4	0.9604			

Here $x = 0.1, h = 0.1$ and $y_0 = 0.9975$ we know that, Newton's forward difference formula

$$[\frac{dy}{dx}]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right]$$

$$= \frac{1}{0.1} - 0.0075 - \frac{1}{2}(-0.0049) + \frac{1}{3}(0.0001)]$$

$$= \textbf{-0.050167 Ans}$$

Example 2

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.1$.

Solution

Since the values are at equidistant and we want to find the value of y at $x = 1.1$

Therefore, we apply Newton's forward difference formula

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	7.989						
		0.414					
1.1	8.403		-0.036				
		0.378		0.006			
1.2	8.781		-0.030		-0.002		
		0.348		0.004		0.001	
1.3	9.129		-0.026		-0.001		-0.002
		0.322		0.003		-0.001	
1.4	9.451		-0.023		-0.002		
		0.299		0.005			
1.5	9.750		-0.018				
		0.281					
1.6	10.031						

We have

$$[\frac{dy}{dx}]_{x=x_0} = \frac{1}{h} \left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 \ldots \ldots \right]$$

Putting $x = 1.1,\ h = 0.1$ and putting the values we get

$$[\frac{dy}{dx}]_{1.1} = \frac{1}{0.1} [0.378 - \frac{1}{2}(-0.030) + \frac{1}{3}(-0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(-0.001)]$$

$$= 10 [0.378 + 0.015 + 0.0013 + 0.00025 - 0.0002]$$

$$= 10 [0.39435] = \mathbf{3.9435} \textbf{ Ans}$$

and
$$[\frac{d^2y}{dx^2}]_{x=x_0} = \frac{1}{h^2} [\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 \dots \dots \dots \dots]$$

$$[\frac{d^2y}{dx^2}]_{1.1} = \frac{1}{(0.1)^2} [-0.030 - 0.004 + \frac{11}{12} \times 0.001 + \frac{5}{6} \times 0.001]$$

$$= 100 [-0.030 - 0.004 - 0.0009 + 0.0008]$$

$$= 10 [-0.0341] = \mathbf{-0.341} \textbf{ Ans}$$

Example 3 Find $f'(1.5)$ from the following table : .

x	0.0	0.5	1.0	1.5	2.0
$f(x)$	0.3989	0.3521	0.2420	0.1245	0.0540

Solution

Here we want to find the derivatives of $f(x)$ at $x = 1.5$, which is near the end of the arguments. Therefore, apply Newton's backward formula

Difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.0	0.3989				
		-0.0468			
0.5	0.3521		-0.0633		
		-0.1101		0.0609	
1.0	0.2420		-0.0024		-0.0215
		-0.1125		0.0394	
1.5	0.1295		0.037		
		0.0755			
2.0	0.0540				

We have

$$[\frac{dy}{dx}]_{x=x_n} = \frac{1}{h} [\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots \cdots \cdots]$$

Putting $x_n = 1.5, \ h = 0.5$

$$f'(1.5) = \frac{1}{0.5} [-0.1125 + \frac{1}{2}(-0.0024) + \frac{1}{4}(0.0609))]$$

$$= 2 [-0.1125 - 0.0012 + 0.0152251]$$

$$= 2 [-0.098475] = -0.19695 \ \textbf{Ans}$$

Example 4 Find the following table, find the values of $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ at $x = 2.03$

x	1.96	1.98	2.00	2.02	2.04
y	0.7825	0.7739	0.7651	0.7563	0.7473

Solution

Difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.96	0.7825				
		-0.0086			
1.98	0.7739		-0.0002		
		-0.0088		0.0002	
2.00	0.7651		0		-0.0004
		-0.0088		-0.0002	
2.02	0.7563		-0.0002		
		-0.0090			
2.04	0.7473				

Here, $x_n = 2.04$, $h = 0.02$ $u = \frac{x - x_n}{h} \Rightarrow \frac{dy}{dx} = \frac{1}{h}$ at $x = 2.03$

$\therefore$ $u = \frac{2.03 - 2.04}{0.02} = -\frac{0.01}{0.02} = -\frac{1}{2}$

Then by Newton's backward formula , we have

$$y_n(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 \ldots \ldots \qquad \ldots\ldots(1)$$

Differentiating w.r.t. x ,   we have

$$y'(x) = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2 + 6u+2}{3!}\nabla^3 y_n + \cdots \ldots \ldots\right] \qquad \ldots\ldots\ldots(2)$$

$$y'(2.03) = \frac{1}{0.02}\left[-0.0090 + 0 + \frac{3\left(-\frac{1}{2}\right)^2 + 6\left(-\frac{1}{2}\right) + 2}{3!}(-0.0002)\right.$$

$$\left.+ \frac{4\left(-\frac{1}{2}\right)^3 + 18\left(-\frac{1}{2}\right)^2 + 22\left(-\frac{1}{2}\right) + 6}{24}(-0.0004)\right]$$

$$y'(2.03) = 50\left[-0.0090 + 0.000008 + 0.000017\right]$$

$$= \mathbf{-0.44875} \quad \textbf{Ans}$$

Again differentiating equation (2) w.r.t. . x

$$y''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6u+6}{6}\nabla^3 y_n + \frac{12u^2 + 36u + 22}{24}\nabla^4 y_n + \cdots \ldots \ldots\right]$$

$$y''(2.03) = \frac{1}{(0.02)^2}\left[-0.0020 + \left(-\frac{1}{2} + 1\right)(-0.0002)\right.$$

$$\left.+ \frac{12\left(-\frac{1}{2}\right)^2 + 36\left(-\frac{1}{2}\right) + 22}{24}(-0.0004)\right]$$

$$y''(2.03) = 2500\left[-0.0002 - 0.0001 - 0.00012\right]$$

$$y''(2.03) = \mathbf{-1.05} \ \textbf{Ans}$$

Check Your Progress

1. Using the given table, find $\frac{dy}{dx}$ at $x = 1.2$

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

[Ans: 3.32015]

7.4   NUMERICAL INTEGRATION

Like numerical differentiation, we need to seek the help of numerical integration techniques in the following situations:

1. Functions do not posses closed from solutions. Example:

$$f(x) = C \int_0^x e^{-t^2} \, dt$$

2. Closed form solutions exist but these solutions are complex and difficult to use for calculations.

3.   Data for variables are available in the form of a table, but no mathematical relationship between them is known as is often the case with experimental data.

GENERAL QUADRATURE FORMULA

Let $y = f(x)$ be a function where y takes the values $y_0, y_1, y_2, \ldots \ldots y_n$, for $x = x_0, x_1, x_2, \ldots, x_n$. We want to find value of

$$I = \int_a^b f(x) \, dx$$

Let the interval of integration (a, b) be divided into n equal subintervals of width

$h = (\frac{b-a}{n})$ so $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \ldots \ldots, x_0 + nh = b$.

$$\therefore \quad I = \int_a^b f(x) \, dx = \int_{x_0}^{x_0+nh} f(x) \, dx \qquad \ldots\ldots\ldots (1)$$

Newton's forward interpolation formula is given by

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$
$$+ \cdots \ldots$$

Where $\quad u = \frac{x - x_0}{h}$

$$\therefore \quad du = \frac{1}{h} \cdot dx \Rightarrow \ dx = h \, du$$

$\therefore$ Equation (1) becomes

$$I = h \int_0^n [\, y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \ldots \quad] \, du$$
$$= nh[\, y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots up \, to \, (n+1) terms \,]$$

$$\therefore \quad \int_{x_0}^{x_n} f(x) \, dx = = nh[\, y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 +$$
$$\cdots up \, to \, (n+1) terms \,] \qquad \ldots\ldots\ldots (2)$$

This is called general quadrature formula.

7.5 TRAPEZOIDAL RULE

TRAPEZOIDAL RULE

Putting $n = 1$ in equation (2) and taking the curve $y = f(x)$ through (x_0, y_0) and (x_0, y_0) as a polynomial of degree one so that differences of order higher than one vanish, we get

$$\int_{x_0}^{x_0+h} f(x)\, dx = h(y_0 + \frac{1}{2}\Delta y_0)$$

$$= \frac{h}{2}[2y_0 + (y_1 - y_0)]$$

$$= \frac{h}{2}[y_0 + y_1]$$

Similarly, for the next sub interval $(x_0 + h, x_0 + 2h)$, we get

$$\int_{x_0+h}^{x_0+2h} f(x)\, dx = \frac{h}{2}(y_1 + y_1) \ldots\ldots \int_{x_0+(n-1)h}^{x_0+nh} f(x)\, dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)\, dx = \frac{h}{2}[(y_n + y_0) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})]$$

which is known as Trapezoidal rule.

7.6 SIMPSON'S ONE-THIRD RULE

Putting $n = 2$ in equation (2) and taking the curve $y = f(x)$ through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a polynomial of degree two so that differences of order higher than two vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\, dx = 2h[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0]$$

$$= \frac{2h}{6}[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 - y_0)]$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly, $\qquad \int_{x_0+2h}^{x_0+4h} f(x)\, dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\, dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[\,(y_0 + y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})\,]$$

Which is known as Simpson/s one-third rule.

Note: Using the formula, the given interval of integration must be divided into an even number of sub-intervals.

7.7 SIMPSON'S THREE-EIGHT RULE

Putting $n = 3$ in equation (2) and taking the curve $y = f(x)$ through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and (x_3, y_3) as a polynomial of degree two so that differences of order higher than two vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h[\,y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0]$$

$$= \frac{3h}{8}[\,8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)\,]$$

$$= \frac{3h}{8}[\,y_0 + 3y_1 + 3y_2 + y_3\,]$$

Similarly, $\quad \int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$

$$\int_{x_0+(n-3)h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = z\frac{3h}{8}[\,(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3})\,]$$

Which is known as Simpson's three-eight rule.

Note: Using the formula, the given interval of integration must be divided into sub-intervals whose number n is a multiple of 3.

7.8 WEDDLE'S RULE

Putting $n = 6$ in equation (2) and neglecting all the differences of order higher than six, we get

$$\int_{x_0}^{x_0+6h} f(x)\,dx = h \int_0^6 [\,y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots + \frac{u(u-1)(u-2)(u-3)(u-4)(u-5)}{6!}\Delta^6 y_0\,]\,du$$

$$= 6h[\, y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{41}{20}\Delta^4 y_0 + \frac{11}{20}\Delta^5 y_0 + \frac{41}{840}\Delta^6 y_0]$$

$$= \frac{6h}{20}[\, 20y_0 + 60\Delta y_0 + 90\Delta^2 y_0 + 80\Delta^3 y_0 + 41\Delta^4 y_0 + 11\Delta^5 y_0 \; \frac{41}{42}\Delta^6 y_0]$$

$$= \frac{6h}{20}[\, 20y_0 + 60(\, y_1 - y_0) + 90(\, y_2 - 2y_1 + y_0) + 80(\, y_3 - 3y_1 + 3y_1 - y_0)$$

$$+ 41(\, y_4 - 4y_3 + 6y_2 - 4y_1 + y_0) + 11(\, y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)] + (\, y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0)]\, [\frac{41}{42} \approx 1]$$

$$= \frac{3h}{10}[\, y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

Similarly,

$$\int_{x_0+6h}^{x_0+12h} f(x)\, dx = = \frac{3h}{10}[\, y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

....

....

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\, dx =$$

$$= \frac{3h}{10}[\, y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

Adding the above integrals, we get

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\, dx = = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2\,y_6$$

$$+5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12} \ldots \ldots \ldots]$$

Which is known as Weddle's rule. Hence n must be a multiple of 6.

Example 1 Evaluate $\int_0^1 \frac{1}{1+x}\, dx$.

Solution- Let $h = 0.125$ (as dividing [0,1] into eight equal parts i.e 1-0 / 8 = 0.125) and $y = f(x) = \frac{1}{1+x}$, then the values of y are given for the arguments which are obtained by dividing the interval [0,1] into eight equal parts are given below:

x	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1.0
$y = \dfrac{1}{1+x}$	1.0	0.889	0.800	0.7273	0.6667	0.6154	0.5714	0.5333	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Now by Trapezoidal rule

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) + y_8]$$

$$= \frac{0.125}{2}[1 + 2(0.889 + 0.800 + 0.7273 + 0.6667 + 0.6154 + 0.5714 + 0.5333) + 0.5]$$

$$= \frac{0.125}{2}[1.5 + 2(4.803)]$$

$$= \frac{0.125}{2}[11.106] = \mathbf{0.69413} \qquad \textbf{Ans}$$

Example 2 Use Trapezoidal rule to evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using

(i)   Simpson's $\frac{1}{3}$ rule taking $h = \frac{1}{4}$

(ii)   Simpson's $\frac{3}{8}$ rule taking $h = \frac{1}{6}$

(iii) Weddle's rule taking $h = \frac{1}{4}$

Hence compute an approximate value of π in each case.

Solution

(i)   The values of $f(x) = \int_0^1 \frac{1}{1+x^2}$ at $x = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$ are given below:

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x)$	1	$\frac{1}{4}$	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpson's $\frac{1}{3}$ rule

$$\int_0^1 \frac{1}{1+x^2} = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2)]$$

$$= \frac{1}{12}[(1 + 0.5) + 4(\frac{16}{17} + .64) + 2(0.8)]$$

$$= 0.78539215$$

Also,
$$\int_0^1 \frac{1}{1+x^2} = [\tan^{-1} x]$$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

$$= \tan^{-1} 1 = \frac{\pi}{4}$$

$\therefore$ $\frac{\pi}{4} = 0.785392156 \Rightarrow \approx 3.1415686.$ Ans

(ii) By Simpson's $\frac{3}{8}$ rule

The values of $f(x) = \int_0^1 \frac{1}{1+x^2}$ at $x = 0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1$ are given below:

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$f(x)$	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\int_0^1 \frac{1}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3)$$

$$= \frac{3(\frac{1}{6})}{8}[(1 + \frac{1}{2}) + 3(\frac{36}{37} + \frac{9}{10} + \frac{9}{13} + \frac{36}{61}) + 2(\frac{4}{5}))$$

$$= \mathbf{0.785395862}$$

Also
$$\int_0^1 \frac{1}{1+x^2} = \frac{\pi}{4}$$

$\therefore$ $\frac{\pi}{4} = 0.785395862 \Rightarrow \pi = 3.141583$ Ans

(iv) By Weddle's rule

$$\int_0^1 \frac{1}{1+x^2} = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

$$= \frac{3(\frac{1}{6})}{10}(1 + 5(\frac{36}{37}) + \frac{9}{10} + 6(\frac{4}{5}) + \frac{9}{13} + 5(\frac{36}{61}) + \frac{1}{2})$$

$$= \mathbf{0.785399611}$$

Also $\qquad \int_0^1 \frac{1}{1+x^2} = \frac{\pi}{4}$

$\therefore \qquad \frac{\pi}{4} = 0.7853999611 \Rightarrow \pi = 3.141598 \qquad$ Ans

Example 3 Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ by using

    (i)   Trapezoidal Rule

    (ii)   Simpson's one-third rule

    (iii)   Simpson's three-eight rule

    (iv)   Weddle's rule

Solution : Divide the interval (0,6) into six parts each of width $h = 1$

          The value of $f(x) = \frac{1}{1+x^2}$ are given below:

    (i)   By Trapezoidal rule

$$\int_0^6 \frac{1}{1+x} dx = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}\left[(1 + \frac{1}{37}) + 2(0.5 + 0.2 + 0.1 + \frac{1}{17} + \frac{1}{26})\right]$$

$$= 1.410798581 \qquad \textbf{Ans.}$$

    (ii)   By Simpson's one-third rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + \frac{1}{37}) + 4(0.5 + 0.1 + \frac{1}{26}) + 2(0.2 + \frac{1}{17})]$$

$$= 1.366173413 \qquad \text{Ans.}$$

    (iii)   By Simpson's three-eight rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3)]$$

$$= \frac{3}{8}[(1 + \frac{1}{37}) + 3(0.5 + 0.2 + \frac{1}{17} + \frac{1}{26}) + 2(0.1))$$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$= 1.35080836 \qquad \textbf{Ans}$$

(iv) By Weddle's rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

$$= \frac{3}{10}(1 + 5(0.5) + 0.2 + 6(0.1) + \frac{1}{17} + 5(\frac{1}{26}) + \frac{1}{37}]$$

$$= \mathbf{1.373447475} \quad \textbf{Ans}$$

Check Your Progress

1. Using Simpson's one-third rule, find $\int_0^6 \frac{dx}{(1+x)^2}$

7.9 SUMMARY

There are number of differential equations which we studied in Calculus to get closed form solutions. But, all differential equations do not possess closed form of finite form solutions. Even if they possess closed form solutions, we do not know the method of getting it. In such situations, depending upon the need of the hour, we go in for numerical solutions of differential equations. In researches, especially after the advent of computer, the numerical solutions of the differential equations have become easy for manipulations. The choice of interpolation formula to be used, will depend on the assigned value of x at which $\frac{dy}{dx}$ is desired. If the value of x are equi-spaced and $\frac{dy}{dx}$, is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula.

7.10 EXERCISE

1. Find $f'(6)$ from the following table

x	0	1	3	4	5	7	9
$f(x)$	150	108	0	-54	-100	-144	-84

[Ans: -23]

2. Use the following data to find $f'(3)$:

x	3	5	11	27	34
f(x)	−13	28	899	17315	35606

[Ans: 1.8828]

3. The table given below reveals the velocity $'v'$ of a body during the time $'t'$ specified. Find its aceeleartion at $t = 1.1$

t	1.0	1.1	1.2	1.3	1.4
v	43.1	47.7	52.1	56.4	60.8

[Ans: 45.167]

4. Find the following table, find the values of $\frac{dy}{dx}$ at $x = 1, x = 3$ and $x = 6$.

x	0	1	2	3	4	5	6
$f(x)$	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506	9.0309

[Ans: 0.3952, 0.3341, 0.2719]

5. A slider in a machine moves along a fixed straight rod. Its distance x (in cm.) along the rod is given at various times t (in secs.)

t	0	0.11	0.2	0.3	0.4	0.5	0.6
x	30.28	31.43	32.98	33.54	33.97	33.48	32.13

Evaluate $\frac{dx}{dt}$ at $at\ t = 0.1$ and $at\ t = 0.5$.

[Ans: 32.44166 cm/sec, -24.05833 cm/sec]

6. Use Trapezoidal rule to evaluate $\int_0^1 x^3\, dx$ consisting five sub-intervals. [Ans 0.26]

7. Evaluate the integral $\int_0^4 e^x\, dx$ by Simpson's one-third rule.

[Ans 53.87]

8. Use Simpson's rule to evaluate $\int_0^1 \frac{1}{1+x}\, dx$ with $h = \frac{1}{6}$.

[Ans 0.69319]

9. Using Weddle's rule to evaluate $\int_0^5 \frac{1}{4x+5}\, dx$ [Ans 0.4023]

10. Using $\frac{3}{8}$ rule Simpson's rule. Evaluate $\int_0^6 \frac{1}{1+x^4}\, dx$

[Ans 1.019286497]

UNIT-8 NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

Structure

8.1   Objective

8.2   Picard's Method

8.3   Euler's Method

8.4   Runge-Kutta Method's

8.5   Summary

8.6   Exercise

8.1   OBJECTIVE

A number of numerical methods are available for the solution of first order differential equations of the form:

$$\frac{dy}{dx} = f(x,y), \text{ given } y(x_0) = y_0$$

These methods yield solutions either as power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. The methods of Picards and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class.

8.2   PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

Consider first order differential equation

$$\frac{dy}{dx} = f(x,y) \qquad\qquad\qquad \text{..........(1)}$$

with the initial condition

$$y = y_0 \text{ at } x = x_0$$

Integrating (1) with respect to x between x_0 and x, we have

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx$$

or $\qquad y = y_0 + \int_{x_0}^{x} f(x,y)\, dx$ $\qquad\qquad$ .... (2)

Now, we solve (2) by the method of successive approximation to find out the solution of (1). The first approximate solution (approximation) y_1 of y is given by

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

Similarly, the second approximation y_2 is given by

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

Similarly, the nth approximation y_n is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx \qquad\qquad ....(3)$$

With $y(x_0) = y_0$.

Hence, this method gives a sequence of approximation $y_1, y_2, ... y_n$ and it can be proved $f(x, y)$ is bounded in some regions containing the point (x_0, y_0) and if $f(x, y)$ satisfies the Lipchitz condition, namely

$$|f(x,y) - f(x, \bar{y})| \le k\, |y - \bar{y}|$$

Where k is a constant and $y_1, y_2, ...$ converges to the solution (2).

Example 1 Use Picard's method to obtain y for $x = 0.2$. Given

$$\frac{dy}{dx} = x - y$$

with the initial condition $y = 1$ at $x = 0$

Solution Here, $f(x,y) = x - y$, $x_0 = 0$, $y_0 = 1$

We have first approximation

$$y_1 = y_0 + \int_{0}^{x} f(x, y_0)\, dx$$

$$= 1 + \int_{0}^{x} (x - 1)\, dx$$

$$= 1 - x + \frac{x^2}{2}$$

Second approximation,

$$y_2 = y_0 + \int_{0}^{x} f(x, y_1)\, dx$$

$$= 1 + \int_{0}^{x} (x - y_1)\, dx$$

$$= 1 + \int_{0}^{x} (x - 1 + x - \frac{x^2}{2})\, dx$$

$$= 1 - x + x^2 - \frac{x^3}{6}$$

Third approximation,

$$y_3 = y_0 + \int_0^x f(x, y_2)\, dx$$

$$= 1 + \int_0^x (x - y_2)\, dx$$

$$= 1 + \int_0^x (x - 1 + x - x^2 + \frac{x^3}{6})\, dx$$

$$= 1 - x + x^2 - \frac{x^3}{6} + \frac{x^4}{24}$$

Fourth approximation,

$$y_4 = y_0 + \int_0^x f(x, y_3)\, dx$$

$$= 1 + \int_0^x (x - y_3)\, dx$$

$$= 1 + \int_0^x (x - 1 - x - x^2 + \frac{x^3}{6} - \frac{x^4}{24})\, dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

Fifth approximation,

$$y_5 = y_0 + \int_0^x f(x, y_4)\, dx$$

$$= 1 + \int_0^x (x - y_4)\, dx$$

$$= 1 + \int_0^x (x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{120})\, dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{720}$$

When $x = 0.2$, we get

$$y_1 = .82, \qquad y_2 = .83867, \qquad y_3 = .83740,$$

$$y_4 = .83746, \qquad y_5 = .83746$$

Thus $y = 0.837$ when $x = 0.2$ **Ans**

Example 2 Find the solution of

$$\frac{dy}{dx} = 1 + xy, \quad y(0) = 1$$

Which passes through (0,1) in the interval (0,0.5) such that the value of y is correct to three decimal places (use the whole interval as one interval only) Take $h = 0.1$.

Solution: The given initial value problem is

$$\frac{dy}{dx} = f(x, y) = 1 + xy; \quad y(0) = 1$$

i.e. $\qquad y = y_0 = 1$ at $\qquad x = x_0 = 0,$

Here,

$$y_1 = 1 + x + \frac{x^2}{2}$$

$$y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

$$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

$$y_4 = y_3 + \frac{x^7}{105} + \frac{x^8}{384}$$

When $\qquad x = 0, \quad y = 1.000$ at

$\therefore \qquad x = 0.1, \quad y_1 = 1.105, \quad y_2 = 1.223 = y_3$ (correct up to 3 decimals)

$\qquad\qquad \therefore \qquad y = 1.223$

$x = 0.3, \quad y = 1.355, \quad y_2 = 1.355 = y_3$ (correct up to 3 decimals)

$$x = 0.4, \quad y = 1.505$$

$$x = 0.5, \quad y = 1.677, \quad y_4 = y_3 = 1.677$$

Thus,

x	0	0.1	0.2	0.3	0.4	0.5
y	1.000	1.105	1.223	1.355	1.505	1.677

We have numerically solved the given differential equation for

$$x = 0.1, 0.2, 0.3, \ 0.4, \ and \ 0.5 \ .$$

Example 3 Use Picard's method to obtain y for $x = 0.1$. Given

$$\frac{dy}{dx} = 3x + y^2$$

with the initial condition $\quad y = 1$ at $x = 0$

Solution

Here, $\quad f(x,y) = 3x + y^2 , \quad x_0 = 0, \quad y_0 = 1$

We have First approximation

$$y_1 = y_0 + \int_0^x f(x, \ y_0) \, dx$$

$$= 1 + \int_0^x (3x + 1) \, dx$$

$$= 1 + x + \frac{3}{2}x^2$$

Second approximation

$$y_2 = 1 + x + \frac{5}{2}x^2 + \frac{4}{2}x^3 + \frac{3}{4}x^4 + \frac{9}{20}x^5$$

Third approximation

$$y_3 = 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{23}{12}x^4 + \frac{25}{12}x^5 + \frac{68}{45}x^6 + \frac{1157}{1260}x^7$$

$$+ \frac{17}{32}x^8 + \frac{47}{240}x^9 + \frac{27}{400}x^{10} + \frac{81}{4400}x^{11}$$

When $x = 0.1$, we have

$$y_1 = 1.115, \qquad\qquad y_2 = 1.1264, \qquad\qquad y_3 = 1.12721$$

Thus $y = 1.127$ when $x = 0.1$ Ans

Check Your Progress

1. Obtain y when $x = 0.1$, $x = 0.2$

    Given that $\frac{dy}{dx} = x + y$, $y(0) = 1$, check the result
    with exact value.

8.3   EULER'S METHOD

The oldest and simplest method was derived by Euler. In this method, we determine the change Δy in y corresponding to small increment in the argument . Consider the differential equation.

$$\frac{dy}{dx} = f(x, y) \qquad\qquad\qquad \text{.......... (1)}$$

with the initial condition

$$y(x_0) = y_0$$

Integrating (1) with respect to x between x_0 and x_1, we have

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\, dx \qquad\qquad \text{------(2)}$$

Now, replacing $f(x, y)$ by the approximation $(x_0,\ y_0)$, we get

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x_0,\ y_0)\, dx$$

$$= y_0 + f(x_0,\ y_0)(x_1 - x_0)$$

$$y_1 = y_0 + h f(x_0,\ y_0) \qquad (\because x_1 - x_0 = \Delta x = h)$$

This is the formula for first approximation y_1 is given by

$$y_2 = y_1 + hf(x_1, y_1)$$

In general,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

EULER'S MODIFIED METHOD

Instead of approximating $f(x, y)$ by $f(x_0, y_0)$ in equation (2). Let the integral is appointed by Trapezoidal rule to obtain.

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

We obtain the iteration formula

$$y_1^{n+1} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{n})] \qquad n = 0,1,2 \ldots \ldots$$

Where, y_1^{n} is the nth approximation to y_1 .

The above iteration formula can be started by $y_1^{(1)}$ from Euler's method.

$$y_1^{(0)} = y_0 + h(x_0, y_0)$$

Example 1 Use Euler's method, compute y(0.5) for differential equation

$$\frac{dy}{dx} = y^2 - x^2$$

with the initial condition $y = 1$ when $x = 0$

Solution Let $\qquad h = \frac{0.5}{5} = 0.1$

$$x_0 = 0, \quad y_0 = 1 , \quad f(x, y) = y^2 - x^2$$

Using Euler's method we have

$$y_{n+1} = y_n + h(x_n, y_n)$$

But considering $n = 0,1,2, \ldots \ldots \ldots$ in succession, we get

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.1(1^2 - 0) = 1.10000$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.10000 + 0.1[(1.10000)^2 - (0.1)^2] = 1.22000$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= 1.22000 + 0.1[(1.22)^2 - (0.2)^2] = 1.36484$$

$$y_4 = y_3 + hf(x_3, y_3)$$

$$= 1.36484 + 0.1[(1.36484)^2 - (0.3)^2] = 1.54212$$

$$y_5 = y_4 + hf(x_4, y_4)$$

$$= 1.54212 + 0.1[(1.54212)^2 - (0.4)^2] = 1.76393$$

Hence, the value of y at $x = 0.5$ is **1.76393**.   Ans

Example 2 Find the solution of differential equation

$$\frac{dy}{dx} = xy \qquad \text{with } y(1) = 5$$

in the interval [1, 1.5] using $h = 0.1$.

Solution   As per given we have

$$x_1 = 1, \quad y_0 = 5, \quad f(x, y) = xy$$

Using Euler's method we have

$$y_{n+1} = y_n + h(x_n, y_n)$$

But considering $n = 0,1,2, \dots \dots \dots$ in succession, we get

For $n = 0$ $y_1 = y_0 + 0.1 f(x_0, y_0)$

$$= 5 + 0.1(1 \times 5) = 5.5$$

For $n = 1$ $y_2 = y_1 + 0.1 f(x_1, y_1)$

$$= 5.5 + 0.1(1.1 \times 5.5) = 6.105$$

For $n = 2$ $y_3 = y_2 + 0.1 f(x_2, y_2)$

$$= 6.105 + 0.1(1.2 \times 6.105) = 6.838$$

For $n = 3$ $y_4 = y_3 + 0.1 f(x_3, y_3)$

$$= 6.838 + 0.1(1.3 \times 6.838) = 7.727$$

For $n = 4$ $y_5 = y_4 + 0.1 f(x_4, y_4)$

$$= 7.727 + 0.1(1.4 \times 7.727) = 8.809$$

Hence, the value of $y(1.5)$ at is **8.809**.   Ans

Example 3 Given

$$\frac{dy}{dx} = x + y \text{ with initial condition } y(0) = 1.$$

Find $y(0.05)$ and $y(0.1)$ correct to 6 decimal places.

Solution Using Euler's method, we have

$$y_1{}^{(0)} = y_1 = y_0 + h(x_0, y_0)$$

$$= 1 + 0.05(0 + 1) = 1.05$$

We improve y_1 by using Euler's modified method

$$y_1{}^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1{}^0)]$$

$$= 1 + \frac{0.05}{2}[(0 + 1) + (0.05 + 1.05)]$$

$$= 1.0525$$

$$y_1{}^{(2)} = 1 + \frac{0.05}{2}[(0+1) + (0.05 + 1.0525)]$$

$$= 1.0525625$$

$$y_1{}^{(3)} = 1 + \frac{0.05}{2}[(0+1) + (0.05 + 1.0525625)]$$

$$= 1.052564$$

$$y_1{}^{(4)} = 1 + \frac{0.05}{2}[(0+1) + (0.05 + 1.0525625)]$$

$$= 1.0525641$$

Since, $y_1{}^{(3)} = y_1{}^{(4)} = 1.0525641$ correct to 6 decimal places. Hence we take $y_1 = 1.052564$

i.e., we have $y(0.05) = 1.052564$

Again, using Euler's method, we obtain

$$y_2{}^{(0)} = y_2 = y_1 + h[f(x_1, \, y_1)]$$

$$= 1.052564 + 0.05(1.052564 + 0.05)$$

$$= 1.1076922$$

We improve y_2 by using Euler's modified method

$$y_2{}^{(1)} = 1.052564 + \frac{0.05}{2}[(1.052564 + 0.05) + (1.1076922 + 0.1)]$$

$$= 1.1120511$$

$$y_2{}^{(2)} = 1.052564 + \frac{0.05}{2}[(1.052564 + 0.05) + (1.1120511 + 0.1)]$$

$$= 1.1104294$$

$$y_2{}^{(3)} = 1.052564 + \frac{0.05}{2}[(1.052564 + 0.05) + (1.1104294 + 0.1)]$$

$$= 1.1103888$$

$$y_2{}^{(4)} = 1.052564 + \frac{0.05}{2}[(1.052564 + 0.05) + (1.1103888 + 0.1)]$$

$$= 1.1103878$$

$$y_2{}^{(5)} = 1.052564 + \frac{0.05}{2}[(1.052564 + 0.05) + (1.1103878 + 0.1)]$$

$$= 1.1103878$$

Since, $y_2{}^{(4)} = y_2{}^{(5)} = 1.1103878$ correct to 7 decimal places. Hence we take $y_2 = 1.1103878$

Therefore, we have $y(0.1) = 1.1103878$, correct to **6 decimal places. Ans**

Example 4 Find $y(2.2)$ using Euler's method for

$$\frac{dy}{dx} = -xy^2 , \text{ where } y(2) = 1. \quad (\text{ Take } h = 0.1)$$

Solution

By Euler's method, we have

$$y_1{}^{(0)} = y_1 = y_0 + h(x_0, y_0)$$

$$= 1 + 0.1(-2)(-1)^2 = 0.08$$

The value of y_1 by using Euler's modified method

$$y_1{}^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1{}^0)]$$

$$= 1 + \frac{0.1}{2}[(-2)(1)^2 + (-2.1)(0.8)^2]$$

$$= 0.8328$$

Similarly

$$y_1{}^{(2)} = 1 + \frac{0.1}{2}[(-2)(1)^2 + (-2.1)(0.8328)^2]$$

$$= 0.8272$$

$$y_1{}^{(3)} = 1 + \frac{0.1}{2}[(-2)(1)^2 + (-2.1)(0.8272)^2]$$

$$= 0.8281$$

$$y_1{}^{(4)} = 1 + \frac{0.1}{2}[(-2)(1)^2 + (-2.1)(0.8281)^2]$$

$$= 0.8280$$

$$y_1{}^{(5)} = 1 + \frac{0.1}{2}[(-2)(1)^2 + (-2.1)(0.8280)^2]$$

$$= 0.8280$$

Since $y_1{}^{(4)} = y_1{}^{(5)} = 0.8280$. Hence, we take $y_1 = 0.828$ at $x_1 = 2.1$

Now, if y_2 is the value of y at $x = 2.2$. Then, we apply Euler's method to compute $y(2.2)$, i.e., we obtain

$$y_2{}^{(0)} = y_2 = y_1 + h(x_1, y_1)$$

$$= 0.828 + 0.1(-2.1)(0.828)^2 = 0.68402$$

Now, using Euler's modified formula, we obtain

$$y_2{}^{(1)} = 0.828 + \frac{0.1}{2}[(-2.1)(0.828)^2 + (-2.2)(0.68402)^2]$$

$$= 0.70454$$

$$y_2{}^{(2)} = 0.828 + \frac{0.1}{2}[(-2.1)(0.828)^2 + (-2.2)(0.70454)^2]$$

$$= 0.70141$$

$$y_2^{(3)} = 0.828 + \frac{0.1}{2}[(-2.1)(0.828)^2 + (-2.2)(0.70141)^2]$$

$$= 0.70189$$

$$y_2^{(4)} = 0.828 + \frac{0.1}{2}[(-2.1)(0.828)^2 + (-2.2)(0.70189)^2]$$

$$= 0.70182$$

$$y_2^{(5)} = 0.828 + \frac{0.1}{2}[(-2.1)(0.828)^2 + (-2.2)(0.70182)^2]$$

$$= 0.70183$$

Since $y_2^{(4)} = y_2^{(5)} = 0.7018$, correct to 4 decimal places.

Hence, we take $y(2.2) = 0.7018$ **Ans.**

Check Your Progress

1.   Given that $\frac{dy}{dx} = \frac{y-x}{y+x}$,   with $y_0 = 1$ find y for $x = 0.1$ in four
     steps by Euler's method.   [Ans $y(0.1) = 1.0932$]

8.4   RUNGE-KUTTA METHOD

The method is very simple. It is named after two german mathematicians Carl Runge (1856-1927) and Wilhelm Kutta (1867-1944). These methods are well-known as Runge-Kutta Method. They are distinguished by their orders in the sense that they agree with Taylor's series solution upto term of h^r where r is a positive integer is the order of the method.

It was developed to avoid the computation of higher order derivations which the Taylors' method may involve. In the place of these derivatives extra values of the given function $f(x, y)$are used.

(i)   **First order Runge-Kutta method**

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) ; \qquad y(x_0) = y_0 \qquad\qquad \text{-------- (1)}$$

By Euler's method, we know that

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy'_0 \qquad\qquad \text{------- (2)}$$

Expanding by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \quad \text{-------(3)}$$

It follows that Euler's method agrees with Taylor's series solution upto the terms in h. Hence Euler's method is the first order Runge-Kutta method.

(ii) First order Runge-Kutta method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) ; \qquad y(x_0) = y_0$$

Let h be the interval between equidistant values of x. Then the second order Runge-Kutta method, the first increment in y is computed from the formulae

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$\Delta y = \frac{1}{2}(k_1 + k_2)$$

Then, $\qquad\qquad x_1 = x_0 + h$

$$y_1 = y_0 + \Delta y = y_0 + \frac{1}{2}(k_1 + k_2)$$

Similarly, the increment in y for the second interval is computed by the formulae,

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf(x_1 + h, y_1 + k_1)$$

$$\Delta y = \frac{1}{2}(k_1 + k_2)$$

and similarly for other intervals.

(iii) Third order Runge-Kutta method

The method agrees with Taylor's series upto the term h^3. The formula is as follows:

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3);$$

$$x_1 = x_0 + h$$

Where, $\qquad\qquad k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$

Similarly for other intervals.

(iv) Fourth order Runge-Kutta method

This method coincides with the Taylor's series solution upto terms of h^4.

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with initial condition } y(x_0) = y_0.$$

Let h be the interval between equidistant vales of x. Then the first increment in y is computed from the formulae.

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3);$$

$$x_1 = x_0 + h$$

Where,
$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$

Similarly for other intervals.

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Then

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and $x_1 = x_0 + h$

Similarly, the increment in y for the second interval is computed by

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

Then,
$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and
$$x_2 = x_1 + h$$

and similary for the next intervals.

Example 1 Apply Runge-Kutta Method to solve

$$\frac{dy}{dx} = xy^{1/3}, \quad y(1) = 1 \text{ to obtain } y(1.1).$$

Solution : Here, $x_0 = 1, \ y_0 = 1$ and $h = 0.1$. Then, we can find

$$k_1 = hf(x_0, \ y_0)$$

$$= 0.1(1)(1)^{\frac{1}{3}} = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\left(1 + \frac{0.1}{2}\right)\left(1 + \frac{0.1}{2}\right)^{\frac{1}{3}} = 0.10672$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\left(1 + \frac{0.1}{2}\right)\left(1 + \frac{0.10672}{2}\right)^{\frac{1}{3}} = 0.10684$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Then

$$= 0.1(1 + 0.1)(1 + 0.10684)^{\frac{1}{3}}$$

$$= 0.11378 = 0.11378$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 2 \times 0.10672 + 2 \times 0.10684 + 0.11378)$$

$$= 1 + 0.10682 = 1.10682 \qquad \textbf{Ans}$$

Example 2 Solve the equation $y' = (x + y)$ with $y_0 = 1$ by Runge-Kutta rule from $x = 0$ to

$$x = 0.4 \text{ with } h = 0.1.$$

Solution : Here, $f(x, y) = x + y$, $h = 0.1$, given $y_0 = 1$ when $x_0 = 0$

We have,

$$k_1 = hf(x_0, \ y_0)$$

$$= 0.1(0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1(0.05 + 1.055) = 0.1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Then

$$= 0.1(0.1 + 1.1105)$$

$$= 0.12105$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105)$$

$$= 1.11034$$

Similarly for finding $y_2 = y(x = 0.2)$, we get

$$k_1 = hf(x_1, y_1)$$

$$= (0.1)[\,(0.1) + 1.11034]$$

$$= 0.121034$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= (0.1)[0.15 + 1.11034 + 0.660517]$$

$$= 0.13208$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= (0.1)[0.15 + 1.11034 + 0.06604]$$

$$= 0.13208$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= (0.1)[0.20 + 1.11034 + 0.13263]$$

$$= 0.14263$$

$\therefore \qquad y_2 = y(x = 0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 1.11034 + \frac{1}{6}[0.121034 + 2(0.13208 + 0.13263 + 0.14429)]$$

$$= 1.2428$$

Similarly for finding $y_3 = y(x = 0.3)$, we get

$$k_1 = hf(x_2, y_2)$$

$$= (0.1)[\,(0.2) + 1.2428]$$

$$= 0.14428$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$$

$$= (0.1)[0.25 + 1.2428 + 0.07214]$$

$$= 0.15649$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$

$$= (0.1)[0.25 + 1.2428 + 0.07824]$$

$$= 0.15710$$

$$k_4 = hf(x_2 + h, y_2 + k_3)$$

$$= (0.1)[0.30 + 1.2428 + 0.15710]$$

$$= 0.16999$$

$\therefore$
$$y_3 = y(x = 0.3) = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.3997$$

Similarly for finding $y_4 = y(x = 0.4)$, we

$$k_1 = (0.1)[(0.3) + 1.3997]$$

$$= 0.16997$$

$$k_2 = (0.1)[0.35 + 1.3997 + 0.08949]$$

$$= 0.18347$$

$$k_3 = (0.1)[0.35 + 1.3997 + 0.9170]$$

$$= 0.18414$$

$$k_4 = (0.1)[0.4 + 1.3997 + 0.18414]$$

$$= 0.19838$$

$\therefore$
$$y_4 = 1.3997 + \frac{1}{6}[0.16997 + 2(0.18347 + 0.18414 + 0.19838)]$$

$$y_4 = 1.5836 \quad \textbf{Ans}$$

Example 3 Given $\frac{dy}{dx} = y - x$ with $y(0) = 2$, find $y(0.1)$ and $y(0.2)$ correct to 4 decimal places.

Solution: We have $x_0 = 0$, $y_0 = 2$, $h = 0.1$

Then, we get

$$k_1 = hf(x_0, y_0)$$

$$= 0.1(2 - 0) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1[2 + \frac{0.2}{2} - (0 + \frac{0.1}{2})] = 0.205$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1[2 + \frac{0.205}{2} - (0 + \frac{0.1}{2})] = 0.20525$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Then
$$= 0.1[2 + 0.20525 - (0 + 0.1)]$$

$$= 0.210525$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2 + 0.2051708 = 2.2051708$$

$\Rightarrow \qquad y(0.1) = 2.2052$ \qquad Correct to 4 decimal places

For $y(0.2)$, we have $x_0 = 0.1$, $y_0 = 2.2052$, we get

$$k_1 = hf(x_0, y_0)$$

$$= 0.1(2.2052 - 0.1) = 0.21052$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1[2.2052 + \frac{0.21052}{2} - (0 + \frac{0.1}{2})] = 0.216046$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1[2.2052 + \frac{0.216046}{2} - (0.1 + \frac{0.1}{2})] = 0.2163223$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Then
$$= 0.1[2.2052 + 0.2163223 - (0.1 + 0.1)]$$

$$= 0.22215223$$

$$y(0.2) = 2.2052 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2.2052 + 0.2162348$$

$\Rightarrow \qquad\qquad y(0.2) = 2.4214$ \quad **Ans**

Example 4 Given $\frac{dy}{dx} = -2xy^2$ with $y(0) = 1$, and $h = 0.1$ on the interval $[0.1]$ using Runge-Kutta fourth order method.

Solution: As per given, we have $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$$k_1 = hf(x_0, y_0)$$

$$= -2(0.2)(0)(1)^2 = 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= -2(0.2)\left(\frac{0.2}{2}\right)(1)^2 = -0.4$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= -2(0.2)\left(\frac{0.2}{2}\right)(0.98)^2 = -0.38416$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= -2(0.2)(0.2)(0.961584)^2 = -0.0739715$$

Hence
$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1 + \frac{1}{6}[0 - 0.0 - 0.076832 - 0.0739715]$$

$$= 0.9615328$$

Now, we have $x_1 = 0.2$, $y_1 = 0.9615328$, $h = 0.2$, we get

$$k_1 = hf(x_1, y_1)$$

$$= -2(0.2)(0.2)(0.9615328)^2 = -0.0739636$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= -2(0.2)(0.3)(0.924551)^2 = 0.1025754$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= -2(0.2)(0.3)(0.9102451)^2 = 0.0994255$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= -2(0.2)(0.4)(0.8621073)^2 = -0.1189166$$

Hence $y(0.4) = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$$= 0.9615328 + \frac{1}{6}[-0.0739636 - 0.2051508 -$$
$0.1988510 - 0.1189166]$

$$= 0.8620525$$

Similarly, we can obtain

$$y(0.6) = 0.7352784$$

$$y(0.8) = 0.6097519$$

$$y(1.0) = 0.500073 \qquad \textbf{Ans}$$

Check Your Progress

1. Apply Runge-Kutta method to find the solution of the differential equation

    $$\frac{dy}{dx} = 3x + \frac{1}{2}y \text{ with } y(0) = 1 \text{ and } x = 0.1.$$

8.5 SUMMARY

Many problems in science and engineering can be reduce to the problem of solving differential equations satisfying certain given conditions. These methods are of even greater importance when we realize

that computing machines are now available which reduce numerical work considerably. A number of numerical methods are available for the solution of first order differential equations. These methods yield solutions either as power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y . The methods of Picards and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta belong to the latter class. However Runge-Kutta forth order method is iterative and gives solution to accuracy.

8.6   EXERCISE

1.   Apply Picard's method to find the third approximation of the solution.

$$\frac{dy}{dx} = x + y^2 \quad \text{with the condition } y(0) = 1.$$

$$[\text{Ans} \; = 1 + x \; + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \; ... ...]$$

2.   Using Picard's method , obtain the solution

$$\frac{dy}{dx} = x(1 + x^3 y); \quad y(0) = 3$$

Compute the value of $y(0.1)$ and $y(0.2)$.

[Ans.   3.005, 3.020]

3.   Solve the following initial value problem by Picard method

$$\frac{dy}{dx} = xe^y \quad y(0) = 0, \text{compute } y(0.1)$$

[Ans.   0.0050125]

4.   Use Picard method to approximate y when $x = 0.2$, given that

$$\frac{dy}{dx} = x - y \quad \text{with } y(0) = 1.$$

[Ans.   0.0837]

5.   Given $\frac{dy}{dx} = 1 + y^2$ where $y = 0,$ when $x = 0,$ find $y(0.2), y(0.4)$ and $y(0.6)$ using Runge-Kutta formula of order four.

[**Ans** $y(0.2) = 0.2027, y(0.4) = 0.4228$ and $y(0.6) = 0.6841$]

6.   Use classical Runge-Kutta method of fourth order to find the numerical solution at $x = 1.4$ for $\frac{dy}{dx} = y^2 + x^2, \; y(1) = 0$. Assume step size $h = 0.2$

[**Ans** $y(1.2) = 0.246326$ and $y(1.4) = 0.622751489$]

7. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{2x-1}{x^2}y + 1$, where $x_0 = 1$, $y_0 = 2$ $h = 0.2$.

Obtain $y(1.2)$ and $y(1.4)$ using Runge-Kutta method.

[**Ans** 2.658913 and 3.432851]

ROUGH WORK