U.P.Rajarshi Tandon Open University, Allahabad

MBA-1.4

Quantitative Techniques for Business Decisions

Block 1

लेखक

Dr. Shiv Bhushan Gupta, Asso. Prof. MMPG College, Kalakankar, Pratapgarh

सम्पादक

Prof. S.A.Ansari Ex. Dean, Director and Head MONIRBA, University of Allahabad.

परिमापक

अनुवाद की स्थिति में

मूल लेखक	अनुवाद
मूल सम्पादक	भाषा सम्पादक
मूल परिमापक	परिमापक

सहयोगी टीम

संयोजक Dr. Gaurav Sankalp, SoMS, UPRTOU, Allahabad.

प्रूफ रीडर

प्रकाशन --उत्तर प्रदेश राजर्षि टण्डन मुक्त विश्वविद्यालय, इलाहाबाद

प्रकाशन- उत्तर प्रदेश राजर्षि टण्डन मुक्त विश्वविद्यालय, प्रयागराज की ओर से डॉ. अरूण कुमार गुप्ता, कुलसचिव द्वारा पुनः मुद्रित एवं प्रकाशित वर्ष-2020।

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Block 1 : Quantitative Techniques for Business Decisions

Block Introduction

This Block (1.4) comprises Four units. The first unit of this block is related with Data Analysis, Measure of Dispersion and Standard Deviation while Second Unit of this section is concerned with various theories of Probability. The Third Unit of this block deals with Prior and Posterior Probability. The last unit of this block is related with Mathematical Expectation and Baye's Theorem.

Unit – 1 Data Analysis, Measure of Dispersion, Standard Deviation

Objectives:

After going through this unit you should be able to know about –

1. Data Analysis
2. Measure of Dispersion
3. Standard Deviation

Structure:

1.1. Introduction

1.2. Editing

1.3. Coding

1.4. Classification

1.5. Tabulation

1.6. Types of Analysis

1.7. Measure of Dispersion

1.8. Objectives of Dispersion

1.9. Different Measures of Dispersion

1.10. Range

1.11. Inter Quartile Range

1.12. Semi Inter-Quartile Range

1.13. Mean Deviation

1.14. Standard Deviation

1.15. Conclusion

1.16. Further Study

1.1. Introduction

The data, after collection, has to be processed and analysed in accordance with the outline laid down for the purpose at the time of developing the research plan. This is essential for a scientific study and for ensuring that we have all relevant data for making contemplated comparisons and analysis. Technically speaking, processing implies editing, coding, classification and tabulation of collected data so that they are amenable to analysis. The term analysis refers to the computation of certain measures along with searching for patterns of relationship that exist among data-groups. Thus, "in the process of analysis, relationships or differences supporting or conflicting with original or new hypotheses should be subjected to statistical tests of significance to determine with what validity data can be said to indicate any conclusions". But there are persons (Selltiz, Jahoda and others) who do not like to make difference between processing and analysis. They opine that analysis of data in a general way involves a number of closely related operations which are performed with the purpose of summari-sing the collected data and organising these in such a manner that they answer the research question(s). We, however, shall prefer to observe the difference between the two terms as stated here in order to understand their implications more clearly.

With this brief introduction concerning the concepts of processing and analysis, we can now proceed with the explanation of all the processing operations.

1.2 Editing

Editing of data is a process of examining the collected raw data (specially ill surveys) to detect errors and omissions and to correct these when possible. As a matter of fact, editing involves a careful scrutiny of the completed questionnaires and/or schedules. Editing is done to assure that the data are accurate, consistent with other facts gathered, uniformly entered, as completed as possible and have been well arranged to facilitate coding and tabulation.

With regard to points or stages at which editing should be done, one can talk of field editing and central editing. *Field editing* consists in the review of the reporting forms by the investigator for completing (translating or rewriting) what the latter has written in abbreviated and/ or in illegible form at the time of recording the respondents' responses. This type of editing is necessary in view of the fact that individual writing styles often can be difficult for others to decipher. This sort of editing should be done as soon as possible after the interview, preferably on the very day or on the next day. While doing field editing, the investigator must restrain himself and must not correct errors of omission by simply guessing what the informant would have said if the question had been asked.

Central editing should take place when all forms or schedules have been completed and returned to the office. This type of editing implies that all forms should get a thorough editing by a single editor in a small study and by a team of editors in case of a large inquiry. Editor(s) may correct the obvious errors such as an entry in the wrong place, entry recorded in months when it should have been recorded in weeks, and the like. In case of inappropriate on missing replies, the editor can sometimes determine the proper answer by reviewing the other information in the schedule. At times, the respondent can be contacted for clarification. The editor must strike out the answer if the same is inappropriate and he has no basis for determining the correct answer or the response. In such a case an editing entry of 'no answer' is called for. All the wrong replies, which are quite obvious, must be dropped from the final results, especially in the context of mail surveys.

Editors must keep in view several points while performing their work: (a) They should be familiar with instructions given to the interviewers and coders as well as with the editing instructions supplied to them for the purpose. (b) While crossing out an original entry for one reason or another, they should just draw a single line on it so that the same may remain legible. (c) They must make entries (if any) on the form in some distinctive colour and that too in a standardised form. (d) They should initial all answers which they change or supply. (e) Editor's initials and the date of editing should be placed on each completed form or schedule.

1.3. Coding

Coding refers to the process of assigning numerals or other symbols to answers so that responses can be put into a limited number of categories or classes. Such classes should be appropriate to the research problem under consideration. They must also possess the characteristic of exhaustiveness (i.e., there must be a class for every data item) and also that of mutual exclusively which means that a specific answer can be placed in one and only one cell in a given category set. Another rule to be observed is that of unidimensionality by which is meant that every class is defined in terms of only one concept.

Coding is necessary for efficient analysis and through it the several replies may be reduced to a small number of classes which contain the critical information required for analysis. Coding decisions should usually be taken at the designing stage of the questionnaire. This makes it possible to precode the questionnaire choices and which in turn is helpful for computer tabulation as one can straight forward key punch from the original questionnaires. But in case of hand coding some standard method may be used. One such standard method is to code in the margin with a coloured pencil. The other method can be to transcribe the data from the questionnaire to a coding sheet. Whatever method is adopted, one should see that coding errors are altogether eliminated or reduced to the minimum level.

1.4. Classification

Most research studies result in a large volume of raw data which must be reduced into homogeneous groups if we are to get meaningful relationships. This fact necessitates classification of data which happens to be the process of arranging data in groups or classes on the basis of common characteristics. Data having a common characteristic are placed in one class and in this way the entire data get divided into a number of groups or classes. Classification can be one of the following two types, depending upon the nature of the phenomenon involved :

(a) **Classification according to attributes :** As stated above, data are classified on the basis of common characteristics which can either be descriptive (such as literacy, sex, honesty, etc.) or numerical (such as weight, height, income, etc.). Descriptive characteristics refer to qualitative phenomenon which cannot be measured quantitatively; only their presence or absence in an individual item can be noticed. Data obtained this way on the basis of certain attributes are known as *statistics of attributes* and their classification is said to be classification according to attributes.

Such classification can be simple classification or manifold classification. In simple classification we consider only one attribute and divide the universe into two classes-one class consisting of items possessing the given attribute and the other class consisting of items which do not possess the given attribute. But in manifold classification we consider two or more attributes simultaneously, and divide that data into a number of classes (total number of classes of final order is given by 2^n, where n = number of attributes considered). Whenever data are classified according to attributes, the researcher must see that the attributes are defined in such a manner that there is least possibility of any doubt/ambiguity concerning the said attributes.

(b) **Classification according to class-intervals :** Unlike descriptive characteristics, the numerical characteristics refer to quantitative phenomenon which can be measured through some statistical units. Data relating to income, production, age, weight, etc. come under this category. Such data are known as *statistics of variables* and are classified on the basis of class intervals. For instance, persons whose incomes, say, are within Rs 201 to Rs 400 can form one group, those whose incomes are within Rs 401 to Rs 600 can form another group and so on. In this way the entire data may be divided into a number of groups or classes or what are usually called, 'class-intervals.' Each group of class-interval, thus, has an upper-limit as well as a lower limit which are known as class limits. The difference between the two class limits is known as class magnitude. We may have classes with equal class magnitudes or with unequal class magnitudes. The number of items which fall in a given class is known as the frequency of the given class. All the classes or groups, with their respective frequencies taken together and put in the form of a table, are described as group frequency distribution or simply frequency distribution. Classification according to class intervals usually involves the following three main problems:

(i) How may classes should be there? What should be their magnitudes?

There can be no specific answer with regard to the number of classes. The decision about this calls for skill and experience of the researcher. However, the objective should be to display the data in such a way as to make it meaningful for the analyst. Typically, we may have 5 to 15 classes. With regard to the second part of the question, we can say that, to the extent possible, class-intervals should be of equal magnitudes, but in some cases unequal magnitudes may result in better classification. Hence the researcher's objective judgement plays an important part in this connection. Multiples of 2, 5 and 10 are generally preferred while determining class magnitudes. Some statisticians adopt the following formula, suggested by H.A. Sturges, determining the size of class interval:

$$i = R/(1 + 3.3 \log N)$$

where

i = size of class interval;

R = Range (i.e., difference between the values of the largest item and smallest item among the given items);

N = Number of items to be grouped.

It should also be kept in mind that in case one or two or very few items have very high or very low values, one may use what are known as open-ended intervals in the overall frequency distribution. Such intervals may be expressed like under Rs 500 or Rs 10001 and over. Such intervals are generally not desirable, but often cannot be avoided. The researcher must always remain conscious of this fact while deciding the issue of the total number of class intervals in which the data are to be classified.

(ii) How to choose class limits?

While choosing class limits, the researcher must take into consideration the criterion that the mid-point (generally worked out first by taking the sum of the upper limit and lower limit of a class and then divide this sum by 2) of a class-interval and the actual average of items of that class interval should remain as close to each other as possible. Consistent with this, the class limits should be located at multiples of 2, 5, 10, 20, 100 and such other figures. Class limits may generally be stated in any of the following forms:

Exclusive type class intervals: They are usually stated as follows:

10–20

20–30

30–40

40–50

The above intervals should be read as under: 10 and under 20

20 and under 30

30 and under 40

40 and under 50

Thus, under the exclusive type class intervals, the items whose values are equal to the upper limit of a class are grouped in the next higher class. For example, an item whose value is exactly 30 would be put in 30–40 class interval and not in 20–30 class interval. In simple words, we can say that under exclusive type class intervals, the upper limit of a class interval is excluded and items with values less than the upper limit, (but not less than the lower limit) are put in the given class interval.

Inclusive type class intervals: They are usually stated as follows:

11–20 , 21–30, 31–40, 41–50

In inclusive type class intervals the upper limit of a class interval is also included in the concerning class interval. Thus, an item whose value is 20 will be put in 11–20 class interval. The stated upper limit of the class interval 11–20 is 20 but the real limit is 20.99999 and as such 11–20 class interval really means 11 and under 21.

When the phenomenon under consideration happens to be a discrete one (i.e., can be measured and stated only in integers), then we should adopt inclusive type classification. But when the phenomenon happens to be a continuous one capable of being measured in fractions as well, we can use exclusive type class intervals.

(iii) How to determine the frequency of each class?

This can be done either by tally sheets or by mechanical aids. Under the technique of tally sheet, the class-groups are written on a sheet of paper (commonly known as the tally sheet) and for each item a stroke (usually a small vertical line) is marked against the class group in which it falls. The general practice is that after every four small vertical lines in a class group, the fifth line for the item falling in the same group, is indicated as horizontal line through the said four lines and the resulting flower (IIII) represents five items. All this facilitates the counting of items in each one of the class groups. An illustrative tally sheet can be shown as under:

Table 1. An Illustrative Tally Sheet for Determining the Number of 70 Families in Different Income Groups.

Income groups (Rupees)	Tally mark	Number of families or (Class frequency)
Below 400	~~IIII~~ ~~IIII~~ III	13
401-800	~~IIII~~ ~~IIII~~ ~~IIII~~ ~~IIII~~	20
801-1200	~~IIII~~ ~~IIII~~ III	12
1201-1600	~~IIII~~ ~~IIII~~ ~~IIII~~ III	18
1601 and above	~~IIII~~ II	7
Total		**70**

Alternatively, class frequencies can be determined, specially in case of large inquires and surveys, by mechanical aids i.e., with the help of machines viz., sorting machines that are available for the purpose. Some machines are hand operated, whereas other work with electricity. There are machines which can sort out cards at a speed of something like 25000 cards per hour. This method is fast but expensive.

1.5 Tabulation

When a mass of data has been assembled, it becomes necessary for the researcher to arrange the same in some kind of concise and logical order. This procedure is referred to as tabulation. Thus, tabulation is the process of summarising raw data and displaying the same in compact form (i.e., in the form of statistical tables) for further analysis. In a broader sense, tabulation is an orderly arrangement of data in columns and rows.

Tabulation is essential because of the following reasons.

1. It conserves space and reduces explanatory and descriptive statement to a minimum.
2. It facilitates the process of comparison.
3. It facilitates the summation of items and the detection of errors and omissions.

4. It provides a basis for various statistical computations.

Tabulation can be done by hand or by mechanical or electronic devices. The choice depends on the size and type of study, cost considerations, time pressures and the availability of tabulating machines or computers. In relatively large inquiries, we may use mechanical or computer tabulation. if other factors are favourable and necessary facilities are available. Hand tabulation is usually preferred in case of small inquiries where the number of questionnaires is small and they are of relatively short length. Hand tabulation may be done using the direct tally, the list and tally pr the card sort and count methods. When there are simple codes, it is feasible to tally directly from the questionnaire. Under this method, the codes are written on a sheet of paper, called tally sheet, and for each response a stroke is marked against the code in which it falls. Usually after every four strokes against a particular code, the fifth response is indicated by drawing a diagonal or horizontal line through the strokes. These groups of five are easy to count and the data are sorted against each code conveniently. In the listing method, the code responses may be transcribed onto a large work-sheet, allowing a line for each questionnaire. This way a large number of questionnaires can be listed on one work sheet. Tallies are then made for each question. The card sorting method is the most flexible hand tabulation. In this method the data are recorded on special cards of convenient size and shape with a series of holes. Each hole stands for a code and when cards are stacked, a needle passes through particular hole representing a particular code. These cards are then separated and counted. In this way frequencies of various codes can be found out by the repetition of this technique. We can as well use the mechanical devices or the computer facility for tabulation purpose in case we want quick results, our budget permits their use and we have a large volume of straight forward tabulation involving a number of cross-breaks.

Tabulation may also be classified as simple and complex tabulation. The former type of tabulation gives information about one or more groups of independent questions, whereas the latter type of tabulation shows the division of data in two or more categories and as such is deigned to give information concerning one or more sets of inter-related questions. Simple tabulation generally results in one-way tables which supply answers to questions about one characteristic of data only. As against this, complex tabulation usually results in two-way tables (which give information about two inter-related characteristics of data), three-way tables (giving information about three interrelated characteristics of data) or still higher order tables, also known as manifold tables, which supply information about several interrelated characteristics of data. Two-way tables, three-way tables or manifold tables are all examples of what is sometimes described as cross tabulation.

Generally Accepted Principles of Tabulation

Such principles of tabulation, particularly of constructing statistical tables, can be briefly states as follows :

1. Every table should have a clear, concise and adequate title so as to make the table intelligible without reference to the text and this title should always be placed just above the body of the table.

2. Every table should be given a distinct number to facilitate easy reference.

3. The column headings (captions) and the row headings (stubs) of the table should be clear and brief.

4. The units of measurement under each heading or sub-heading must always be indicated.

5. Explanatory footnotes, if any, concerning the table should be placed directly beneath the table, along with the reference symbols used in the table.

6. Source or sources from where the data in the table have been obtained must be indicated just below the table.

7. Usually the columns are separated from one another by lines which make the table more readable and attractive. Lines are

always drawn at the top and bottom of the table and below the captions.

8. There should be thick lines to separate the data under one class from the data under another class and the lines separating the sub-divisions of the classes should be comparatively thin lines.

9. The columns may be numbered to facilitate reference.

10. Those columns whose data are to be compared should be kept side by side. Similarly, percentages and/or averages must also be kept close to the data.

11. It is generally considered better to approximate figures before tabulation as the same would reduce unnecessary details in the table itself.

12. In order to emphasise the relative significance of certain categories, different kinds of type, spacing and indentations may be used.

13. It is important that all column figures be properly aligned. Decimal points and (+) or (–) signs should be in perfect alignment.

14. Abbreviations should be avoided to the extent possible and ditto marks should not be used in the table.

15. Miscellaneous and exceptional items, if any, should be usually placed in the last row of the table.

16. Table should be made as logical, clear, accurate and simple as possible. If the data happen to be very large, they should not be crowded in a single table for that would make the table unwieldy and inconvenient.

17. Total of rows should normally be placed in the extreme right column and that of columns should be placed at the bottom.

18. The arrangement of the categories in a table may be chronological, geographical, alphabetical or according to magnitude to facilitate comparison. Above all, the table must suit the needs and requirements of an investigation.

Some Problems in Processing

We can take up the following two problems of processing the data for analytical purposes :

(a) **The problem concerning "Don't know" (or DK) responses :** While processing the data, the researcher often comes across some responses that are difficult to handle. One category of such responses may be 'Don't Know Response' or simply DK response. When the DK response group is small, it is of little significance. But when it is relatively big, it becomes a matter of major concern in which case the question arises: Is the question which elicited DK response useless? The answer depends on two points viz., the respondent actually may not know the answer or the researcher may fail in obtaining the appropriate information. In the first case the concerned question is said to be alright and DK response is taken as legitimate DK response. But in the second case, DK response is more likely to be a failure of the questioning process.

How DK responses are to be dealt with by researchers? The best way is to design better type of questions. Good rapport of interviewers with respondents will result in minimising DK responses. But what about the DK responses that have already taken place? One way to tackle this issue is to estimate the allocation of DK answers from other data in the questionnaire. The other way is to keep DK responses as a separate category in tabulation where we can consider it as a separate reply category if DK responses happen to be legitimate, otherwise we should let the reader make his own decision. Yet another way is to assume that DK responses occur more or less randomly and as such we may distribute them among the other answers in the ratio in which the latter have occurred. Similar results will be achieved if all DK replies are excluded from tabulation and that too without inflating the actual number of other responses.

(b) **Use or percentages :** Percentages are often used in data presentation for they simplify numbers, reducing all of them to a a to 100 range. Through the use of percentages; the data are reduced in the standard form with base equal to 100 which fact facilitates relative comparisons. While using percentages, the following rules should be kept in view by researchers:

1. Two or more percentages must not be averaged unless each is weighted by the group ,size from which it has been derived.

2. Use of too large percentages should be avoided, since a large percentage is difficult to understand and tends to confuse, defeating the very purpose for which percentages are used.

3. Percentages hide the base from which they have been computed. If this is not kept in view, . the real differences may not be correctly read.

4. Percentage decreases can never exceed 100 per cent and as such for calculating the percentage of decrease, the higher figure should invariably be taken as the base.

5. Percentages should generally be worked out in the direction of the causal-factor in case of two-dimension tables and for this purpose we must select the more significant factor out of the two given factors as the causal factor.

1.6. Types of Analysis

As stated earlier, by analysis we mean the computation of certain indices or measures along with searching for patterns of relationship that exist among the data groups. Analysis, particularly in case of surveyor experimental data, involves estimating the values of unknown parameters of the population and testing of hypotheses for drawing inferences. Analysis may, therefore, be categorised as descriptive analysis and inferential analysis (Inferential analysis is often known as statistical analysis). "*Descriptive analysis* is largely the study of distributions of one variable. This study provides us with profiles of companies, work groups, persons and other subjects on any of a multiple of characteristics such as size. Composition, efficiency, preferences, etc." this sort of analysis may be in respect of one variable (described as unidimensional analysis), or in respect of two variables (described as bivariate analysis) or in respect of more than two variables (described as multivariate analysis). In this context we work out various measures that show the size and shape of a distribution(s) along with the study of measuring relationships between two or more variables.

We may as well talk of correlation analysis and causal analysis. *Correlation analysis* studies the joint variation of two or more variables for determining the amount of correlation between two or more variables. *Causal analysis* is concerned with the study of how one or more variables affect changes in another variable. It is thus a study of functional relationships existing between two or more variables. This analysis can be termed as regression analysis. Causal analysis is considered relatively more important in experimental researches, whereas in most social and business researches our interest lies in understanding and controlling relationships between variables then with determining causes per *se* and as such we consider correlation analysis as relatively more important.

In modem times, with the availability of computer facilities, there has been a rapid development of *multivariate analysis* which may be defined as "all statistical methods which simultaneously analyse more than two variables on a sample of observations". Usually the following analyses are involved when we make a reference of multivariate analysis:

(a) **Multiple regression analysis :** This analysis is adopted when the researcher has one dependent variable which is presumed to be a function of two or more independent variables. The objective of this analysis is to make a prediction about the dependent variable based on its covariance with all the concerned independent variables.

(b) **Multiple discriminant analysis :** This analysis is appropriate when the researcher has a single dependent variable that cannot be measured, but can be classified into two or more groups on the basis of some attribute. The object of this analysis happens to be to predict an entity's possibility of belonging to a particular group based on several predictor variables.

(c) **Multivariate analysis of variance (or multi-ANOVA) :** This analysis is an extension of two-way ANOVA, wherein the ratio of among group variance to within group variance is worked out on a set of variables.

(d) **Canonical analysis :** This analysis can be used in case of both measurable and non-measurable variables for the purpose of simultaneously predicting a set of dependent variables from their joint covariance with a set of independent variables.

Inferential analysis is concerned with the various tests of significance for testing hypotheses in order to determine with what validity data can be said to indicate some conclusion or conclusions. It is also concerned with the estimation of population values. It is mainly on the basis of inferential analysis that the task of interpretation (i.e., the task of drawing inferences and conclusions) is performed.

1.7. Measures of Dispersion

Introduction : Averages fail to reveal the Full details of the distribution. Two or Three distributions may have the same average but still they may differ from each other in many ways. In such cases, rather statistical analysis of the data is necessary so that these differences between various series can be studied and accounted for. Such analysis will make our results more accurate and we shall be more confident of our conclusions.

Suppose, there are three series of nine items each as follows:

Series A	Series B	Series C
40	36	1
40	37	9
40	38	20

40	39	30
40	40	40
40	41	50
40	42	60
40	43	70
40	44	80
Total 360	360	360
Mean 40	40	40

In the first series, the mean is 40 and the value of all the items is identical. The items are not at all scattered, and the mean fully discloses the characteristics of this distribution. However, in the second case, though the mean is 40 yet all the items of the series have different values. But the items are not very much scattered as the minimum value of the series is 36 and the maximum is 44 in the range. In this case also, mean is a good representative of the series because the difference between the mean and other items is not very significant. In the third series also, the mean is 40 and the values of different items are also different, but here the values are very widely scattered and the mean is 40 times of the smallest value of the series and half of the maximum value. Though the mean is the same in all the three series, yet the series differ widely from each other in their formation. Obviously, the average does not satisfactorily represent the individual items in this group and to know about the series completely, further analysis is essential. The scatter among the items in the first case is nil, in the second case it varies within a small range, while in the third case the values range between a very big span and they are widely scattered. It is evident from the above, that a study of the extent of the scatter around average should also be made to throw more light on the composition of a series. **The name given to this scatter is dispersion.**

Definition

Some important definitions of dispersion are given below:

(i) "Dispersion or spread is the degree of the scatter or variation of the variable about a central value." – *Brooks and Dick*

(ii) "Dispersion is the measure of the variations of the items."
– *A.L. Bowley*

(iii) "The degree to which numerical data tend to spread about an average value is called the variation or dispersion of the data." – *Spiegel*

From the above definitions, it is clear that in a general sense the term dispersion refers to the variability in the size of items. If the variation is substantial, dispersion is said to be considerable and if the variation is very little, dispersion is insignificant.

Usually, in a precise study of dispersion the deviations of size of items from a measure of central tendency are found out and then these deviations are averaged to give a single figure representing the dispersion of the series. This figure can be compared with similar figures representing other series. Such comparisons give a better idea about the formation of series than a mere comparison of their averages.

Averages of second order : For a precise study of dispersion, we have to average deviations of the values of the various items, from their average. We have seen earlier that arithmetic mean, median, mode, geometric mean and harmonic mean, etc., are all averages of the first order. Since in the calculation of measures of dispersion, the average values are derived by the use of the averages of the first order, the measures of dispersion are called averages of the second order.

1.8. Objects of Measuring Dispersion

Measures of variations are calculated to serve the following purposes:

(i) To judge the reliability of measures of central tendency.

(ii) To make a comparative study of the variability of two series.

(iii) To identify the causes of variability with a view to control it.

Spur and **Bonimi** have very rightly observed that, "in matters of health, variations in body temperature, pulse beats and blood pressure are basic guides to diagnosis. Prescribed treatment is designed to control their variation. In industrial production, efficient operation requires control of quality variation, the causes of which are sought through inspection and quality control programs."

In Social Sciences where we have to study problems relating to inequality in income and wealth, measures of dispersion are of immense help.

(iv) To serve as a basis for further statistical analysis.

Characteristics of A Good Measure of Dispersion

The properties of a good measure of dispersion are the same as the properties of a good measure of central tendency. Precisely, they are:

(i) It should be rigidly defined.

(ii) It should be based on all the observations of the series.

(iii) It should be capable of further algebraic treatment.

(iv) It should be easy to calculate and simple to follow.

(v) It should not be affected by fluctuations of sampling.

1.9. Different Measures of Dispersion

Absolute and relative "dispersion : Dispersion or variation can be expressed either in terms of the original units of a series or as an abstract figure like a ratio or percentage. If we calculate dispersion of a series relating to the income of a group of persons in absolute figures, it will have to be expressed in the unit in which the original data are, say, rupees. Thus, we can say that the income of a group of persons is Rs. 5000 per month and the dispersion is Rs. 500. This is called Absolute Dispersion. If, on the other hand, dispersion is measured as a percentage or ratio of the average, it is called Relative Dispersion. Since the relative dispersion is a ratio, it has no units. In the above case, the average income would be referred to as Rs. 5000 per month and the relative dispersion $\frac{500}{5000} = 0.1$ or 10%. In a comparison of the variability of two or more series, it is the relative dispersion that has to be taken into account as the absolute dispersion may be erroneous or unfit for comparison if the series are originally in different units.

1.10. Range

Range is the simplest possible measure of dispersion. It is the difference between the values of the extreme items of a series. Thus, if in a series relating to the weight measurements of a group of students, the lightest student has a weight of 40 kg. and the heaviest, of 110 kg. The value of range would be 110 − 40 = 70 kg. This figure indicates the variability in the weight of students.

Symbolically,

$$\text{Range (R)} = L - S$$

where, L is the largest value and S the smallest value in a series.

Range as calculated above is an absolute measure of dispersion which is unfit for purposes of comparison if the distributions are in different units. For example, the range of the weights of students cannot be compared with the range of their height measurements as the range of weights would be in kg. and that of heights in centimetres. Sometimes, for purpose of comparison, a relative measure of range is calculated. If range is divided by the sum of the extreme items, the resulting figure is called "The Coefficient of the Range" or "The Coefficient of the Scatter."

Symbolically,

The Ratio of Range or the Coefficient of the scatter (or Range)

$$= \frac{\text{Max. value} - \text{Min. value}}{\text{Max. value} + \text{Min. value}} = \frac{L - S}{L + S}$$

$$= \frac{\text{Absolute range}}{\text{Sum of the extreme values}}$$

The following illustration would illustrates the use of the above formulae

Example 2. The profits of a Company for the last 8 years are given below. Calculate the Range and its Coefficient:

Year	1975	1976	1977	1978	1979	1980	1981	1982
Profits (in '000 Rs.)	40	30	80	100	120	90	200	230

Solution.

Here, L = 230 and S = 30

Range = L – S = 230 – 30 = 200

Coefficient of Range = $\frac{L-S}{L+S}$ or $\frac{230-30}{230+30}$ or $\frac{200}{260} = 0.77$

Example 3. Calculate Co-efficient of Range from the following data :

Weekly Wages (Rs.)	No. of Labourers
50–60	50
60–70	45
70–80	45
80–90	40
90–100	35
100–110	30
110–120	30

Solution.

Coefficient of Range (first method) = $\frac{L-S}{L+S}$

Here, L = 120 and S = 50. such the

Coefficient of Range = $\frac{120-50}{120+50} = \frac{70}{170} = 0.41$

Coefficient of Range (second method) = $\frac{L-S}{L+S}$ Here, L = 115 and S = 55

Co-efficient of Range = $\frac{115-55}{115+55} = \frac{60}{70} = 0.35$

Example 4. Find the Range and the Co-efficient of range for the following observations 65, 70, 59, 81, 76,57,60,55, and 50.

Solution. Highest value = 82

Lowest value = 50

Range = 82 – 50 = 32

Coefficient of Range = $\frac{82-50}{82+50} = \frac{32}{132}$ 0.2424

1.11. Inter-Quartile Range

Third quartile (Q_3)

$$IR = Q_3 - Q_1$$

Inter-quartile range — First quartile (Q_1)

Just as in a case of range the difference of extreme items is found, similarly, if the difference in the values of two quartiles is calculated, it would give us what is called the Inter-Quartile Range. Inter-Quartile range is also a measure of dispersion. It has an advantage over range, in as much as, it is not affected by the values of the extreme items. In fact, 50% of the values of a variable are between the two quartiles and as such the inter-quartile range gives a fair measure of variability. However, the inter-quartile range suffers from the same defects from which range suffers. It is also affected by fluctuations of sampling and IS not based on all the observations of a series.

1.12. Semi-Inter-Quartile Range

Semi-inter-quartile range, as the name suggests is the midpoint of the inter- quartile range. In other words, it is one-half of the difference between the third quartile and the first quartile.
Symbolically,

Semi-inter-quartile range or quartile deviation = $\frac{Q_3 - Q_1}{2}$

where Q_3 and Q_1 are the upper and lower quartiles respectively.

In a symmetrical series median lies halfway on the scale form Q_1 to Q_3. In a symmetrical distribution Q_3 – median = Median – Q_1 or median $= \frac{Q_3 - Q_1}{2} = Q_3 - \left(\frac{Q_3 - Q_1}{2}\right) = Q_1 + \left(\frac{Q_3 - Q_1}{2}\right)$. If, therefore, the value of the quartile deviation is added to the lower quartile or subtracted from the upper quartile in a symmetrical series, the resulting figure would be the value of the median. But, generally series are not symmetrical and in a moderately asymmetrical series Q_1 + quartile deviation or Q_3 – quartile deviation would not give the value of the median. There would be a difference between the two figures and the greater the difference, the greater would be the extent of departure from normality.

Quartile deviation is an absolute measure of dispersion. If it is divided by the average value of the two quartiles, a relative measure of dispersion is obtained. It is called the Co-efficient of Quartile deviation.

$$\text{Co-efficient of a quartile deviation} = \frac{\frac{Q_3 - Q_1}{2}}{\frac{Q_3 + Q_1}{2}} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

Example 5. Find the Quartile Deviation and its Coefficient from the following data, relating to the weekly wages of seven labourers :

Weekly Wages (Rs.)	50	70	80	60	65	40	90

Solution

Calculation of Quartile Deviation

Wages arranged in ascending order would be as follows :

40 50 60 65 70 80 90

$$Q_1 = \text{the value of} \left(\frac{N+1}{4}\right)^{th} \text{or} \left(\frac{7+1}{4}\right)^{th} \text{or } 2^{nd} \text{ item} = \text{Rs. } 50$$

$$Q_3 = \text{the value of } 3\left(\frac{N+1}{4}\right)^{th} \text{or} \left(\frac{7+1}{4}\right)^{th} \text{or } 6^{th} \text{ item} = \text{Rs. } 80$$

$$\text{Quartile Deviation} = \frac{Q_3 - Q_1}{2} = \frac{80-50}{2} = 15$$

$$\text{Coefficient of Quartile Deviation} = \frac{Q_3 - Q_1}{Q_2 + Q_1} = \frac{30}{130} = 0.23$$

Discrete Series

Example 6. Calculate Quartile Deviation and its Coefficient from the following data:

Weight (in pounds)	120	122	124	126	130	140	150	160
No. of Student	1	3	5	7	10	3	1	1

Solution

Computation of Quartile Deviation and its Coefficient

Weight (in pounds)	120	122	124	126	130	140	150	160
Frequency	1	3	5	7	10	3	1	1
Cumu. Frequency	1	4	9	16	26	29	30	31

$$Q_1 = \text{Size of}\left(\frac{N+1}{4}\right)^{th} \text{ or } \left(\frac{31+1}{4}\right)^{th} \text{ or } 8^{th} \text{ item} = 124$$

$$Q_3 = \text{Size of}\left(\frac{31+1}{4}\right)^{th} \text{ or } 24^{th} \text{ item} = 130$$

$$\text{Quartile Deviation} = \frac{Q_3 - Q_1}{2} = \frac{130-124}{2} = 2\ \textit{pounds.}$$

$$\text{Coefficients of Q.D.} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{130-124}{130+124} = \frac{6}{254} = 0.0236$$

Continuous Series (Exclusive)

Example 7. Calculate Semi-Inter-Quartile Range and its Coefficient of Q.D. from the following data:

Marks	No. of Students
0–10	11
10–20	18
20–30	25
30–40	28
40–50	30
50–60	33
60–70	22
70–80	15
80–90	22

Solution

Calculation of Quartile Deviation

Marks (X)	Frequency (f)	Cumulative Frequency (cf)
0–10	11	11
10–20	18	29
20–30	25	54
30–40	28	82
40–50	30	112
50–60	33	145
60–70	22	167
70–80	15	182
80–90	22	204

Q_1 = the value of $\left(\frac{204}{4}\right)^{th}$ or 51st item which is in 20–30 group.

Q_3 = the value of $\left(\frac{204}{4}\right)^{th}$ or 153rd item which is in 60–70 group.

$$\text{The value of } Q_1 = l_1 + \frac{l_2 - l_1}{f_1}\left(\frac{N}{4} - c\right) = 20 + \frac{10}{25}(51 - 29) = 2.8$$

$$\text{The value of } Q_3 = l_1 + \frac{l_2 - l_1}{f_1}\left(\frac{3N}{4} - c\right) = 60 + \frac{10}{25}(153 - 145) = 63.64$$

$$\text{Q.D.} = \text{marks } \frac{Q_3 - Q_1}{2} = \frac{63.64 - 28.8}{2} = 17.42\,marks$$

$$\text{Coefficient of Q.D.} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{63.64 - 28.8}{63.64 + 28.8} = 0.37$$

1.13. Mean Deviation

The range, the inter-quartile range and the quartile deviation suffer from a common defect, I.e., they are calculated by taking into account only two values of a series--either the extreme values as in case of range, or the values of the quartiles as in case of quartile deviation. This method of studying dispersion (by location of limits) is also called the "Method of Limits". As we have seen earlier, such measures of dispersion suffer from many limitations and have many demerits.

It is, therefore, always better to have such a measure of dispersion which takes into account all the observations of a series and is calculated in relation to a central value. The method of calculating dispersion by calculating the deviations of all the values about a central value is called the method of averaging deviations. In this method, the deviations of items from a measure of central tendency are averaged to study the dispersion of the series. Mean deviation is such a measure of dispersion, "Mean deviation of a series is the arithmetic average of the absolute deviations of various items from a measure of central tendency (either mean, median or mode).

Theoretically, deviations can be taken from any of the three averages mentioned above but in actual practice, mean deviation is calculated either from mean or from median. Mode is usually not considered, as its value is indeterminate, and it gives erroneous conclusions. Between mean and median the median is supposed to be better than the mean, because the sum of the Absolute deviations from the medians is always less than the sum of the absolute deviations from the mean. Mean deviation is also known as the first moment of dispersion. Symbolically,

(i) $$\partial\overline{x} = \frac{\sum \partial\overline{x}}{N}$$

where $\partial\overline{x}$ stands for the mean deviation from mean, $\partial\overline{x}$ for the deviations from the mean, and N for the number of items.

(ii) $$\partial m = \frac{\sum \partial m}{N}$$

where stands for the mean deviation from median, for the deviations from the median, and N for the number of items.

(iii) $\partial z = \frac{\sum \partial z}{N}$

where ∂z stands for the mean deviation from mode, ∂z for deviations from the mode, and N for the number of items.

Mean deviation or first moment of dispersion, as calculated above would be an absolute measure of dispersion, expressed in the same units in which the original data are. In order to transform it into a relative measure, it is divided by the average from which it has been calculated. It is then known as the Mean coefficient of dispersion.

Thus, mean co-efficient of dispersion from mean, and median would be repetitively: $\frac{\delta \overline{X}}{X}, \frac{\delta M}{M}$ *and* $\frac{\delta Z}{Z}$

Calculation of Mean Deviation and its Coefficient

1. Series of Individual observations: There are two methods of calculating mean deviation from a series of individual observation: one is the Direct Method and the other is the Short-cut Method.

Direct method: In this method, the mean deviation is calculated by totalling the deviations from the mean, median or mode (plus, minus signs ignored) and dividing the total by the number of items. Thus,

$$\text{Mean Deviation} = \frac{\sum |d|}{N}$$

Short-cut method: In this method. mean or median is calculated and the total of the values of the items below the mean or median and above it are found out. The former is subtracted from the latter and divided by the number of items. The resulting figure is the Mean Deviation:

Symbolically,

(i) $\delta M = \frac{1}{N}(My - Mx)$

where δM stands for the mean deviation from median, My for the total of the values above the actual median. and Mx for total of the values below the median and N for the number of items.

(ii) $$\delta\overline{X} = \frac{1}{N}(\overline{X}y - \overline{X}x)$$

where δM stands for the mean deviation from mean, $\overline{X}y$ stands for the total of the values above the actual arithmetic average and $\overline{X}x$ for the total of the values below the actual A.M. The following examples would illustrate these formulae :

Example 8. The following are the marks obtained by a batch of 9 students in a certain test:

Si. No.	Marks (Out of 100)	Sl. No.	Marks (Out of 100)
1	68	6	38
2	49	7	59
3	32	8	66
4	21	9	41
5	54		

Calculate the mean deviation of the series.

Solution.

Direct method. Calculation of mean deviation of the series of marks of 9 students (arranged in ascending order of magnitude).

Students	Marks(X)	Deviations from median (49) (+and– signs ignored)(dM)
1	21	28
2	32	17
3	38	11
4	41	8

5	49	0
6	54	5
7	59	10
8	66	17
9	68	19
		$\sum \| Dm \| = 115$

Median = value of $\left(\frac{N+1}{2}\right)^{th}$ item = 49 marks.

Mean deviation from the median or $\delta M = \frac{\sum |dM|}{N}$

where $\sum |dM|$ represents the summation of the deviations from the median, and N, the number of items.

$$\delta M = \frac{115}{9} \text{ marks } = 12.8 \text{ marks.}$$

Short-cut Method : Marks arranged in ascending order of magnitude.

Marks (X)	21	32	38	41	49	54	59	66	68

Sum of items above median (with values less than median)=

$(21 + 32 + 38 + 41) = 132$ (Mx)

Sum of items below median (with values more than median) =

$(54 + 59 + 66 + 68) = 247$ (My)

Mean Deviation = $\frac{1}{N}(My - Mx) = \frac{1}{9}(247 - 132) = \frac{1}{9} \times 115 = 12.8$ *Marks*

Example 9. Calculate the mean deviation (from mean) of the following

data:

Marks	No. of Students
5	5
15	8

25	15
35	16
45	6

Solution

Calculation of mean deviation

Marks (X)	Step Deviation from as av. (25) (d′)	No. of students (f)	(fd′)	Deviation from actual average (27) (dX)	(fd)
5	-2	5	-10	22	110
15	-1	8	-8	12	96
25	0	15	0	2	30
35	+1	16	+16	8	128
45	+2	6	+12	18	108
	$\sum f = 50$	$\sum fd = +10$		$\sum \lvert fd\overline{X} \rvert = 472$	

Arithmetic average of = $\overline{X} = \left(\frac{10}{50} \times 10\right) = 27.$

Mean deviation = $\frac{\sum \lvert fd\overline{X} \rvert}{\sum f} = \frac{472}{50}$ marks = 9.44 marks.

2. Calculation of mean deviation in continuous series: The calculation of mean deviation in continuous series is done by the same procedure by which it is done in discrete series. In the short-cut method also the same procedure is followed provided the assumed mean or median is in the same class-interval in which the actual mean or median is. If the assumed average is in a different class interval, further adjustments are necessary. The following examples would illustrate these procedures:

Example 10. Calculate the mean deviation (from median) from the following data :

Class interval	Frequency	Class interval	Frequency
1 - 3	6	9 --11	21
3 - 5	53	11 - 13	26
5 - 7	85	13 - 15	4
7 - 9	56	15 - 17	4

Solution. Direct and short-cut method : The median of the above series is 6.5.

Class Interval	Mid-points	Deviation from Actual median (6.5)	Freq-uency (f)	Dev. x Freq-uency	Dev. from assumed median (g)	Dev. x Freq-uency
X	m	(dM)	(f)	(fdM)	(dM)	(fdM)
1-3	2	4.5	6	27.0	4	24
3-5	4	2.5	53	132.5	2	106
5-7	6	0.5	85	42.5	0	0
7 - 9	8	1.5	56	84.0	2	112
9-11	10	3.5	21	73.5	4	84
11-13	12	5.5	16	88.0	6	96
13-15	14	7.5	4	30.0	8	32
15-17	16	9.5	4	38.0	10	40
Total			**245**	**515.5**		**494**

Direct Method: Mean deviation $= \frac{\sum fdM}{N} = \frac{515.5}{245} = 2.1$

Short-cut Method : Total of deviations from assumed median = 494

No. of items with values less than the actual median (6.5) = (6+53+85) = 144

No. of items with values more than the actual median = (56+21+16+4+4) = 101

Difference between actual and assumed median = (6.5 – 6) = 0.5

Total deviation from actual median (when, actual and assumed medians are in the same class interval)

$$= 494 + (144 \times 0.5) - (101 \times 0.5) = 494 + 72 - 50.5 = 515.5$$

$$\text{Mean deviation} = \frac{515.5}{245} = 2.1$$

Example 11. Calculate the mean deviation (from mean) from the following data :

Marks :	0–10	10–20	20–30	30–40	40–50	50–60	60–70
No. of Students	6	5	8	15	7	6	3

Solution.

Calculation of Mean Deviation Direct and Short-cut Methods

Marks (X)	Mid value (mv)	No. of students (f)	Dev. from assumed average dx (35)	Step Devia-tion dx/i	Total Dev. from a.av. fdx	Dev. from actual mean d (+– signs ignored)	Total Dev. from actual fd
(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
0-10	5	6	-30	-3	-18	28.4	170.4
10-20	15	5	-20	-2	-10	18.4	92.0
20-30	25	8	-10	-1	-8	8.4	67.2
30-40	35	15	0	0	0	1.6	24.0
40-50	45	7	+10	+1	+7	11.6	81.2
50-60	55	6	+20	+2	+12	21.6	129.6
60-70	65	3	+30	+3	+ 9	31.6	94.8
		$\sum f = 50$			-8		$\sum fdX$=6592

Arithmetic Average = $A+\left(\frac{\sum dx}{N}\times i\right)=35+\left(\frac{-8}{50}\times 10\right)=33.4$

Mean Deviation (Direct Method) = $\frac{\sum|fd\overline{X}|}{N}=\frac{659.2}{50}=13.18\,marks$

1.14. Standard Deviation

The concept of standard deviation was first used by Karl Pearson in the year 1893. It is the most commonly used measure of dispersion. It satisfies most of the properties laid down for an ideal measure of dispersion.

Meaning: The technique of the calculation of mean deviation is mathematically illogical as in its calculation, the algebraic signs are ignored. This drawback is removed in the calculation of standard deviation, where squares of the deviations from the mean one used. Standard deviation is the square root of the arithmetic average of the squares of the deviations measured from the mean. The standard deviation is conventionally represented by the Greek letter Sigma s.

Symbolically, $\sigma=\sqrt{\frac{\Sigma(X-\overline{X})^2}{N}}$

Where s stands for the standard deviation,

Calculation of Standard Deviation

1. Series of individual observation: In such a series the standard deviation can be calculated in any of the following ways:

Direct Method No. 1 : In this method, the following steps are involved:

(i) Find the arithmetic average of the series.

(ii) Find the deviations of each item from the arithmetic average and denote it by (d) i.e., find $(X-\overline{X})$ for each X.

(iii) Square these deviations and total them to find $\sum d^2$.

(iv) Divide Sd² by the number of items to find $\frac{\Sigma d^2}{N}$. This figure is called the second moment about N the Mean.

(v) Standard Deviation = VN = $\sqrt{\frac{\Sigma d^2}{N}}=\sqrt{\frac{\Sigma(X-\overline{X})^2}{N}}$

Example 12. Calculate the standard deviation of the heights of 10 students given below :

Heights (in cms.) : 160, 160, 161, 162, 163, 163, 163, 164, 164, 170

Solution. Calculation of Standard Deviation of heights.

Height in centimeters	**Deviations from mean 163**	**Deviations squared (d^2)**
160	-3	9
160	-3	9
161	-2	4
162	-1	1
163	0	0
163	0	0
163	0	0
164	+1	1
164	+2	1
170	+7	49
		$\sum d^2 = 74$

$$\text{Arithmetic average or } X = \frac{\Sigma X}{N} = \frac{1630}{10} = 163 \text{ Cms.}$$

$$\text{Standard Deviation or } s = \sqrt{\frac{\Sigma d^2}{N}} = \sqrt{\frac{74}{10}} = \sqrt{7.4} = 2.27 \text{ Cms.}$$

Second method for Calculating S.D.

Calculation of Standard Deviation

Discrete Series. In discrete series also standard deviation can be calculated by

(i) The Direct Method as well as by

(ii) The Short-cut Method. Further it is possible to have step deviations, both in the Direct and Short-cut Methods :

(a) **Direct Method:**

1. Calculate the arithmetic mean.

2. Find out the deviations (d) of the various values, from the mean value. Square these deviations (d^2).

3. Multiply d^2 with the respective frequencies (f) against various values and add all such values ($\sum fd^2$).

4. Divide Sfd^2 by the number of items (N) and find out the square root of the figure so obtained i.e., find $\sqrt{\frac{\Sigma fd^2}{N}}$. This will be the value of the Standard Deviation.

(b) **Short-cut Method**

1. Assume a mean (A) and take deviations (dx) from it and square them up (dx^2).

2. Multiply dx^2 with the respective frequencies or f to get (fdx^2). Total them to get ($\sum fdx^2$).

3. Divide ($\sum fdx^2$) by the number of items or N to get $\frac{(\Sigma fdx^2)}{N}$.

4. From $\left(\frac{\Sigma fdx^2}{N}\right)$ subtract the square of the difference between actual and assumed average $(\overline{X}-A)^2$ to get $\left(\frac{\Sigma fdx^2}{N}\right)-(\overline{X}-A)^2$.

5. Find out the square root of the above value or

$\sqrt{\frac{(\Sigma fdx^2)}{N}-(\overline{X}-A)^2}$ and it will be the value of the standard

deviation. This formula can also be written as :

$$\sigma=\sqrt{\frac{(\Sigma fdx^2)}{N}-\left(\frac{\Sigma fdx}{N}\right)^2} \quad \text{as} \quad \overline{X}-A=\frac{\Sigma fdx}{N}$$

Example 13. Calculate standard deviation for the following distribution :

Values	10	20	30	40	50	60	70
Frequency	1	5	12	22	17	9	4

Solution

Calculation of standard deviation (Step Deviation Method)

Values (X)	Freq-uency (f)	Dev. from assumed av. 40 $\frac{x-40}{10}=dx$	Total Step Dev. fdx	dx²	fdx²
10	1	-3	3	9	9
20	5	-2	-10	4	20
30	12	-1	-12	1	12
40	22	0	0	0	0
50	17	+1	+17	1	17
60	9	+2	+18	4	36
70	4	+3	+12	9	36
	N = 70		Sfdx = +22		$\sum$ fdx² =130

$$\sigma = \sqrt{\frac{\Sigma fdx^2}{N} - \left(\frac{\Sigma fdx}{N}\right)^2} \times i = \sqrt{\frac{130}{70} - \left(\frac{+22}{70}\right)^2} \times 10 = \sqrt{1.757} \times 10$$

$= 1.326 \,'\, 10 = 13.26$

Thus, the standard deviation is 13.26.

Example 14. Calculate the standard deviation for the following table giving the age distribution of 542 members of the House of Common :

Age	No. of Members
20–30	3
30–40	61
40–50	132
50–60	153
60-70	140
70-80	51
80-90	2
Total	**542**

Solution. Calculation of the standard deviation of the age distribution of 542 members of the House of Commons :

Age Group (X)	Mid-Value (mv)=x	Frequency (f)	Deviations from the assumed av. (55) dx	fdx	Square of deviations dx^2	Frequency x square of deviations fdx^2
20–30	25	3	– 30	– 90	900	2700
30–40	35	61	– 20	– 1220	400	24400
40–50	45	132	– 10	– 1320	100	13200
50–60	55	153	0	0	0	0
60–70	65	140	+ 10	+ 1400	100	14000
70–80	75	51	+ 20	+ 1020	400	20400
80–90	85	2	+ 30	+ 60	900	1800
			N = 542	$\sum fdx=150$		$\sum fdx^2=76500$

$$\sigma = \sqrt{\frac{\sum fdx^2}{N} - \left(\frac{\sum fdx}{N}\right)^2} = \sqrt{\frac{76500}{542} - \left(\frac{-150}{542}\right)^2}$$

$= \sqrt{141.07} = 11.9$ years.

In the above example, step deviations were not taken, so the calculation become cumbersome. In the following example, step deviation method has been illustrated.

1.15. Conclusion

Editing of data is a Process of Examining the collected raw data to detect errors and omissions and to correct these when possible. It also involve careful scrutiny of completed questionnaires and/or schedules.

Dispersion is the degree of the scatter or variation of the variable about a central value.

1.16. Further Study

1. Monga, G.S., Elementary Statistics.

2. Gupta S.B. Principal of Statistics.

3. Agrawal, B.N., Elhance, D.N. Elhance, Veina, Fundamental of Statistics.

4. Anand Vinod, Chand Mahesh, Economic Theory of Mathematical Approach.

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Unit – 2 Theory of Probability, Basic Concept, Simple Joint Conditional and Marginal Probabilities, Addition and Multiplication Theorems

Objectives : After going through this unit you should be able to know about

1. Concept, Definition and Theory of Probability.
2. Simple, Joint and Conditional Probabilities.
3. Addition and Multiplication Theorems.

Structure :

2.1. Concept of Probability

Probability is the measure of the likeliness that an event will occur. Probability is quantified as a number between 0 and 1 (where 0 indicates impossibility and 1 Indicates certainly. The higher the probability of an event, the more certain we are that the event will occur A simple example is the toss of a fair coin. Since the two outcomes are equally probable, the probability of "heads" equals the probability of "tails", so the probability IS 1/2 (or 50%) chance of either "heads" or "tails".

These concepts have been given an axiomatic mathematical formalization in probability theory (see probability axioms), which is used widely in such areas of study as mathematics, statistics, finance, gambling, science (in particular physics), artificial intelligence/machine learning, computer science, and philosophy to, for example, draw inferences about the expected frequency of events. Probability theory is also used to describe the underlying mechanics.

Interpretations

When dealing with experiments that are random and well-defined in a purely theoretical setting (like tossing a fair coin), probabilities can be numerically described by the statistical number of outcomes considered favorable divided by the total number of all outcomes (tossing a fair coin twice will yield head-head with probability 1/4, because the four outcomes head-head, head-tails. tails-head and tails-tails are equally likely to occur). When it comes to practical application however there are two major competing categories of **probability interpretations,** whose adherents possess different views about the fundamental nature of probability:

1. Objectivists assign numbers to describe some objective or physical state of affairs. The most popular version of objective probability is frequentist probability, which claims that the probability of a random event denotes the relative frequency of occurrence of an experiment's outcome. when repeating frequency "in the long run" of outcomes. A modification of this is propensity probability, which interprets probability as the tendency of some experiment to yield a certain outcome, even if it is performed only once.

2. Subjectivists assign numbers per subjective probability, i.e. as a degree of belief. The degree of belief has been interpreted as, "the price at which you would buy or sell a bet that pays 1 unit of utility if E, 0 if not E. The most popular version of subjective probability is Bayesian probability, which includes expert knowledge is represented by some (subjective) prior

probability distribution. The data is incorporated in a likelihood function. The product of the prior and the likelihood. normalized. results in a posterior probability distribution that incorporates all the information known to date. Starting from arbitrary, subjective probabilities for a group of agents, some Bayesians claim that all agents will eventually have sufficiently similar assessments of probabilities, given enough evidence (see Cromwell's rule).

The word probability derives from the Latin probabilitas, which can also mean "probity", a measure of the authority of a witness in a legal case in Europe, and often correlated with the witness's nobility. In a sense, this differs much from the modern meaning of probability, which, in contrast, is a measure of the weight of empirical evidence, and is arrived at from inductive reasoning and statistical inference.

2.2. History of Probability

The scientific study of probability is a modern development. Gambling shows that there has been an interest in quantifying the ideas of probability for millennia, but exact mathematical descriptions arose much later. There are reasons of course, for the slow development of the mathematics of probability. Whereas games of chance provided the impetus for the mathematical study of probability, fundamental issues are still obscured by the superstitions of gamblers.

According to Richard Jeffrey, "Before the middle of the seventeenth century, the term 'probable' (Latin sense, univocally, to opinion and to action. A probable action or opinion was one such as sensible people would undertake or hold, in the circumstances." However. in legal contexts especially, 'probable' could also apply to propositions for which there was good evidence.

The sixteenth century polymath Gerolamo Cardano demonstrated the efficacy of defining odds as the ratio of favourable to unfavourable outcomes (which implies that the probability of an event is given by the ratio of favourable outcomes to the total number of possible outcomes. Aside from the elementary work by Cardano, the doctrine of probabilities dates to the correspondence of Pierre de Fermat and Blaise Pascal (1654).

Christiaan Huygens (1657) gave the earliest known scientific treatment of the subject. Jakob Bernoulli's *Ars Conjectandi* (posthumous, 1713) and Abraham mathematics. See Ian Hacking's *The Emergence of Probability* and James Franklin's *The Science of Conjecture* for histories of the early development of the very concept of mathematical probability.

The theory of errors may be traced back to Roger Cotes's *Opera Miscellanea* (posthumous, 1722), but a memoir prepared by Thomas Simpson in 1755 (printed 1756) first applied the theory to the discussion of errors of observation. The reprint (1757) of this memoir lays down the axioms that positive and negative errors are equally probable, and that certain assignable limits define the range of all errors. Simpson also discusses continuous errors and describes a probability curve.

The first two laws of error that were proposed both originated with Pierre-Simon Laplace. The first law was published in 1774 and stated that the frequency of an error could be expressed as an exponential function of the numerical magnitude of the error, disregarding sign. The second law of error was proposed in 1778 by Laplace and stated that the frequency of the error is an exponential function of the square of the error. The second law of error is called the normal distribution or the Gauss law. "It is difficult historically to attribute that law to Gauss, who in spite of his well-known precocity had probably not made this discovery before he was two years old."

Daniel Bernoulli (1778) introduced the principle of the maximum product of the probabilities of a system of concurrent errors.

Adrien Marie Legendre (1805) developed the method of least squares, and introduced it in his *Nouvelles methodes pour la determination des orbites des cometes* (New Methods for Determining the Orbits of Comets). In ignorance of Legendre's contribution, an Irish-American writer, Robert Adrain, editor of "The Analyst" (1808), first deduced the law of facility of error,

$$\phi(x) = ce^{h^2x^2},$$

where h, is a constant depending on precision of observation, and c is a scale factor ensuring that the area under the curve equals 1. He gave two proofs, the second being essentially the same as John Herschel's (1850). Gauss gave the first proof that seems to have been known in Europe (the third after Adrain's) in 1809. Further proofs were given by Laplace (1810, 1812), Gauss (1823), James Ivory (1825, 1826). Hagen (1837), Friedrich Bessel (1838), W.F. Donkin (1844, 1856), and Morgan Crofton (1870). Other contributors were Ellis (1844), De Morgan (1864), Glaisher (1872), and Giovanni Schiaparelli (1875). Peter's (1856) formula for r, the probable error of a single observation, is well known.

In the nineteenth century authors on the general theory included Laplace, Sylvestre Lacroix (1816), Littrow (1833), Adolphe Quetelet (1853), Richard Dedekind (1860), Helmert (1872), Hermann Laurent (1873), Liagre, Didion, and Karl Pearson. Augustus De Morgan and George Boole improved the exposition of the theory.

Andrey Markov introduced the notion of Markov chains (1906), which played an important role in stochastic processes theory and its applications. The modern theory of probability based on the measure theory was developed by Andrey Kolmogorov (1931).

On the geometric side (see integral geometry) contributors to *The Educational Times* were influential (Miller, Crofton, McColl, Wolstenholme, Watson, and Artemas Martin).

2.3. Theory of Probability

Like other theories, the theory of probability is a representation of probabilistic concepts in formal terms-that is, in terms that can be considered separately from their meaning. These formal terms are manipulated by the rules of mathematics and logic, and any results are interpreted or translated back into the problem domain.

There have been at least two successful attempts to formalize probability, namely the Kolmogorov formulation and the Cox formulation. In Kolmogorov's formulation (see probability space), sets are interpreted as events and probability itself as a measure on a class of sets. In Cox's theorem, probability is taken as a primitive (that is, not further analyzed) and the emphasis is on constructing a consistent assignment of probability values to propositions. In both cases, the laws of probability are the same, except for technical details.

There are other methods for quantifying uncertainty, such as the Dempster-Shafer theory or possibility theory, but those are essentially different and not compatible with the laws of probability as usually understood.

2.4. Applications of Probability

Probability theory is applied in everyday life in risk assessment and in trade on financial markets. Governments apply probabilistic methods in environmental regulation, where it is called pathway analysis. A good example is the effect of the perceived probability of any widespread Middle East conflict on oil prices which have ripple effects in the economy as a whole. An assessment by a commodity trader that a war is more likely vs. less likely sends prices up or down, and signals other traders of that opinion. Accordingly, the probabilities are neither assessed independently nor necessarily very rationally. The theory of behavioural finance emerged to describe the effect of such groupthink on pricing, on policy, and on peace and conflict.

The discovery of rigorous methods to assess and combine probability assessments has changed society. It is important for most citizens to understand how probability assessments are made, and how they contribute to decisions.

Another significant application of probability theory in everyday life is reliability. Many consumer products, such as automobiles and consumer electronics, use reliability theory in product design to reduce the probability of failure. Failure probability may influence a manufacturer's decisions on a product's warranty.

The cache language model and other statistical language models that are used in natural language processing are also examples of applications of probability theory.

2.5. Mathematical Treatment

Consider an experiment that can produce a number of results. The collection of all results is called the sample space of the experiment. The power set of the sample space is formed by considering all different collections of possible results. For example, rolling a dice can produce six possible results. One collection of possible results gives an odd number on the dice. Thus, the subset {1, 3, 5} is an element of the power set of the sample space of dice rolls.

These collections are called "events." In this case, {1 ,3,5} is the event that the dice falls on some odd number. If the results that actually occur fall in a given event, the event is said to have occurred.

A probability is a way of assigning every event a value between zero and one, with the requirement that the event made up of all possible results (in our example, the event {1, 2, 3, 4, 5, 6}) is assigned a value of one. To qualify as a probability, the assignment of values must satisfy the requirement that if you look at a collection of mutually exclusive events (events with no common results, e.g. the events {1,6}, {3}, and {2,4} are all mutually exclusive), the probability that at least one of the events will occur is given by the sum of the probabilities of all the individual events.

The probability of an event A is written as P(A), p(A) or Pr(A). This mathematical definition of probability can extend to infinite sample spaces, and even uncountable sample spaces, using the concept of a measure.

The opposite or complement of an event A is the event [not A] (that is, the event of A not occurring); it probability is given by P(not A) = 1 – P(A). As an example, the chance of not rolling a six on a six-sided die is 1 – (chance of rolling a six) $= 1 - \frac{1}{6} = \frac{5}{6}$, See Complementary event for a more complete treatment.

If two events A and B occur on a single performance of an experiment, this is called the intersection or joint probability of A and B, denoted as P (AÇB).

Independent Events

If two events, A and B are independent then the joint probability is

$$P(A \text{ and } B) = P(A \cap B) = P(A)P(B).$$

for example, if two coins are flipped the chance of both being heads is

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

Mutually Exclusive Events

If either event A or event B or both events occur on a single performance of an experiment this is called the union of the events A and B denoted as $P(A \cup B)$. If two events are mutually exclusive then the probability of either occurring is

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B).$$

For example, the chance of rolling a 1 or 2 on a six-sided die is

$$P(1 \text{ or } 2) = P(1) + P(2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Not Mutually Exclusive Events

If the events are not mutually exclusive then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

For example, when drawing a single card at random from a regular deck of cards, the chance of getting a heart or a face card (J, Q, K) (or one that is both) is $\frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{11}{26}$, because of the 52cards of a deck 13 are hearts, 12 are face cards, and 3 are both: here the possibilities included in the "3 that are both" are included in each of the "13 hearts" and the "12 face cards" but should only be counted once.

Conditional Probability

Conditional probability is the probability of some event A, given the occurrence of some other event B. Conditional probability is written P(A | B), and is read "the probability of A, given B". It is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

If P(B) = 0 then P(A | B) is formally undefined by this expression. However, it is possible to define a conditional probability for some zero probability events using a s algebra of such events (such as those arising from a continuous random variable).

For example, in a bag of 2 red balls and 2 blue balls (4 balls in total), the probability of taking a red ball is ½; however, when taking a second ball, the probability of it being either a red ball or a blue ball depends on the ball previously taken, such as, if a red ball was taken, the probability of picking a red ball again would be 1/3 since only 1 red and 2 blue balls would have been remaining.

Inverse Probability

In probability theory and applications, Bayes' rule relates the odds of event A_1 to event A_2, before (prior to) and after (posterior to) conditioning on another event B. The odds on A_1 to event. A_2 is simply the ratio of the probabilities of the two events. When arbitrarily many events A are of interest, not just two, the rule can be rephrased as posterior is proportional to prior times likelihood, P(A | B) μ P(A) P(B | A) where the proportionality symbol means that the left hand side is proportional to (i.e., equals a constant times) the right hand side as A varies, for fixed or given B (Lee, 2012; Bertsch McGrayne, 2012). In this form it goes back to Laplace (1774) and to Cournot (1843), see Fienberg (2005). See Inverse probability and Bayes' rule.

Summary of Probabilities

Summary of probabilities

Event	Probability
A	P(A) Î [0, 1]
not A	$P(A^c) = 1 - P(A)$
A or B	P(AÈB) = P(A) + P(B) − P(AÇB)P(AÈB) = P(A) + P(B) if A and B are mutually exclusive
A and B	P(AÇB) = P(A | B) P(B) = P(B | A) P(A)P(AÇB) = P(A) P(B) if A and B are independent
A given B	$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B\|A)P(A)}{P(B)}$

Relation to Randomness

In a deterministic universe based on Newtonian concepts, there would be no probability if all conditions were known (Laplace's demon), (but there are situations in which sensitivity to initial conditions exceeds our ability to measure them, i.e. know them). In the case of a roulette wheel, if the force of the hand and the period of that force are known, the number on which the ball will stop would be a certainty (though as a practical matter, this would likely be true only of a roulette wheel that had not been exactly levelled — as Thomas A. Bass' Newtonian Casino revealed). Of course, this also assumes knowledge of inertia and friction of the wheel, weight, smoothness and roundness of the ball, variations in hand speed during the turning and so forth. A probabilistic description can thus be more useful than Newtonian mechanics for analyzing the pattern of outcomes of repeated rolls of a roulette wheel. Physicists face the same situation in kinetic theory of gases, where the system, while deterministic *in principle,* is so complex (with the number of molecules typically the order of magnitude of Avogadro constant 6.02.10) that only a statistical description of its properties is feasible.

Probability theory is required to describe quantum phenomena. A revolutionary discovery of early 20th century physics was the random character of all physical processes that occur at sub-atomic scales and are governed by the laws of quantum mechanics. The objective wave function evolves deterministically but, according to the Copenhagen interpretation, it deals with probabilities of observing, the outcome being explained by a wave function collapse when an observation is made. However, the loss of determinism for the sake of instrumentalism did not meet with universal approval. Albert Einstein famously remarked in a letter to Max Born: "I am convinced that God does not play dice". Like Einstein, Erwin Schrödinger, who discovered the wave function, believed quantum mechanics is a statistical approximation of an underlying deterministic reality. In modern interpretations, quantum decoherence accounts for subjectively probabilistic behaviour.

2.6. Simple Probability

SIMPLE PROBABILITY is the likelihood that a specific event will occur, represented by a number between 0 and 1.

There are two categories of simple probabilities.

THEORETICAL PROBABILITY is calculated probability. If every event is equally likely, it is the ratio of the number of ways the event can occur to the total number of possible outcomes.

$$\text{theoretical probability} = \frac{\text{number of ways to get want you want}}{\text{total number of possible outcomes}}$$

EXPERIMENTAL PROBABILITY is the probability based on data collected in experiments.

$$\text{experimental probability} = \frac{\text{number of times the event occurred}}{\text{total number of outcomes}}$$

Example 1

There are three pink pencils, two blue pencils, and one green pencil. If one pencil is picked randomly, what is the theoretical probability it will be blue?

* Find the total number of possible outcomes, that is, the total number of pencils. 3 + 2 + 1 = 6

* Find the number of specified outcomes, that is, how many pencils are blue? 2

  Find the theoretical probability. P(blue pencil) = $\frac{2}{6} = \frac{1}{3}$.

  (You may reduce your answer).

Example 2

Jayson rolled a die twelve times. He noticed that three of his rolls were fours.

a. What is the theoretical probability of rolling a four'?

   Because the six sides are equally likely and there is only one four, $P(4) = \frac{1}{6}$.

b. What is the experimental probability of rolling a four?

There were three fours in twelve rolls. The experimental probability is:

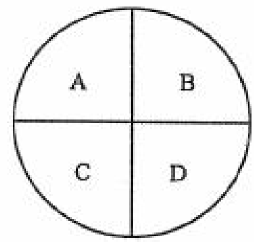
$$P(4) = \frac{3}{12} = \frac{1}{4}.$$

Example 3

On the spinner, what is the probability of spinning an A or a B ?

The probability of an A is $\frac{1}{4}$. The probability of a B is $\frac{1}{4}$. Add the two probabilities for the combined total.

$$\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2} \qquad P(A \text{ or } B)=\frac{1}{2}$$

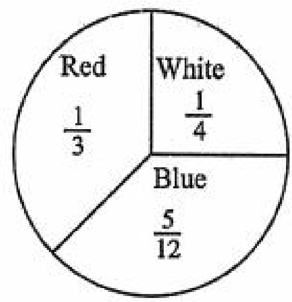


Example 4

What is the probability of spinning red or white ?

* We know that $P(R)=\frac{1}{3}$ and $P(W)=\frac{1}{4}$
* Add the probabilities together, $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$

$$P(R \text{ or } W)=\frac{7}{12}$$



Problems

1. There are five balls in a bag: 2 red, 2 blue, and 1 white. What is the probability of randomly choosing a red ball?

2. In a standard deck of cards, what is the probability of drawing an ace ?

3. A fair die numbered 1, 2, 3, 4, 5, 6 is rolled. What is the probability of rolling an odd number ?

4. In the word "probability", what is the probability of selecting a vowel ?

5. Anna has some coins in her purse: 5 quarters, 3 dimes, 2 nickels, and 4 pennies.

   a. What is the probability of selecting a quarter?

   b. What is the probability she will select a dime or a penny?

6. Tim has some gum drops in a bag: 20 red, 5 green, and 12 yellow.

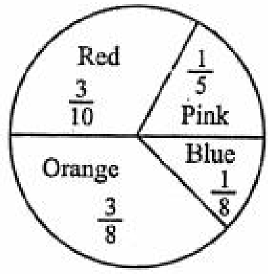
   a. What is the probability of selecting a green?

   b. What is the probability of not selecting a red?

7. What is the probability of spinning :

   a. pink or blue ? b. orange or pink ?

   c. red or orange ? d. red or blue ?

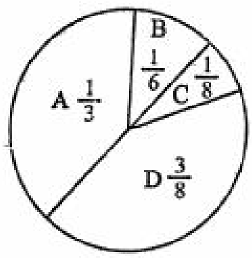


8. What is the probability of spinning :

   a. A or C? b. B or C?

   c. A or D? d. B or D?

   e. A, R, or C?



Answers

1. $\frac{2}{5}$ 2. $\frac{4}{52}=\frac{1}{13}$ 3. There are 3 odd numbers. $\frac{3}{6}=\frac{1}{2}$

4. $\frac{4}{11}$ 5. a. $\frac{5}{14}$ b. $\frac{1}{2}$ 6. a. $\frac{5}{37}$ b. $\frac{17}{37}$

7. a. $\frac{13}{40}$ b. $\frac{23}{40}$ c. $\frac{27}{40}$ d. $\frac{17}{40}$

8. a. $\frac{11}{24}$ b. $\frac{7}{24}$ c. $\frac{17}{24}$ d. $\frac{13}{24}$ e. $\frac{5}{8}$

Dependent and Independents

Two events are DEPENDNT if the outcome of the first event affect the outcome of the second event. For example, if you draw a card from a deck and **do not** replace it for the next draw, the two events — drawing one card without replacing it, then drawing a second card are dependent.

Two events are **INDEPENDENT** if the outcome of the first event does not affect the outcome of the second event. For example, if you draw a card from a deck but replace it before you draw again, the two events are independent.

Example 1

Aiden pulls an ace from a deck of regular playing cards. He does not replace the card. What is the probability of pulling out a second ace?

First draw : $\frac{4}{52}$ Second draw : $\frac{3}{51}$ *aces left* / *cards left to pull from*

This is an example of a dependent event – the probability of the second draw has changed.

Example 2

Tharon was tossing coins. He tossed a head. What is the probability of tossing a second head? $\frac{1}{2}$

The probability for the second event has not changed. This is an independent event.

Problems

1. You throw a die twice. What is the probability of throwing a six and then a second six? Are these independent or dependent events?

2. You have a bag of candy filled with pieces of the same size and shape. Four are gumballs and six are sweet and sours. You draw a gumball out, decide you don't like it put it back, then select another piece of candy. What is the probability of selecting another gumball? Are these independent or dependent events?

3. Joey has a blocks with eight alphabet blocks and four plain red blocks. He gave an alphabet block to his sister. What is the probability his next selection will be another alphabet block? Are these independent or dependent events?

4. In your pocket you have three nickels and two dimes.

   a. What is the probability of selecting a nickel ?

   b. What is the probability of selecting a dime ?

   c. If you select a nickel and place it on a table, what is the probability the next coin selected is a dime ? Are these independent or dependent events ?

d. If all the coins are back in your pocket, what is the probability that the next coin you lake out is a dime? Is this event independent or dependent in relation to the previous events?

5. How do you tell the difference between dependent and independent events?

Answers

1. $\frac{1}{36}$, independent (the probability doesn't change).

2. $\frac{4}{10} or \frac{2}{5}$, independent.

3. $\frac{7}{11}$, dependent.

4. a. $\frac{3}{5}$ b. $\frac{2}{5}$ c. $\frac{2}{4}=\frac{1}{2}$, dependent d. $\frac{2}{5}$, independent

5. For dependent events, the second probability changes because there is no replacement.

2.7. Compound Probability

When multiple outcomes are specified (as in problem 7(a) in the first section, pink or blue), and either outcome may occur but not both, find the probability of each specified outcome and add their probabilities.

If the desired outcome is a compound event, that is. it has more than one characteristic (as in Example 1, an I and a T), find the probability of each outcome (in Example 1, the probability of "I", then the probability of "T") and multiply the probabili*ies.

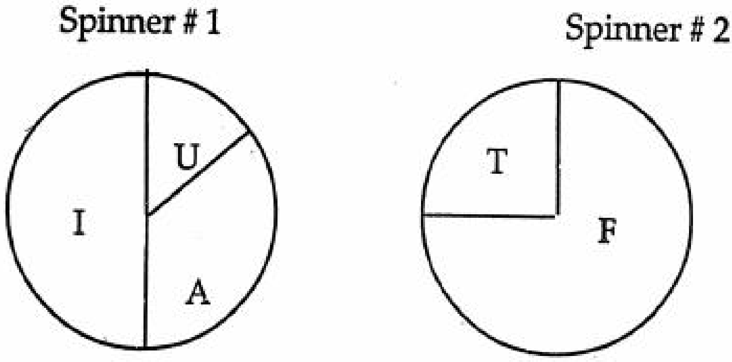
Example 1

To find the P(I and T) :

* Find the probability of I: $\frac{1}{2}$

- Find the probability of T : . $\frac{1}{4}$
- Multiply them together.

$$\frac{1}{2}\times\frac{1}{4}=\frac{1}{8} \qquad P(I \text{ and } T) = \frac{1}{8}$$



This can also be shown with an area diagram.

Divide the top of a square proportionally to correspond with the areas of the sectors in spinner #1. Divide the side of the square proportionally to correspond with the areas of the sectors in spinner #2. Drawing the created interior rectangles yields all possible outcomes for the two spinners. Multiplying the dimensions of the interior rectangles determines its area and also the probability of the compound event.

The P(I and T) = area of rectangle IT = $\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$

Spinner #1

Spinner #2	I $\left(\frac{1}{2}\right)$	A $\left(\frac{1}{3}\right)$	U $\left(\frac{1}{6}\right)$
T $\left(\frac{1}{4}\right)$	IT	AT	UT
F $\left(\frac{3}{4}\right)$	IF	AF	UF

Area diagrams will be especially useful in solving conditional probability problems, which are covered in the final section of this probability practice.

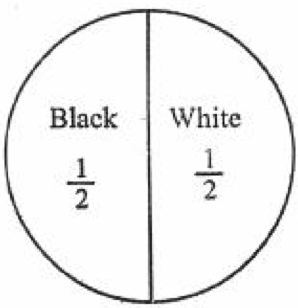
Example 2

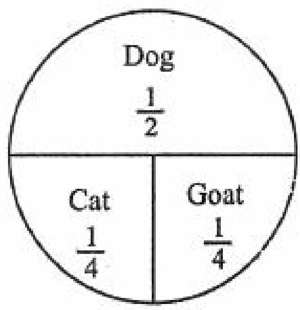
What is the probability of spinning a black dog or a black cat?

P (B Cat) = $\frac{1}{2},\frac{1}{4}=\frac{1}{8}$

P (B Dog) = $\frac{1}{2},\frac{1}{2}=\frac{1}{4}$

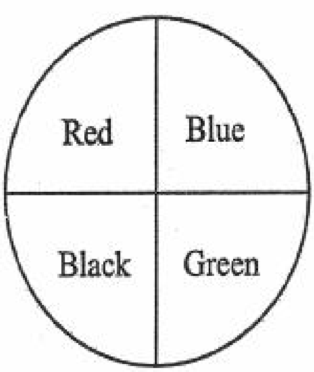
P (B Cat or B Dog) = $\frac{1}{8}+\frac{1}{4}=\frac{1}{8}+\frac{2}{8}=\frac{3}{8}$

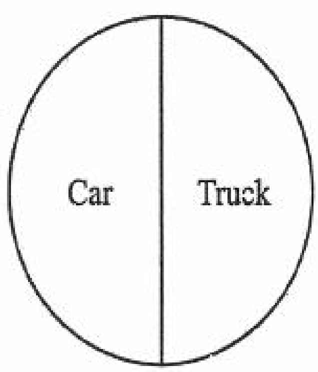




Problems

1. If each section in each spinner is the same size, what is the probability of getting a black truck'?





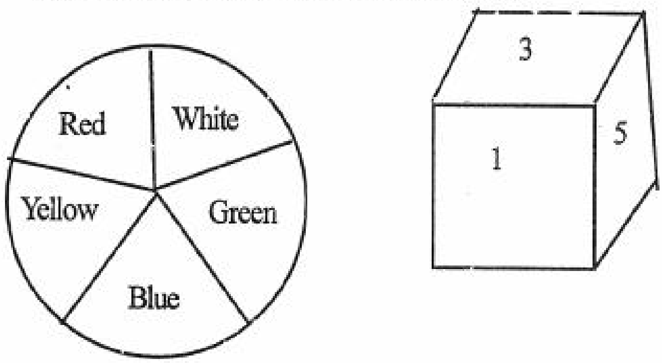
2. Bipasha loves purple, pink, turquoise and black. and has a house in each color, She has two pair of black pants and a pair of khaki

pants. If she randomly chooses one blouse and one pair of pants, what is the probability she will wear a purple blouse with black pant?

3. The spinner at right is divided into five equal regions. The die is numbered from 1 to 6. What is the probability of :

   a. getting red and rolling a 5 ?

   b. white or blue and an even number.

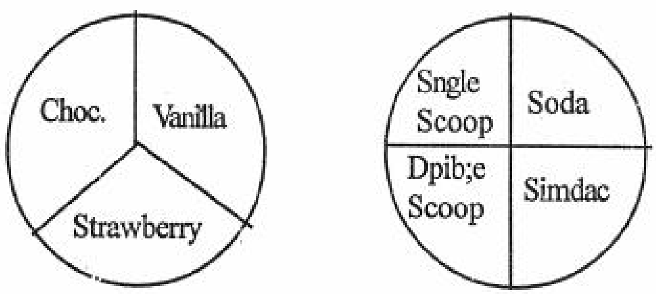


4. What is the probability Joanne will win a :

   a. chocolate double scoop ?

   b. chocolate or strawberry sundae ?

   c. chocolate double scoop or chocolate sundae ?



5. Jay is loc!:ing in his closet, trying to decide what to wear. He has 2 red t-shirts, 2 black t-shirts, and 3 white t-shirts. He has 3 pair of blue jeans and 2 pair of black pants.

   a. What is the probability that he randomly chooses a red shirt with jeans ?

   b. What is the probability that he randomly chooses an all black outfit ?

c. Which combination out of all his possible choices has the greatest probability of being randomly picked? How can you tell ?

6. What is the probability of spinning a :

a. Red or Green ?

? b. Blue or Yellow ?

c. Yellow of Green ?

Answers

1. $\frac{1}{4}.\frac{1}{2}=\frac{1}{8}$ 2. $\frac{1}{4}.\frac{2}{3}=\frac{2}{12}=\frac{1}{6}$ 3. a. $\frac{1}{5}.\frac{1}{5}=\frac{1}{30}$

b. $\left(\frac{1}{5}+\frac{1}{5}\right).\frac{3}{6}=\frac{1}{5}$

4. a. $\frac{1}{3}.\frac{1}{4}=\frac{1}{12}$ 5. a. $\frac{2}{7}.\frac{3}{5}=\frac{6}{35}$ 6. a. $\frac{13}{24}$ c. $\frac{17}{24}$

b. $\left(\frac{1}{3}+\frac{1}{3}\right).\frac{1}{4}=\frac{1}{6}$ b. $\frac{2}{7}.\frac{2}{5}=\frac{4}{35}$ b. $\frac{11}{24}$

c. $\frac{1}{3}.\frac{1}{4}+\frac{1}{3}.\frac{1}{4}=\frac{1}{6}$ c. white shirt, blue jeans; most selections in both categories

2.8. Conditional Probability

Conditional probability is the probability that a second event occurred knowing that a first event occurred. The probability that event A occurred knowing that event B occurred is written P (A | B) and is verbalized "the probability of A, given B."

Example 1

Two dice are thrown. What is the probability of getting 10 points ?

Since all possibilities are equally likely we could just make a list of ways to make 10 points : (4, 6), (5, 5) and (6, 4). Three ways out of 36 possible outcomes is $\frac{1}{12}$.

If we knew we made 10 points what was the probability that each die showed five points ?

Of those three ways, only one has two-fives so the conditional probability is $\frac{1}{3}$.

It can also be viewed as an area diagram where each cell has area $\frac{1}{36}$.

$$P(10) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{3}{36} = \frac{1}{12}$$

$$P(\text{two-fives} \mid 10) = \frac{1}{36} \text{ out of } \frac{3}{36} = \frac{1}{3}.$$

	1	2	3	4	5	6
1						
2						
3						
4						X
5					X	
6				X		

Example 2

A spinner comes up red 25% of the time and green 25% of the time. The rest of the time it lands on blue.

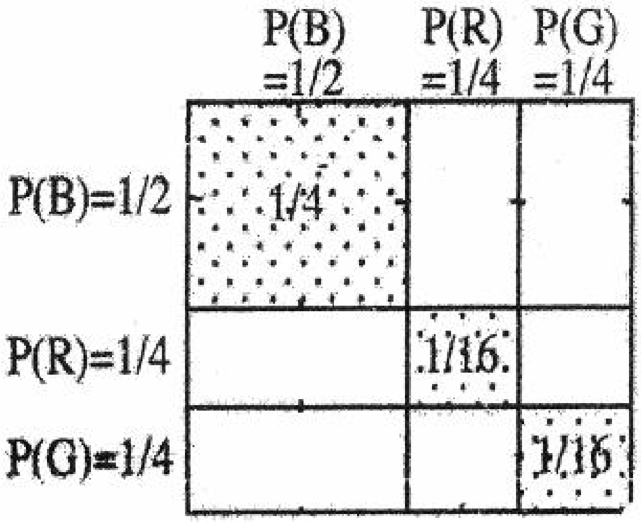
Draw an area diagram for spinning twice, and shade the region on your area diagram corresponding to getting the same color twice. See diagram at right.

Since the probabilities of each color are not the same, just making a list will not work.

What is the probability that both spins land on the same color ?
Add the areas or probabilities of the shaded squares :

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{16}=\frac{3}{8}.$$

If you know that you got the same color twice, what is the conditional probability the color was blue? The possible region is $\frac{3}{8}$ of the area. The desired area is $\frac{1}{4}$ of the region. $\frac{1}{4}$ out of $\frac{3}{8}$ means $\frac{1}{4}\div\frac{3}{8}=\frac{4}{6}=\frac{2}{3}$.



Problems

1. Two dice are thrown and the sum is seven points.

   a. Shade the squares on an area diagram where this outcome could occur.

   b. What is the probability of getting seven points ?

   c. If the total was seven points, what is the probability that one of the dice was a six?

2. Suppose you roll two dice and the sum is more than 8.

   a. Shade the squares on an area diagram where this outcome could occur.

   b. What is the probability that both dice show the same number?

c. What is the probability that exactly one 6 is showing?

d. What is the probability that at least one 5 is showing?

3. A spinner comes up blue, red, and green with a probability of 1/3 for each colour.

a. Sketch an area diagram for spinning twice.

b. Shade the region on your area diagram that corresponds to landing on the same color twice.

c. What is the probability that both spins land on the same colour ?

d. If you know that you got the same colour twice, what is the probability the colour was red.

4. Another spinner. This time the spinner lands on red half of the time and on green one third of the time. The rest of the time it lands on blue.

a. Draw an area diagram for spinning twice and shade the region that corresponds to getting the same color on both spins.

b. Suppose you know that the spinner landed on the same colour twice. What is the probability that colour was green ?

5. In the children's game Build a Farm each player firs spins a spinner. Half of the time the spinner comes up red. Half of the time the spinner comes up blue. If the spinner is red, you reach into the red box. If the pinner is blue, you reach into the blue box. The red box has 10 chicken counters, 10 pig counters, and 10 cow counters, while the blue box has 5 chicken counter, 4 pig counter and 1 cow counter.

a. Sketch an area diagram for the situation where a child spins and then draws. Note that the parts corresponding to the two boxes of animal markers will be quite different.

b. Shade the pmts of the diagram corresponding to getting a pig counter. What is P(pig)?

c. Find the probability of getting a cow counter.

d. Find the probability that if you got a cow counter you also spun red.

e. Find the probability that if you got a cow counter you also spun blue.

6. A spinner has two colours, red and blue. The probability the spinner will land one blue is x.

a. What is the probability it will land on red ?

b. Sketch an area diagram for spinning twice ?

c. If the spinner is spun twice, what is the probability it will land on the same color both times?

d. Given that the spinner lands on the same colour twice, what is the probability that it landed on blue both times ?

Answers

2.9 Theorems of probability

There are two important theorems of Probability, viz. ....,

(A) The Additional Theorem

(B) The Multiplication Theorem

(A) Additional Theorem

If A and B are any two events, then the probability that at least one of them occurs is denoted by $P(A \cup B)$ and as given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

where $P(A)$ = Probability of the occurrence of the user event.

$P(B)$ = Probability of the occurrence of event B

$P(A \cap B)$ = Probability of simultaneous occurrence of event A & B.

Manually exclusive events have no sample point common to them therefore, it A & B are two mutually exclusive event then, A & B = Q i.e. the enter-section of two mutually exclusive event is a null set and in this case $P(A \cap B) = 0$

In case of mutually exclusive events.

$$P(A \cup B) = P(A) + P(B)$$

if there are three events A, B & C the Probability of the occurrence of at least one of them is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

In case the Finite number, say n of mutually exclusive events,

$$P(A_1 \cup A_2 \cup A_n \ldots\ldots P(A) + P(A_2) + \ldots\ldots P(A_n)$$

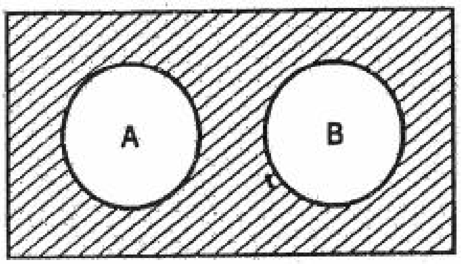
Note : (i) If a number of events $A_1, A_2 \ldots A_n$ are mutually exclusive and exhaustive, then the sum of the individual probabilities of their happening in equal to 1, i.e.,

$$P(A_1) + P(A_2) + \ldots + P(A_n) = 1$$

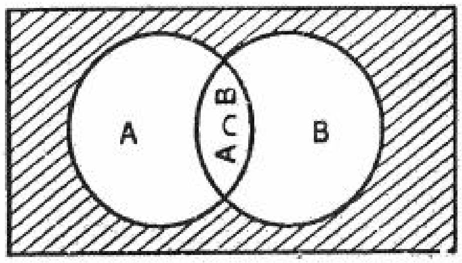
(ii) If the events are finite and mutually exclusive, then the probability of the occurrence of at least one of them is equal to the sum of their individual probabilities.

(iii) The event A and its compliment $\bar{A}$ can be considered as mutually exclusive and exhaustive.

$$P(A)+P(\bar{A})=1 \Rightarrow P(\bar{A})=1-P(A)$$



Mutually exclusive events
Mutually exclusive events



Events are not mutually exclusive
Events are not mutually exclusive

2.10. Conditional Probability

Two events A and B are said to be dependent when B can occur only when A is known to have occurred and vice versa. The probabilities associated with such events are called conditional probabilities.

The probability of the occurrence of the event A when the event B has already occurred is called the conditional probability of the occurrence of A given that the event B has already occurred and is denoted by P(A/B). If P(A/B) = 1. Then occurrence of B implies the occurrence of A.

Suppose, in a college 51% of students are girls; out of total students 35% are left handed, 10% of girls are left handed. Then the probability that a student randomly selected is left handed or a girl

= P (left handed or a girl)

= P (left handed) + P (girl) – P [Both left handed and girl]

= 0.35 + 0.51 – 0.10 = 0.76

P (left handed given that she is a girl) = P (Left handed among girls)

$$=\frac{10}{51}=\frac{P(\text{left handed girl})}{P(\text{girl})}$$

P [The student is left handed and a girl]

$$= 0.35 \times \frac{0.1}{0.35} = 0.51 \times \frac{0.1}{0.51}$$

2.11. Conclusion

Like other theories, the theory of Probability is a representation of Probabilities Concept in Formal terms that is, in terms that can be considered separately from there meaning.

2.12. Further study

1. Monga, G.S., Elementary Statistics.

2. Agrawal, B.M., Elhance, D.N., Elhance, Fundamental of Statistics.

3. Gupta, S.B., Principal of Statistics.

Unit – 3 Prior and Posterior Probabilities, Random Variables in Probability Functions

Objectives : After going through this unit you should be able to know about –

1. Basic Concept of Probability
2. Prior and Posterior Probabilities
3. Random Variables

Structure :

3.1. Introduction

The theory of probability provides a numerical measure of the element of uncertainty. It enables us to the decisions under conditions of uncertainty with a calculated risks.

Today the theory of Probability has been very extensively developed and there is hardly any disciplines – physical or social where it is not being extensively used. In the fiscal of Business and Economics it is very widely used for quantitative analysis of various problems and it forms the very basis of modern theory of Decision making i.e. decision making under conditions of uncertainty with calculated risks.

3.2. Simple Probability

The event could be either simple or compound. An event is called simple if it corresponds to a single possible outcome. Thus in tossing a dice, the chance of getting 3 is a simple event because 3 occurs in the dice only once. However, the chance of getting odd number is a compound event because odd numbers are more than one i.e. 1, 3 and 5.

A compound event can further be decomposed in to simple event e.g. if a dice is shown, getting an odd number in compound event but it can be decomposed into single event as getting (1, 3 or 5) i.e., there are three simple events for the above stated compound event.

3.3. Classical or a Priori Probability

This is the earliest approach to the theory of probability. *Laplace*, the French mathematician who belongs to this school defined probability as the 'ratio of the number of favourable cases to the total number of equally likely cases'.

This theory assumes that various outcomes of an event are 'equally likely' the device with which the experiment is performed is supposed to be a fair device and so the probability of their happening is also equal. In a way, the theory determines the probability of an event even before the event has taken place. It is for this reason that these probabilities are called a priori probabilities.

This theory is suitable for calculating the probabilities of various events in games of chance, where various events', are equally likely to happen. Thus, while tossing a dice, there are six possible ways in which an event can happen. They are supposed to be equally likely and the probability of getting 6 on a single throw of a dice would be 1/6. Similarly, the probability of getting a head in a single throw of a coin would be 1/2. Here, the presumption is that a coin can fall only in 2 ways each of which is equally likely. It precludes the possibility of a coin falling on its rim.

Thus, according to this theory, the probability of getting 6 in a single throw of a dice is 1/6 and the probability that dice will not throw $6 = 1 - \frac{1}{6} = \frac{5}{6}$ this.

It means that $p + q = 1$ or $1 - q = p$ or $1 - p = q$.

3.4. Relative Frequency Theory of Probability or Posterior Probabilities

The relative frequency approach to theory of probability is not based on a *priori* probabilities. In many situations it is not possible to have equally likely events, on which the classical theory of probability is based. For example, whether the prices of the shares of a company would go up or down are not two events which may be called equally likely. No doubt, there are three alternatives here: (i) the prices may remain constant, (ii) the prices may go up, and (iii) the prices may go down: but these events are not equally likely to happen. Similarly, whether a machine would turn out an unacceptable article or an acceptable article are two alternatives but they are not equally likely. Therefore, a priori probability theory cannot be applied here.

In such a situation, the probability of the happening of an event is determined on the basis of past experience or on the basis of relative frequency of success in the past. Thus, if a machine has been turning out 10% unacceptable articles in the past, the relative frequency for unacceptable articles would be 10% of the total items. However, relative frequency should always be estimated on the basis of a large number of readings in the past. The larger the number of past readings, the greater will be the accuracy of the result. Since in relative frequency approach probabilities are calculated on the basis of past experience, these probabilities are called posterior probabilities as contrasted with a priori probabilities calculated through the classical approach.

It should be understood that a *priori* probabilities are generally calculated in games of chance and *posterior probabilities* in problems relating to various types of economic and social phenomena where a *priori* probabilities are not constant. A *priori* probabilities are deductive in nature, and are based on theory instead of evidence or experience or experimentation. Posterior probabilities, also called **Empirical Probabilities,** are based on experience of the past and on experiments conducted. It is, thus, empirical in nature. For example, mortality tables, tables relating to expectation of life at various ages, birth rates, death rates, of depreciation of machinery etc., are all based on past experience.

The posterior probability or empirical probability (P) of an event is, thus, given by :

$$P = \frac{\text{Relative frequency}}{\text{Total Number of the items}}$$

If out of 1000 items produced by a machine in the past, 50 were found to be defective, the probability of a defective article to be produced by this machine would be $\frac{50}{1000}$ or 0.05.

3.5. Random Variable

The outcome of an experiment need not be a number, for example, the outcome when a coin is tossed can be 'heads' or 'tails'. However, we often want to represent outcomes as numbers. A random variable is a function that associates a unique numerical value with every outcome of an experiment. The value of the random variable will vary from trial to trial as the experiment is repeated.

There are two types of random variable – discrete and continuous.

A random variable has either an associated probability distribution (discrete random variable) or probability density function (continuous random variable).

1. A coin is tossed ten times. The random variable X is the number of tails that are noted X can only take the values 0, 1. ... 10, so X is a discrete random variable.

2. A light bulb is burned until it burns out. The random variable Y is its lifetime in hours. Y can take any positive real value, so Y is a continuous random variable.

Expected Value

The expected value (or population mean) of a random variable indicates its average or central value. It is a useful summary value (a number) of the variable's distribution.

Stating the expected value gives a general impression of the behaviour of some random variable without giving full details of its probability distribution (if it is discrete) or its probability density function (if it is continuous).

Two random variables with the same expected value can have very different distributions. There are other useful descriptive measures which affect the shape of the distribution, for example variance.

If X is a discrete random variable with possible values $x_1, x_2, x_3, \ldots, x_n$, and p(xi) denotes P(X = xi), then the expected value of X is defined by :

$$\mu = E(X) = \Sigma xi\ p(xi)$$

where the elements are summed over all values of the random variable X.

If X is a continuous random variable with probability density function f(x), then the expected value of X is defined by:

$$\mu = E(X) = \int x\ f(x) dx$$

Example

Discrete case : When a die is thrown, each of the possible faces 1 , 2, 3, 4, 5, 6 (the xi's) has a probability of 1/6 (the p(xi)'s) of showing.

The expected value of the face showing is therefore:

$$\mu = E(X) = (1 ' 1/6) + (2 ' 1/6) + (3 ' 1/6) + (4 ' 1/6) + (5 ' 1/6) + (6 ' 1/6) = 3.5$$

Notice that in this case. E(X) is 3 5, which is not a possible value of X.

Variance

The (population) variance of a random variable is a non-negative number which gives an idea of how widely spread the values of the random variable are likely to be; the larger the variance, the more scattered the observations on average.

Stating the variance gives an impression of how closely concentrated round the expected value the distribution is; it is a measure of the 'spread' of a distribution about its average value.

Variance is symbolised by V(X) or Var(X) or s^2

The variance of the random variable X is defined to be :

$$V(X) = \sigma^2 = E(X - E(X))^2 = E(X^2) - E(X)^2$$

where E(X) is the expected value of the random variable X.

Notes

a. the larger the variance, the further that individual values of the random variable (observations) tend to be from the mean, on average;

b. the smaller the variance, the closer that individual values of the random variable (observations) tend to be to the mean, on average;

c. taking the square root of the variance gives the standard deviation, i.e.: $\sqrt{V(X)} = \sqrt{\sigma^2} = \sigma = s$

d. the variance and standard deviation of a random variable are always non-negative.

3.6. Probability Distribution

The Probability distribution of a discrete random variable is a list of probabilities associated with each of its possible values. It is also sometimes called the probability function or the probability mass function.

More formally, the probability distribution of a discrete random variable X is a function which gives the probability p(xi) that the random variable equals xi, for each value xi :

$$p(xi) = P(X = xi)$$

It satisfies the following conditions :

a. $0 \le p(xi) \le 1$

b. $\Sigma p(xi) = 1$

3.7. Cumulative Distribution Function

All random variables (discrete and continuous) have a cumulative distribution function. It is a function giving the probability that the random variable X is less than or equal to x, for every value x.

Formally, the cumulative distribution function F(x) is defined to be : $F(x) = P(X \le x)$

for

$$-\infty < x < \infty$$

For a discrete random variable, the cumulative distribution function is found by summing up the probabilities as in the example below.

For a continuous random variable, the cumulative distribution function is the integral of its probability density function.

Example

Discrete case: Suppose a random variable X has the following probability distribution p(xi):

xi	0	1	2	3	4	5
p(xi)	1/32	5/32	10/32	10/32	5/32	1/32

This is actually a binomial distribution: Bi(5, 0.5) or B(5, 0.5). The cumulative distribution function F(x) is then:

xi	0	1	2	3	4	5
F(xi)	1/32	6/32	16/32	26/32	31/32	32/32

F(x) does not change at intermediate values. For example:

$F(1.3) = F(1) = 6/32$

$F(2.86) = F(2) = 16/32$

3.8. Probability Density Function

The probability density function of a continuous random variable is a function which can be integrated to obtain the probability that the random variable takes a value in a given interval.

More formally, the probability density function, f(x), of a continuous random variable X is the derivative of the cumulative distribution function F(x):

$$f(x) = \frac{d}{dx}F(x)$$

Since $F(x) = P(X \leq x)$ it follows that:

$$\int f(x)dx = F(b) - F(a) = P(a < X < b)$$

If f(x) is a probability density function then it must obey two conditions:

a. that the total probability for all possible values of the continuous random variable X is 1:

$$\int f(x)dx = 1$$

b. that the probability density function can never be negative : $f(x) > 0$ for all x.

3.9. Discrete Random Variable

A discrete random variable is one which may take on only a countable number of distinct values such as 0, 1, 2, 3, 4, ... Discrete random variables are usually (but not necessarily) counts. If a random variable can take only a finite number of distinct values, then it must be discrete. Examples of discrete random variables include the number of children in a family, the Friday night attendance at a cinema, the number of patients in a doctor's surgery, the number of defective light bulbs in a box of ten.

Continuous Random Variable

A continuous random variable is one which takes an infinite number of possible values. Continuous random variables are usually measurements. Examples include height, weight, the amount of sugar in an orange, the time required to run a mile.

Independent Random Variables

Two random variables X and Y say, are said to be independent if and only if the value of X has no influence on the value of Y and vice versa.

The cumulative distribution functions of two Independent random variables X and Y are related by

$$F(x, y) = G(x).H(y)$$

where

G(x) and H(y) are the marginal distribution functions of X and Y for all pairs (x, y).

Knowledge of the value of X does not effect the probability distribution of Y and vice versa. Thus there is no relationship between the values of independent random variables.

For continuous independent random variables, their probability density functions are related by

$$f(x, y) = g(x).h(y)$$

where

g(x) and h(y) are the marginal density functions of the random variables X and Y respectively, for all pairs (x, y).

For discrete independent random variables, their probabilities are related by

$$P(X = x_i; Y = y_j) = P(X = x_i).P(Y = y_j)$$

for each pair $(x_i y_j)$.

Probability-Probability (P–P) Plot

A probability-probability (P-P) plot is used to see if a given set of data follows some specified distribution. It should be approximately linear if the specified distribution is the correct model.

The probability-probability (P-P) plot is constructed using the theoretical cumulative distribution function, F(x), of the specified model. The values in the sample of data, in order from smallest to largest, are denoted $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$. For i = 1, 2, ..., n, $F(x_{(1)})$ is plotted against (i-0.5)/n.

Quantile-Quantile (Q-Q) plot

A quantile-quantile (Q-Q) plot is used to see if a given set of a data follows some specified distribution. It should be approximately linear if the specified distribution is the correct model.

The quantile-quantile (Q-Q) plot is constructed using the theoretical cumulative distribution function. F(x), of the specified model. The values in the sample of data, in order from smallest to largest, are denoted $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$. For i = 1, 2, ..., n, $x_{(i)}$ is plotted against $F^{-1}((i-0.5)/n)$.

3.10. Normal Distribution

Normal distributions model (some) continuous random variables. Strictly, a Normal random variable should be capable of assuming any value on the real line, though this requirement is often waived in practice. For example, height at a given age for a given gender in a given racial group is adequately described by a Normal random variable even though heights must be positive.

A continuous random variable X, taking all real values in the range (–¥, ¥) is said to follow a Normal distribution with parameters m and s if it has probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

we write

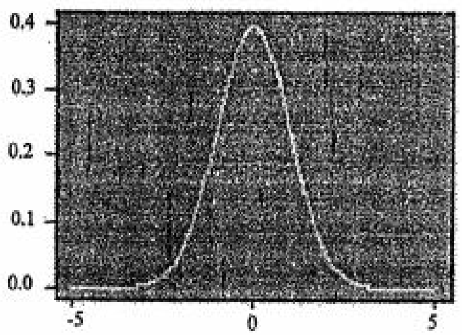
$$X \sim N(m, s^2)$$

This probability density function (p.d.f.) is a symmetrical, bell-shaped curve, centred at its expected value m. The variance is s^2.

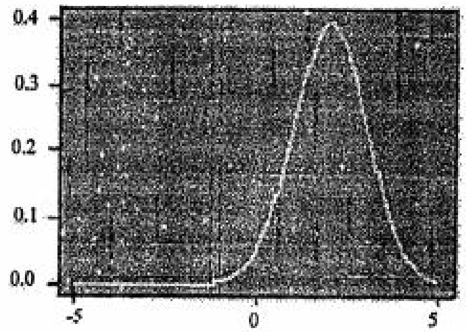
Many distributions arising in practice can be approximated by a Normal distribution. Other random variables may be transformed to normality.

The simplest case of the normal distribution, known as the Standard Normal Distribution, has expected value zero and variance one. This is written as N(0, 1).

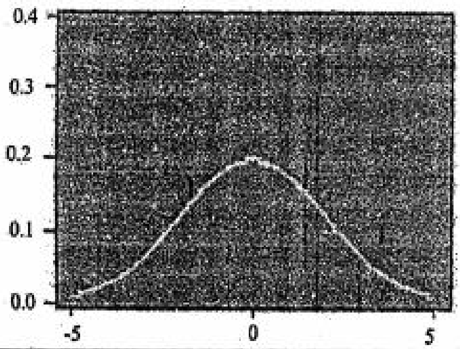
Normal (0, 1)
Probability Density Function



Normal (2, 1)
Probability Density Function



Normal (0, 2)
Probability Density Function



3.11. Poisson Distribution Steps

Poisson distributions model (some) discrete random variables. Typically, a Poisson random variable is a count of the number of events that occur in a certain time interval or spatial area. For example, the number of cars passing a fixed point in a 5 minute interval, or the number of calls received by a switchboard during a given period of time.

A discrete random variable X is said to follow a Poisson distribution with parameter m, written X ~ Po(m), if it has probability distribution

$$P(X = x) = \frac{m^x}{x!} e^{-m}$$

where

$$x = 0, 1, 2, ... n$$

$m > 0$.

The following requirements must be met :

a. the length of the observation period is fixed in advance:

b. the events occur at a constant average rate;

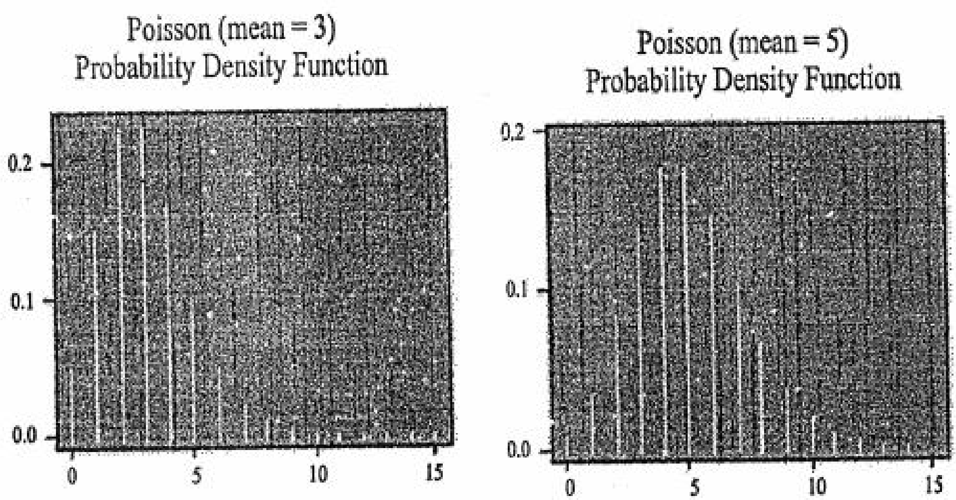
c. the number of events occurring in disjoint intervals are statistically independent.

The Poisson distribution has expected value $E(X) = m$ and variance $V(X) = m$; i.e. $E(X) = V(X) = m$.

The Poisson distribution can sometimes be used to approximate the Binomial

distribution with parameters n and p. When the number of observations n is large, and the success probability p is small, the Bi(n, p) distribution approaches the Poisson distribution with the parameter given by $m = np$. This is useful since the computations involved in calculating binomial probabilities are greatly reduced.

Examples



3.12. Binomial Distribution

Binomial distributions model (some) discrete random variables.

Typically, a binomial random variable is the number of successes in a series of trials, for example, the number of 'heads' occurring when a coin is tossed 50 times.

A discrete random variable X is said to follow a Binomial distribution with parameters n and p, written $X \sim Bi(n, p)$ or $X \sim B(n, p)$, if it has probability distribution

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where

x	=	0, 1, 2, ..., n
n	=	1, 2, 3, ...
p	=	success probability; $0 < p < 1$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

The trials must meet the following requirements :

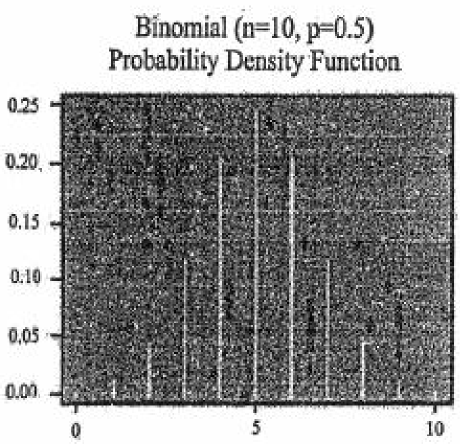
a. the total number of trials is fixed in advance;

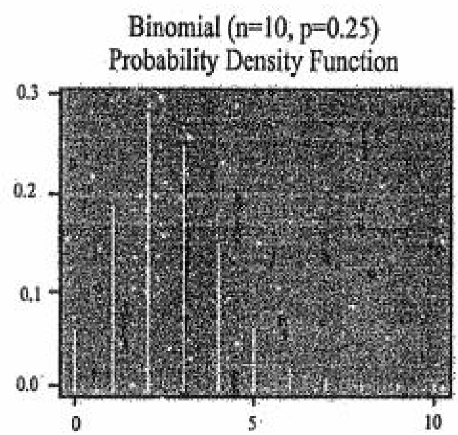
b. there are just two outcomes of each trial; success and failure.

c. the outcomes of all the trials are statistically independent:

d. all the trials have the same probability of success.

The Binomial distribution has expected value $E(X) = np$ and variance $V(X) = np(1-p)$.





3.13. Geometric Distribution

Geometric distributions model (some) discrete random variables. Typically, a Geometric random variable is the number of trials required to obtain the first failure, for example, the number of tosses of a coin until! the first 'tail' is obtained, or a process where components from a production line are tested, in turn, until the first defective item is found.

A discrete random variable X is said to follow a Geometric distribution with parameter p, written X ~ Ge(p), if it has probability distribution

$$P(X = x) = p^{x-1} (1 - p)^{x}$$

where

x = 1,2,3, ...

P = success probability; $0 < p < 1$

The trials must meet the following requirements :

a. the total number of trials is potentially Infinite;

b. there are just two outcomes of each trial; success and failure,

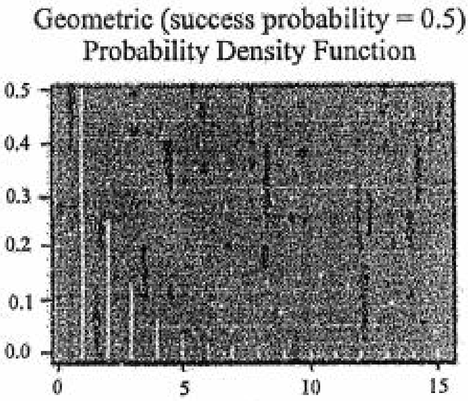
c. the outcomes of all the trials are statistically independent;

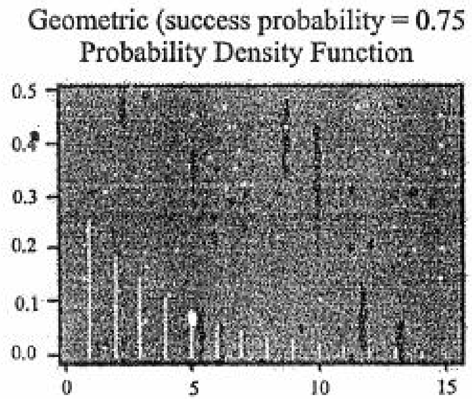
d. all the trials have the same probability of success.

The Geometric distribution has expected value $E(X) = 1/(1-p)$ and variance $V(X) = p/\{(1-p)^2\}$.

The Geometric distribution is related to the Binomial distribution in that both are based on independent trials in which the probability of success is constant and equal to p. However, a Geometric random variable is the number of trials until the first failure, whereas a Binomial random variable is the number of successes in n trials.

Examples





3.14. Uniform Distributions

Uniform distributions model (some) continuous random variables and (some) discrete random variables. The values of a uniform random variable are uniformly distributed over an interval. For example, if buses arrive at a given bus stop every 15 minutes, and you arrive at the bus stop at a random time, the time you wait for the next bus to arrive could be described by a uniform distribution over the interval from 0 to 15.

A discrete random variable X is said to follow a Uniform distribution with parameters a and b, written X ~ Un(a, b), if it has probability distribution

$$P(X = x) = 1/(b-a)$$

where

$$x = 1, 2, 3, \ldots\ldots, n.$$

A continuous random variable X is said to follow a Uniform distribution With parameters a and b, written X ~ Un(a, b). If its probability density function is constant within a finite interval [a, b], and zero outside this interval (with a less than or equal to b).

The Uniform distribution has expected value E(X) = (a+b)/2 and variance $\{(b-a)^2\}/12$.

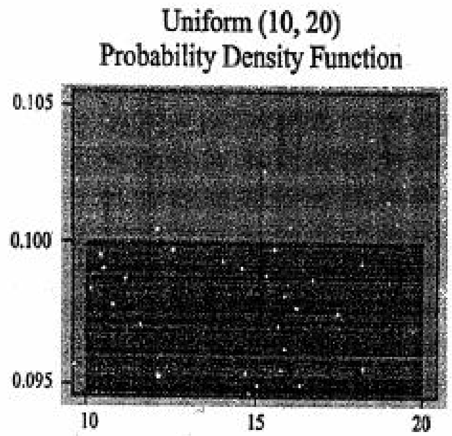
Example

0.2

0.1

0.0

0



3.15. Central Limit Theorem

The Central Limit Theorem states that whenever a random sample of size n is taken from any distribution with mean m and variance s^2, then the sample mean $\bar{x}$ will be approximately normally distributed with mean m and variance s^2/n. The larger the value of the sample size n, the better the approximation to the normal.

This is very useful when it comes to inference. For example, it allows us (if the sample size is fairly large) to use hypothesis tests which assume normality even if our data appear non-normal. This is because the tests use the sample mean, which the Central Limit Theorem tells us will be approximately normally distributed.

3.16. Conclusion

Today the theory of Probability has been very extensible developed and there is hardly any disciplines – Physical or Social where it is not being extensible used.

3.17. Further study

1. Abrams, William, A Brief History of Probability.

2. Ross Sheldom, A First Course in Probability.

3. Hogg, Robert, Introduction to Mathematical Statistics.

—x—x—

Unit – 4 Mathematical Expectation and Baye's Theorem

Objectives: After going through this unit you should be able to know about –

1. Concept of Mathematical Expectation.
2. Addition and Multiplication Laws of Expectation.
3. Baye's Theorem.

Structure:

4.1. Concept of Mathematical Expectation

4.2. Addition and Multiplication Law of Expectation.

4.3. Baye's Theorem

4.4. Second Method

4.5. Inverse Probability

4.6. Conclusion

4.7. Further Study

4.1. Concept of Mathematical Expectation

Mathematical expectation is the weighted arithmetic mean of values of a random variable with their respective probabilities as weights. The mean is weighted in the sense that the various values of a variable are multiplied by their respective probabilities.

If X is a random variable which can assume anyone of the values x_1, x_2, x_3 .... x_n with respective probabilities of $P_1, P_2, P_3 .... P_n$ then the mathematical expectation of X (or the expected value of X) which is denoted by E(x) would be:

$$E(X) = P_1x_2 + p_2x_2 + P_3x_3 + ... + P_nx_n$$

Thus, the expected value is the sum of the products of the values of X and the probabilities associated with each value. The term expected value is not very appropriate. For example, when we say that the expected value of the number of heads in 3 tosses of a coin is 3/2 or the expected value of a white ball being drawn from a bag containing 7 white and 4 red balls is $\frac{5}{4}$ the results are unrealistic.

In games of chance if a player would gain a sum 'a' if he wins and would lose a sum 'b' if he loses, then the mathematical expectation would be :

$(a \times p) + (-b \times q)$ here loss is regarded as a negative gain.

or $ap - bq$ where p is the Probability of gain and q the Probability of loss.

If the mathematical expectation of a game is zero, it is a fair game. If it is more than unity, it is biased to the player, i.e., the game is in favour of the player and if the mathematical expectation is a negative figure the game would obviously be biased against the player. In fact, mathematical expectation is the average in the long run.

Pl. Note that if the game is to be a fair game, the mathematical expectation is zero.

4.2. Addition and Multiplication Laws of Expectation

If X and Y are two random variable, then the expected value of X and Y together or $E(X+Y) = E(X) + E(Y)$

If X and Yare two independent random variable, then the expected value of (XY) or

$$E(XY) = E(X)\,'\,(Y).$$

Variability of X in Terms of Expectation

$$\sigma_x^2 = \text{variance of } X = \frac{\Sigma(X-\bar{X})^2}{N} = \frac{\Sigma X^2}{N} + \frac{\Sigma N\bar{X}^2}{N} - \frac{2\Sigma X\bar{X}}{N} = \frac{\Sigma X^2}{N} - \bar{X}^2$$

$$= \text{Arithmetic average of } X^2 - \left(\frac{\Sigma X}{N}\right)^2$$

= Arithmetic average of X^2 – [Arithmetic average of X]^2

= $E(X^2)-[E(X)]^2$

Since the expected value of X is the arithmetic mean of X series over a period of time,

Variance of $X = E[X-E(X)]^2 = E(X^2)-[E(X)]^2$

Some More Results

E (ax + b) = aE (x) + b

Variance (ax) =a^2 Var (x)

Variance (a + bx) = b^2 Var (x) [\ Variance (a) = 0]

Variance (ax + by) =a^2 Var (x) + b^2 Var (y)

where a and b are constants, and var. (a) = 0, var (b) = 0, var (ab) = 0

Variance (ax – by) = a^2 var (x) + b^2 var (y).

Example. A throws a coin 3 times. If he gets a head all the three times, he is to get a prize of Rs. 160. The entry fee for the game is Rs. 16. What is the mathematical expectation of A ?

Solution. A's probability of winning = $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

A's probability of losing is $1-\frac{1}{8}=\frac{7}{8}$

Mathematical Expectation of A = a x p + (– b x q)

[where a represents the prize money, b represents the entry fee and p and q refer to probabilities of winning and losing.]

$A=144\times\frac{1}{8}+\left(-16\times\frac{7}{8}\right)=18-14=\text{Rs. }4$

Thus, A's expectation is Rs. 6. The game is biased to the player. If A plays this game over a long period of time he stands to gains on an average, Rs. 6 per game.

$$P(R / Buy) = \frac{P(R).P(Buy / R)}{P(Buy)}$$

$$P(Buy) = P(TV).P(Buy / TV) + P(R).P(Buy / R) + P(News).P(Buy / News)$$

$0.18 + 0.15 + 0.02 = 0.35$ = Probability that the customer buys the product.

$$P(R/Buy) = \text{Required Probability} = \frac{0.50(0.15)}{0.35} = \frac{3}{14}$$

4.3. Baye's Theorem And Inverse Probability

Bayes' theorem is based on the proposition that probabilities should be revised when new information is available. The idea of revising probabilities is used by all of us in daily life even though we may not be knowing anything about probability. For example, a student while going to the college may start without taking a raincoat or umbrella, but as soon as he comes out of his home and sees a large mass of cloud in the sky, he may decide to take a raincoat or umbrella with him. In a way, he has revised his earlier decision of going to college without a raincoat or umbrella after he has seen the clouds in the sky. Thus, probabilities are revised as soon as some new information is available about the probable concerned.

The need of revising probabilities arises from a need to make better use of available information, and thereby reducing the element of risk involved in decision-making.

This idea of revising probabilities on the basis of new information was propounded by *Reverend Thomas Bayes* (1702-61). His theorem was actually published in 1763 after his death in the form of a short paper. Since then, it has evoked a lot of interest among mathematicians and has even become a subject of controversy. Some people feel that empirical tests or additional information may be misleading and may camouflage the truth. This is partially true and, sometimes, the probabilities revised on the basis of additional information may be misleading and may lead to wrong decision-making.

However, in general, the idea behind Bayes theorem is sound, and for this reason, it is very widely used in decision theory. On the basis of this theorem, probabilities of various outcomes are revised upwards or downwards, depending on the evidence obtained. Thus, if the probability of finding oil in Mumbai High was supposed to be 1/10, but after some explorations there was an indication of oil in Mumbai High, the probability of finding oil there would be revised upwards. This appears to be sensible. However, sometimes, such additional information may be misleading also.

The probabilities before revision are called *prior probabilities* and those after revision are called *posterior probabilities.*

Bayes' Theorem

Imagine a situation where two uncertain events (A) and (not A) are possible. Suppose, we know their probability, i.e., we know the probability of A's happening and also the probability of A's not happening. These probabilities are *prior probabilities* because they are probabilities before any further information is available. Suppose, an investigation is conducted. The investigation may have several outcomes which would be dependent on event A. For any particular outcome (which may be called B), the conditional probabilities P(B/A) and P(B not A) are available.

The result itself serves to revise the probabilities for event (A) and event (not A). The resulting values would be the *posterior probabilities* since they have been obtained after the results of the investigation.

The posterior probabilities are actually conditional probabilities of the form P(A/B) or P(not A/B).

Thus, according to Bayes' theorem, the posterior probability of event (A) for a particular result of an investigation (B) may be found from :

$$P(A/B)=\frac{P(A)\ P(B/A)}{P(A)\ P(B/A)+P(\text{Not A})\ P(B/\text{Not A})}$$

Example. Your favourite team is in the final playoffs. You have assigned a probability of 60% that they will win the championship. Past records indicate that when teams win the championship, they win the first game of the series 70% of the time. When they lose the series, they will win the first game 25% of the time. The first game is over; your team has lost. What is the probability that they will win the series.

Solution. Let B_1 be the event that the team wins the championship and B_2 the event that the team does not win the championship. Let A be the event that the team loses the first game.

$P(B_1) = 0.6, \quad P(B_2) = 0.4$

$P(A/B_1) = 0.3. \qquad P(AIB_2) = 0.75$

$$P(B_1/A) = \frac{P(B_1)\ P(A/B_1)}{P(B_1)\ P(A/B_1)+P(B_2)\ P(A/B_2)}$$

$$=\frac{0.6\times0.3}{0.6\times0.3+0.4\times0.75}=\frac{0.18}{0.18+0.3}=\frac{0.18}{0.48}=0.375$$

Example. Suppose, the probability of A's winning a prize in a Lottery is $\frac{1}{10}$.

Suppose, C, who speaks truth 2 times out on, informs A that he has won a prize. What is the probability that A has actually won a prize after C's information?

Solution. Let B_1 be the event that A wins a prize and B_2 the event that A does not win the prize. Let S be the event that C reports that A has won the prize $P(B_1)=\frac{1}{10}$,

$$P(B_2)=\left(1-\frac{1}{10}\right)=\frac{9}{10}$$

$P(S/B_1)$ = Conditional Probability that C reports that A has won when A has actually won $=\frac{2}{3}$.

$P(S/B_2)$ = Conditional Probability that C reports that A has won when A has not actually won $=\frac{1}{3}$.

$P(B_1/S)$ = Conditional Probability that A has won when C is known to have reported that A has won

$$=\frac{P(B_1)\,P(S/B_1)}{P(B_1)\,P(S/B_1)+P(B_2)\,P(S/B_2)}$$

$$=\frac{\frac{1}{10}\times\frac{2}{3}}{\frac{1}{10}\times\frac{2}{3}+\frac{9}{10}\times\frac{1}{3}}=\frac{\frac{1}{15}}{\frac{11}{30}}=\frac{2}{11}$$ **Ans.**

Example. A certain production process produces items that are 10% defective. Each item is inspected before being supplied to customers but the inspector incorrectly classifies an item 10% of the time. Only items classified as good are supplied. If 820 items in all are supplied, how many of them are expected to be defective.

Solution. There are two cases:

(i) The item is good and the inspector also declared it good for which the Probability

$$=\frac{[\text{illegible}]}{10}\times\frac{9}{10}=\frac{18}{100} \qquad \text{.... (A)}$$

(ii) The item is defective and the inspector declares it good for which the Probability

$$=\frac{1}{10}\times\frac{1}{10}=\frac{1}{100} \qquad \text{... (B)}$$

The number of defective items supplied in a lot of 820 items.

$$=820\times\frac{\frac{1}{100}}{\frac{81}{100}+\frac{1}{100}}=10 \quad \textbf{Ans.}$$

4.4. Second Method

This question can be done by Bayes' theorem also. Let B_1, B_2 be the events that item is acceptable and not acceptable, then,

$P(B_1)=\frac{9}{10}$ = Probability that the item is acceptable, $P(B_2)=\frac{1}{10}$ = Probability that the item is defective or unacceptable.

Let A be the event that the inspector declares the item as good.

$P(A/B_1)=$ The conditional probability that the item is declared as good by the inspector when the item is actually good =

$P(A/B_2)=$ Conditional Probability that the inspector declares the item as good when the item is actually defective.

$P(B_2/A) = \frac{1}{10} =$ Conditional Probability that item is supplied (declared good by the inspector) when the item is defective

$$= \frac{P(B_2)\, P(A/B_2)}{P(B_1)\, P(A/B_1) + P(B_2)\, P(A/B_2)}$$

$$= \frac{\frac{1}{10} \times \frac{1}{10}}{\frac{9}{10} \times \frac{9}{10} + \frac{1}{10} \times \frac{1}{10}} = \frac{1}{82}$$

= No. of defective items supplied as good in 820 items

$= \frac{1}{82} \times 820 = 10$ **Ans.**

Example. (a) The probability that a certain event happened was 10' and A, who is accurate in 49 cases out of 50, said that it happened. B agrees with A in stating that the event happened. B is accurate in 9 cases out of 10. What is the probability that it actually did occur?

(b) Suppose, if C, who is accurate in 7 cases out of 10, denies that the event mentioned above happened, what is the probability that it happened?

Solution. (a) Probability that the event happened and that A and B were right

$$= \frac{1}{10} \times \frac{49}{50} \times \frac{9}{10} = \frac{441}{5000}$$

Probability that the event did not happen and that A and B were wrong,

i.e., both said that the event happened $= \frac{9}{10} \times \frac{1}{50} \times \frac{1}{10} = \frac{9}{5000}$

Probability that the event actually did occur when it was reported to have happened by both A and B

$$= \frac{441/5000}{(441/5000) + (9/5000)} = \frac{441/5000}{450/5000} = \frac{441}{450} = \frac{49}{50}$$

(b) Probability that the event happened and A and B were right and C wrong

$$=\frac{1}{10}\times\frac{49}{50}\times\frac{9}{10}\times\frac{3}{10}=\frac{1323}{50000}$$

Probability that the event did not happen and A and B were wrong and C right

$$=\frac{9}{10}\times\frac{1}{50}\times\frac{1}{10}\times\frac{7}{10}=\frac{63}{50,000}$$

$\therefore$ Probability that the event actually did occur

$$=\frac{1323/50,000}{(1323/50,000)+(63/50,000)}=\frac{1323/50,000}{1386/50,000}=\frac{1323}{1386}=\frac{21}{22}$$

4.5. Inverse Probability

Inverse Probability is the probability of the happening of an event as a result of factor (a) if the event could have happened as a result of factors (a) or (b) or (c) .... (n).

This is another way of putting the Bayes' theorem. In Bayes' theorem, we actually find an inverse probability that the event happened and it was involved with the probability of one of the n factors associated with the event.

With this concept of inverse probability, Bayes' Theorem can be stated as under:

An event A can occur only if any one of the set of exhaustive and mutually exclusive events B_1, B_2 ...... B_n occurs. The probabilities $P(B_1)$, $P(B_2)$ .... $P(B_n)$ and the conditional probabilities $P(A/B_i)$ where i = 1,2, 3, ... n are known, then the conditional probability $P(B_i/A)$ when A has actually occurred is given by:

$$P(B_i / A) = \frac{P(B_i)\, P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\, P(A/B_i)} \qquad \text{where } i = 1, 2, 3, \ldots\ldots n$$

In practice, this formula is more useful.

Example. Suppose, a black ball has been drawn from one of the three bags, the first containing three black balls and seven white, the second five black balls and three white, the third eight black balls and four white: what is the probability that it was drawn from the first bag?

Let B_1, B_2 and B_3 are the events that first, second and third bags respectively are selected. Let A be the event that ball drawn is Black.

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

Then, $P(A/B_1)$ = Probability of drawing a black ball given that first bag is selected = $\frac{3}{10}$

$$P(A/B_2) = \frac{5}{8}, \quad P(A/B_3) = \frac{8}{12}$$

$$P(B_1 / A) = \frac{P(B_1)\, P(A/B_1)}{P(B_1)\, P(A/B_1) + P(B_2)\, P(A/B_2) + P(B_3)\, P(A/B_3)}$$

$$= \frac{\frac{1}{3} \times \frac{3}{10}}{\frac{1}{3} \times \frac{3}{10} + \frac{1}{3} + \frac{5}{8} + \frac{1}{3} \times \frac{8}{12}} = \frac{36}{191}$$

4.6. Conclusion

Mathematical Expectation is the weighted arithmetic mean of values of a Random Variables with their respective Probabilities as weight while Baye's theorem is based on the Proposition that Probabilities should revised when new information is available.

4.7. Further study

1. Jeffrey, R.C. Probability and the Art of Judgement.

2. Franklin, Evidence and Probability.

3. Gupta, S.B., Principal of Statistics.

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U.P.Rajarshi Tandon Open
University, Allahabad

MBA-1.4
Quantitative Techniques for
Business Decisions

Block
2

MBA-1.4/97

विशेषज्ञ -समिति

1. **Dr. Omji Gupta, Director SoMS UPRTOU, Allahabad**
2. **Prof. Arvind Kumar, Prof., Deptt. of Commerce, Lucknow University, Lucknow**
3. **Prof. Geetika, HOD, SoMS, MNNIT, Allahabad**
4. **Prof. H.K. Singh, Prof., Deptt. of Commerce, BHU, Varanasi**

लेखक

Dr. Sanjay Mishra, **Asso. Prof. MPJ Rohilkhand University, Bareilly.**

सम्पादक

Prof. S.A. Ansari, **Ex-Dean, Director & Head, MONIRBA, University of Allahabad.**

परिमापक

अनुवाद की स्थिति में

मूल लेखक	**अनुवाद**
मूल सम्पादक	**भाषा सम्पादक**
मूल परिमापक	**परिमापक**

सहयोगी टीम

संयोजक **Dr. Gaurav Sankalp, SoMS, UPRTOU, Allahabad.**

प्रकाशन --उत्तर प्रदेश राजर्षि टण्डन मुक्त विश्वविद्यालय, इलाहाबाद

प्रकाशन- उत्तर प्रदेश राजर्षि टण्डन मुक्त विश्वविद्यालय, प्रयागराज की ओर से डॉ. अरूण कुमार गुप्ता, कुलसचिव द्वारा पुनः मुद्रित एवं प्रकाशित वर्ष-2020।

मुद्रक- चन्द्रकला यूनिवर्सल प्राइवेट लिमिटेड 42/7 जवाहर लाल नेहरू रोड, प्रयागराज-211002

Block 2 : Quantitative Techniques for Business Decisions

Block Introduction

Block Two contains four units. Unit four deals with probability distribution, Binomial, Poisson and Normal distribution. Unit Six deals with normal compling distribution and unit. seven deals probability and decision making while the eight unit deals with decision making under uncertainity.

Unit – 5 Probability Distributions : Binomial, Poisson and Normal Distributions

Objectives :

After going through this unit you should be able to know about –

1. random experiment and random variable.
2. probability distributions.
3. Binomial distribution
4. conditions under which Binomial distribution is applicable
5. Poisson distribution.
6. conditions under which Poisson distribution is applicable; and
7. Normal distribution.

Structure :

5.1. Introduction

In an experiment related with tossing of a coin on throwing of a dice nobody knows exactly what we will get in a particular toss or throw, If we are tossing a coin we do not exactly know whether we will get head or tail upward in a particular toss. In a similar manner when we are throwing a dice we do not exactly know whether we will get 1, 2, 3, 4, 5 or 6 in a particular throw. Such types of experiments are called random experiments.

In these experiments, tossing a coin or throwing the dice is known as **trial** and getting head or tail in a particular toss or getting 1, 2, 3, 4, 5 or 6 in a particular throw are known as **events** or **outcomes**.

A **random variable** is a numerical measure of the outcomes or events of a random experiment, so its value is determined by chance. For example, when two coins are tossed at a time, the different possible outcomes are HH, HT, TH and TT. The number of heads appearing in the outcomes – 0, 1 and 2 could be taken as random variable.

In certain cases the outcomes or events are quantitative or numerical like the number of mobile phones sold during a particular day or the number of people, aged 50 years in a group, who will remain alive to see their 51st birthday, etc. In these cases the outcome number like the number of mobile phones sold or the number of people who will remain alive could be taken as random variable.

In certain other cases the outcomes or events are not quantitative or numerical, like an item manufactured by a machine is defective or not defective. In this case we associate with each outcome a real number like 0 if item is defective and 1 if item is not defective. And this real number could be taken as random variable.

There are two types of random variables – discrete random variables and continuous random variables.

A **discrete random variable** takes only a finite or countable number of values. For example, the number of light bulbs that burn out in a room of 10 light bulbs in the next one year or the number of children in a family, etc.

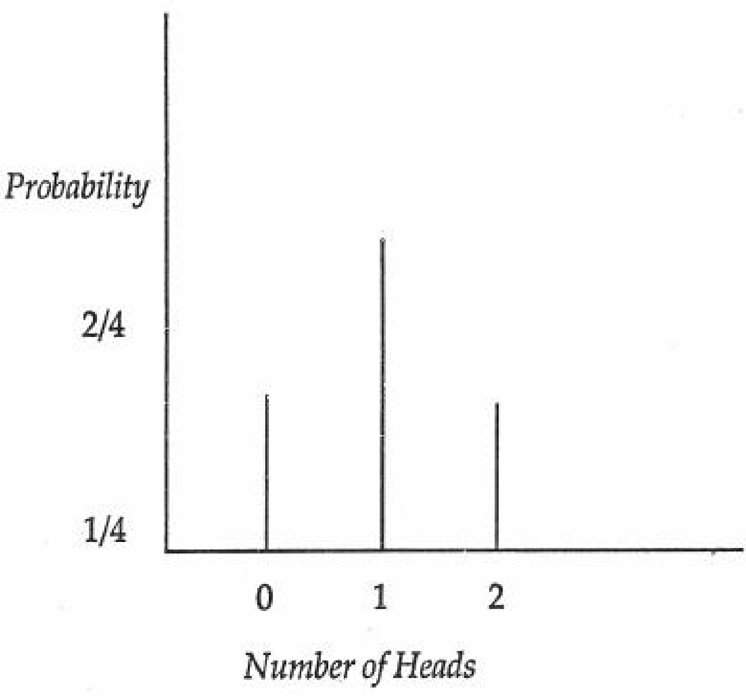
A **continuous random variable** takes all possible values in its range or it is a random variable that results from measurement. For example, the length of time between calls to 100 or life time of tubelights in hours, etc.

The **probability distribution** of a random variable provides the possible values of the random variable and their corresponding probabilities. The probability distribution could be in the form of a table, graph, or mathematical formula.

If we take the example of tossing two coins simultaneously and noting down number of heads appearing in the outcomes (random variable) – 0, 1 and 2. Then this information could be represented through table given below:

No of Heads	Probability
0	¼
1	½
2	¼

This information could also be depicted with the help of graph given below:



This information could also be represented with the help of mathematical formula which is referred to as probability mass function or probability density function.

The graph of probability distribution gives us an idea about the shape, central tendency of distribution (called mean or expected value) and the amount of variation present in distribution.

Expected Value of a Probability Distribution

The expected value of probability distribution gives us an idea about the central tendency of the distribution. It is calculated as follows:

$$E(X) = X_1 P(X_1) + X_2 P(X_2) + X_3 P(X_3) + \dots\dots\dots\dots + X_n P(X_n)$$

$$= \text{“}\; X_i P(X_i)$$

Where X_i is the value of random variable and $P(X_i)$ is the probability of the occurrence of that value of random variable.

Some important properties of mathematical expectation are:

1) $E(c) = c$, where c is any constant.

2) $E(cX) = c\,E(X)$, where c is any constant.

3) $E(aX + b) = a\,E(X) + b$ where a & b are constants.

In case of tossing of two coins simultaneously, the expected value is:

$$E(X) = \text{“}\; X_i P(X_i)$$

$$= X_1 P(X_1) + X_2 P(X_2) + \dots\dots\dots\dots + X_n P(X_n)$$

$$= X_1 P(X_1) + X_2 P(X_2) + X_3 P(X_3)$$

$$= 0 * 1/4 + 1 * 2/4 + 2 * 1/4$$

$$= 1$$

Variance of a Probability Distribution

The variance of probability distribution gives us an idea about the variation or spread of the distribution. It is calculated as follows:

$$\text{Variance}(X) = E\{X_i - E(X)\}^2$$

$$= \text{“}\{X_i - E(X)\}^2 P(X_i)$$

Some important properties of variance are:

1) $Var(X + c) = Var(X)$ where c is a constant.

2) $Var(aX) = a^2\, Var(X)$

3) Var (a X +b) = a^2 Var(X)

4) Var (c) = 0 where c is a constant

In case of tossing of two coins simultaneously, the variance is:

$$\text{Variance}(X) = E\{X_i - E(X)\}^2$$
$$= "\{X_i - E(X)\}^2 P(X_i)$$
$$= \{0-1\}^2 * 1/4 + \{1-1\}^2 * 2/4 + \{2-1\}^2 * 1/4$$
$$= 1/4 + 0 + 1/4$$
$$= 2/4 = 1/2$$

5.2 Binomial Distribution

The letters 'Bi' at the beginning of any word signifies two and binomial distribution is no exception to it. A random variable is said to follow a binomial distribution when it can take only two values in each of the trial. For example, tossing a coin can result in either head or tail; a new born baby could be either male or female; a person selected randomly could be either employed or unemployed, etc. These outcomes are often labeled as "success" or "failure." Also the outcome of one trial does not influence the outcome of other trials and probability of success remains same in all the trials.

Historically speaking, binomial distribution was given by James Bernoulli in the year 1700 and was first published in 1713. An experiment with two mutually disjoint outcomes - success/failure - is called a Bernoulli trial or Bernoulli experiment. The binomial distribution is the discrete probability distribution of the number of successes in a sequence of 'n' independent Bernoulli experiments/trials (where p = probability of success and q = probability of failure = 1 – p remains same in all trials); when $n = 1$, the binomial distribution is a Bernoulli distribution.

The essential characteristics which any distribution must possess in order to be treated as binomial distribution are:

- The experiment is repeated 'n' number of times under identical conditions.

- Each trial results in only one of the two possible outcomes, namely, success or failure; hit or miss; head or tail; defective or non defective; etc.
- The probability of success (p) or failure (q = 1-p) remains constant in each trial.
- The trials are independent of each other i.e. the outcome of one trial does not influence the outcome of other trials.

Under these conditions 'x,' the number of successes in 'n' trials represents a binomial variable. We can associate a probability with each value which 'x' can take and derive a probability distribution known as binomial distribution. This binomial distribution can be represented with the help of a mathematical formula. The mathematical formula used for computing the probability of getting 'x' successes in 'n' trials is as follows:

$$f(x) = {}^{n}C_{x} p^{x} q^{n-x}; \qquad x=0,1,\ldots\ldots,n$$

The sum of probabilities P(x=0), P(x=1), P(x=2),................. P(x=n) is always equal to one. i.e. $\sum {}^{n}C_{x} p^{x} q^{n-x} = 1$

The two parameters of the binomial distribution are n and p and with the help of these we can find the probabilities for different values of 'x.'

Mean and Variance of Binomial Distribution

The mean of binomial distribution is the product of its two parameters, namely, n and p.

$$\mu = E(X_i) = \sum x f(x)$$

$$= \sum x\,{}^{n}C_{x} p^{x} q^{n-x}$$

$$= np$$

The variance of binomial distribution is defined as follows:

$$\text{Variance}(X) = E\{X_i - E(X)\}^2$$

$$= \sum \{X_i - E(X)\}^2 P(X_i)$$

$$= \sum \{x - np\}^2 f(x)$$

$$= \sum \{x - np\}^2\,{}^{n}C_{x} p^{x} q^{n-x}$$

$= npq$

And Standard Deviation $= \sqrt{npq}$

Shape of Binomial Distribution

The shape of binomial distribution depends on the values of n and p. If the value of $p = 0.5$ then the distribution is symmetrical, if the value of $p < 0.5$ then the distribution is positively skewed and if $p > 0.5$ then the distribution is negatively skewed.

In general as the number of trials increases (i.e. n increases) the skewness decreases. As a thumb rule, if $np(1 - p) > 10$, then the probability distribution will be approximately symmetrical or bell-shaped.

5.3 Examples of Binomial Distribution

5.3.1 Example On an average one telephone number out of fifteen is busy. What is the probability that out of six randomly selected telephone numbers not more than three will be busy?

Solution

Here $n = 6$, $p = 1/15$ and $q = 14/15$. We know that the probability of getting 'x' successes in 'n' trials is given by the formula

$$P(x) = f(x) = {}^{n}C_{x} p^{x} q^{n-x}$$

The probability of getting atmost 3 busy telephone numbers is

$= P(0) + P(1) + P(2) + P(3)$

$= f(0) + f(1) + f(2) + f(3)$

$= {}^{6}C_{0} (1/15)^{0}(14/15)^{6} + {}^{6}C_{1}(1/15)^{1}(14/15)^{5} + {}^{6}C_{2} (1/15)^{2}(14/15)^{4} + {}^{6}C_{3} (1/15)^{3}(14/15)^{3}$

$= (14)^{3}/(15)^{6} \{(14)^{3} + 6(14)^{2} + 15(14) + 20\}$

$= (2744 * 4150) / 11390625$

$= 0.998$

5.3.2 Example In an experiment seven coins are tossed simultaneously. Find the probability of getting

(a) exactly 3 heads?

(b) atleast 3 heads?

(c) atmost 2 heads?

Solution

Here n = 7, p = 1/2 and q = 1/2. We know that the probability of getting 'x' successes in 'n' trials is given by the formula

$$P(x) = f(x) = {}^{n}C_{x} p^{x} q^{n-x}$$

(a) Therefore the probability of getting exactly 3 heads is

$P(3) = f(3) = {}^{7}C_{3} (1/2)^{3} (1/2)^{4}$

$= \{(7*6*5)/(3*2*1)\}*\{1/8\}*\{1/16\}$

$= 35/128$

(b) The probability of getting atleast 3 heads is

$= P(3) + P(4) + P(5) + P(6) + P(7)$

$= f(3) + f(4) + f(5) + f(6) + f(7)$

$= 1 - \{f(0) + f(1) + f(2)\}$

$= 1 - \{{}^{7}C_{0} (1/2)^{0} (1/2)^{7} + {}^{7}C_{1} (1/2)^{1} (1/2)^{6} + {}^{7}C_{2} (1/2)^{2} (1/2)^{5}\}$

$= 1 - \{1*1*(1/128) + 7*(1/2)*(1/64) + [(7*6)/(2*1)]*(1/4)*(1/32)\}$

$= 1 - \{(1/128) + (7/128) + (21/128)\}$

$= 1 - \{29/128\}$

$= 99/128$

(c) The probability of getting atmost 2 heads is

$= P(0) + P(1) + P(2)$

$= f(0) + f(1) + f(2)$

$= (1/128) + (7/128) + (21/128)$

$= 29/128$

5.3.3 Example According to a survey it was found that 20% of all households in town have one or more cars. In a random sample of 5 households what is the probability that

(a) exactly 2 have one or more cars?

(b) atmost 1 has one or more cars?

(c) atleast 2 have one or more cars?

Solution

Here n = 5, p = 1/5 and q = 4/5. We know that the probability of getting 'x' successes in 'n' trials is given by the formula

$P(x) = f(x) = {}^{n}C_{x} p^{x} q^{n-x}$

(a) Therefore the probability that exactly 2 households have one or more cars is

$P(2) = f(2) = {}^{5}C_{2} (1/5)^{2} (4/5)^{3}$

$= \{10\}*\{1/25\}*\{64/125\}$

$= 28/625$

(b) The probability that atmost 1 household has one or more cars is

$= P(0) + P(1)$

$= f(0) + f(1)$

$= {}^{5}C_{0} (1/5)^{0} (4/5)^{5} + {}^{5}C_{1} (1/5)^{1} (4/5)^{4}$

$= 1* 1* (1024/3125) + 5* (1/5)* (256/625)$

$= 2304/3125$

(c) The probability that atleast 2 households have one or more cars is

$= P(2) + P(3) + P(4) + P(5)$

$= f(2) + f(3) + f(4) + f(5)$

$= 1 - \{ f(0) + f(1)\}$

$= 1 - \{ 1* 1* (1024/3125) + 5* (1/5)* (256/625)\}$

$= 1 - \{2304/3125\}$

$= 821/3125$

5.3.4 Example Twenty percent of the items manufactured by a machine are defective. If 200 items are randomly picked, what is the expected number of items which will be found to be defective? Also calculate the variance for the number of defective items.

Solution

Here n = 200, p = 0.20 and q = 0.80.

Therefore

E(X) = np

= 200*0.20

= 40 is the expected number of items which will be found to be defective.

Variance (X) = npq

= 200*0.20*0.80

= 32

5.4 Poisson Distribution

The Poisson distribution was originally given by Simeon Denis Poisson (1781-1840) in 1838. It is a discrete probability distribution for the counts of events that occur randomly in a given interval of time, area, volume, length etc. i.e. it is the distribution of rare events in which the probability of occurrence of events (p) is very small but the number of trials (n) is very large so that the product 'np' is finite. For example,

- The number of particles emitted by a radioactive source in a given time.
- The number of car accidents on the road in front of Parliament House.
- The number of defects on a piece of cloth.
- The Number of telephone calls in a week.

- The Number of industrial accidents per month in a manufacturing plant.
- The number of printing mistakes per page in a book.

The Poisson distribution can be represented with the help of a mathematical formula. The mathematical formula used for computing the probability of 'x' occurrences in 'n' trials is as follows:

$$f(x) = \frac{e^{-m} m^x}{x!}; \qquad x=0,1,.......,"$$

where

'p' is the probability of occurrence

'n' is the number of trials

'm' is the expected number of events

'e' is mathematical constant approximated by 2.71828

'x' is the number of occurrences

There is only one parameter in Poisson distribution and that is 'm.' If 'm' is known then all the probabilities for different values of 'x' can be calculated.

The essential characteristics which any distribution must possess in order to be treated as Poisson distribution are:

- The variable i.e. number of occurrences should be a discrete variable.
- The probability of occurrence of events (p) should be very small.
- Theoretically an infinite number of occurrences must be possible.
- The occurrences should be independent of each other.

Mean and Variance of Poisson Distribution

The mean of Poisson distribution is defined as

$$\mu = E(X_i) = \text{“} x f(x)$$

$$= \sum x \frac{e^{-m} m^x}{x!}$$

$$= m$$

The variance of binomial distribution is defined as follows:

$$\text{Variance}(X) = E\{X_i - E(X)\}^2$$
$$= \sum \{X_i - E(X)\}^2 P(X_i)$$
$$= \sum \{x - m\}^2 f(x)$$
$$= \sum \{x - m\}^2$$
$$= m$$

And Standard Deviation $= \sqrt{m}$

5.5 Examples of Poisson Distribution

5.5.1 Example If a random variable 'x' follows a Poisson distribution with mean 2, find f(x=0) and f(x=2).

Solution

For a Poisson distribution the probability of getting 'x' occurrences is given by the formula

$$P(x) = f(x) = \frac{e^{-m} m^x}{x!}$$

Here 'm' is given to be equal to 2 i.e. m = 2

(a) Therefore the probability of getting zero i.e. P(0) is

$$P(0) = f(x=0) = \frac{e^{-2} 2^0}{0!}$$
$$= \{0.13534*1\}/\{1\}$$
$$= 0.13534$$

(b) The probability of getting 2 i.e. P(2) is

$$= P(2) = f(2)$$
$$= \frac{e^{-2} 2^2}{2!}$$
$$= \{0.13534*4\}/\{2\}$$
$$= 0.27068$$

5.5.2 Example On an average one blade out of 400 is found to be defective. If the blades are packed in boxes of 100, what is the probability that a box selected at random will contain less than two defective blades?

Solution

For a Poisson distribution the probability of getting 'x' occurrences is given by the formula

$$P(x) = f(x) = \frac{e^{-m} m^x}{x!}$$

Here, p = 1/400; n = 100; and

m = np = 100/400 = 0.25

Therefore the probability that a box selected at random will contain less than two defective blades is

= P(0) + P(1)

= f (0) + f (1)

$$= \frac{e^{-0.25} 0.25^{\circ}}{0!} + \frac{e^{-0.25} 0.25^1}{1!}$$

$= e^{-0.25}\ (1 + 0.25)$

= 0.7788 (1.25)

= 0.9735

5.5.3 Example The daily production of factory on an average contains three defective items. What is the probability that the production on a particular day will contain

(a) no defective item?

(b) at the most two defective items?

(c) atleast three defective items?

(Given = 0.0498)

Solution

For a Poisson distribution the probability of getting 'x' occurrences is given by the formula

$$P(x) = f(x) = \frac{e^{-m} m^x}{x!}$$

Here 'm' is given to be three i.e. m = 3

(a) Therefore the probability of getting zero defective items is

$$P(0) = f(0) = \frac{e^{-3}3^{\circ}}{0!}$$

$= \{0.0498*1\}/\{1\}$

$= 0.0498$

(b) The probability of getting atmost two defective items is

$= P(0) + P(1) + P(2)$

$= f(0) + f(1) + f(2)$

=

$= 0.0498 + 0.1494 + 0.2241$

$= 0.4233$

(c) The probability of getting atleast three defective items is

$= P(3) + P(4) + P(5) + \ldots\ldots\ldots\ldots\ldots\ldots$

$= f(3) + f(4) + f(5) + \ldots\ldots\ldots\ldots\ldots\ldots$

$= 1 - \{f(0) + f(1) + f(2)\}$

$$\frac{e^{-3}3^{\circ}}{0!} + \frac{e^{-3}3^{3}}{1!} + \frac{e^{-3}3^{2}}{2!}$$

$= 1 - \{0.0498 + 0.1494 + 0.2241\}$

$= 1 - \{0.4233\}$

$= 0.5767$

5.5.4 Example If for a Poisson distribution P(X=2) = (1/3) P(X=4), then find the mean and variance of X.

Solution

For a Poisson distribution the probability of getting 'x' occurrences is given by the formula

$$P(x) = f(x) = \frac{e^{-m}m^{x}}{x!}$$

$$\text{Now } P(2) = f(2) = \frac{e^{-m}m^{2}}{2!}$$

And $P(4) = f(4) = \dfrac{e^{-m}m^4}{4!}$

Given $P(X=2) = (1/3)\,P(X=4)$

Or $= \dfrac{e^{-m}m^2}{2!} = (1/3)\dfrac{e^{-m}m^4}{4!}$

Or $m^2 = (3)\dfrac{4!}{2!}$

Or $m^2 = 3*4*3*$

$= 36$

Or $m = 6$

Therefore

mean of X = m = 6

And variance of X = m = 6

5.6 Normal Distribution

The normal distribution is the most widely known and commonly used continuous probability distribution. It is also referred to as Gaussian distribution in honour of famous German Mathematician Carl Gauss (1777 – 1855). Gauss published a book on the Mathematics of Planetary Orbits in which he further developed the theory of least squares regression by analyzing the errors. The analysis of these errors led to the discovery that errors follow a normal distribution.

The distribution of errors could be understood by taking the example of shooting competition in Olympics. In this competition each shooter tries to hit the target. But chances are that some of the times the shooters will be missing the target and there will be deviations of different magnitudes from the target. These deviations from the target, which will be on both sides, are also referred to as errors. The frequency of these errors will be less for larger deviations. The curve representing the distribution of these errors will have mode corresponding to the true position of target and will taper off gradually at same rate on both sides of the target position. Some of the situations which are quite akin to the distribution of errors and where normal distribution is applicable are:

- Download time for a page from internet.
- Daily usage of raw material over a period of year where the production plan is uniform.
- Time for completion of a job by different clerks in an organization.
- IQ scores of management students in a state.
- Weight of children at birth.
- Weekly sales of an item over a period of year.
- Average time spent by users on a social networking site in India.
- Mileage per litre of gasoline of 100 cars of same type in a particular district.

The normal distribution can be represented with the help of a mathematical function. The mathematical function used for representing normal distribution, the normal probability density function is given as follows:

$$f(X)=\frac{1}{\sigma\sqrt{2\pi}}e\left(-\frac{1}{2}\right)\left(\frac{x-\mu}{\sigma}\right)^2;-\infty<x<\infty$$

where

'x' is any value of continuous variable

'μ' is the mean of the variable

'ó' is the standard deviation of the variable

'' is mathematical constant approximated by 3.14159

'e' is mathematical constant approximated by 2.71828

There are two parameters in normal distribution, namely, μ and ó.' If these two are known then all the probabilities for different values of 'x' can be calculated.

5.7 Key Words

1. Random Experiment – An experiment in which we do not know exactly what we will get in a particular trial. These are called random experiments.
2. Events or Outcomes – The results of the trials of random experiments are known as events or outcomes.
3. Random Variable – A random variable is a numerical measure of the outcomes or events of a random experiment and its value is determined by chance.
4. Discrete Random Variable – A discrete random variable takes only a finite or countable number of values. For example, the number of rooms in a building, the number of children in a family, etc.
5. Continuous Random Variable – A continuous random variable takes all possible values in its range or it is a random variable that results from measurement. For example, the length of time between calls to 100 or life time of tubelights in hours, etc.
6. Probability Distribution – It is a mathematical function of random variable that provides the possible values of the random variable and their corresponding probabilities.
7. Mathematical Expectation – The mathematical expectation of probability distribution gives us an idea about the central tendency of the distribution.
8. Variance – The variance of probability distribution gives us an idea about the variation or spread of the distribution.
9. Bernoulli Experiment – An experiment with two mutually disjoint outcomes success/failure is called a Bernoulli trial or Bernoulli experiment.
10. Binomial Distribution – The binomial distribution is the discrete probability distribution of the number of successes in a sequence

of 'n' independent Bernoulli experiments/trials; when $n = 1$, the binomial distribution is a Bernoulli distribution.

11. Mean and Variance – The mean of binomial distribution is 'np' and variance is 'npq.'

12. Poisson Distribution – It is a discrete probability distribution for the counts of events that occur randomly in a given interval of time, area, volume, length etc. i.e. it is the distribution of rare events in which the probability of occurrence of events (p) is very small but the number of trials (n) is very large so that the product 'np' is finite.

13. Mean and Variance – The mean of Poisson distribution is 'm' and variance is also 'm,' where 'm' is equal to 'np.'

14. Normal Distribution --The normal distribution is the most widely known and commonly used continuous probability distribution. It is also referred to as Gaussian distribution.

5.8 Self Assessment Test

5.8.1 Explain the terms events and random variable with examples.

5.8.2 A player tosses 3 fair coins. He wins Rs.5 if 3 heads appear Rs.3 if 2 heads appear and Re.1 if 1 head occurs. On the other hand he losses Rs.15/- if all tails occur. Find the expected gain.

5.8.3 Define binomial distribution and state the conditions which must be satisfied for the application of binomial distribution.

5.8.4 A pair of dice is thrown 12 times. If getting a doublet is considered a success find the probability of

(i) 4 success

(ii) no success

5.8.5 If on an average 1 ship out of 10 do not arrive safely to ports. Find the mean and the standard deviation of ships returning safely out of a total of 500 ships.

5.8.6 The overall percentage of failures in an examination is 30%. If five candidates appeared in the examination, what is the probability that

(i) three candidates will clear the examination, and

(ii) atleast three candidates will be able to clear the examination?

5.8.7 In a binomial distribution with p = ½, how many trials will be required to guarantee a standard deviation equal to 3% of expected value.

5.8.8 Define Poisson distribution and state the conditions which must be satisfied for the application of Poisson distribution.

5.8.9 The probability that a man aged 50 years will die within a year is 0.05. What is the probability that out of 15 men

(i) four will die within a year,

(ii) atmost four will die within a year, and

(iii) atleast five will die within a year.

5.8.10 It is found that the number of accidents occurring near the market complex of a colony follows a Poisson distribution with a mean of 2 accidents per week. Find the probability that

(i) no accident occurs in a week, and

(ii) number of accidents in a week exceeds 2.

5.8.11 A car rental agency has six cars for hiring out. If the demand for cars follows a Poisson distribution with mean 3.0 then find the probability of

(i) no car being hired, and

(ii) some of the demand being not met.

5.8.12 What is a Normal Distribution? Explain with examples. Also state the conditions which must be satisfied for the application of Normal Distribution

5.9 Further Reading

1. Levin , R.I. : Statistics for Management(PHI)

2. Gupta , S.P. & Gupta , M.P. : Business Statistics

UNIT 6 NORMAL SAMPLING DISTRIBUTION

Objectives

After studying this unit, you would be able to understand:

- Normal distribution;
- Standard Normal distribution;
- The properties of Normal distribution; and
- How to use the area table of standard normal variate.

Unit Structure

6.1 Introduction

6.2 Relationship among Binomial, Poisson and Normal Distribution

6.3 Properties of Normal Distribution

6.4 Area Property of Normal Curve

6.5 Examples of Normal Distribution

6.6 Key Words

6.7 Self Assessment Test

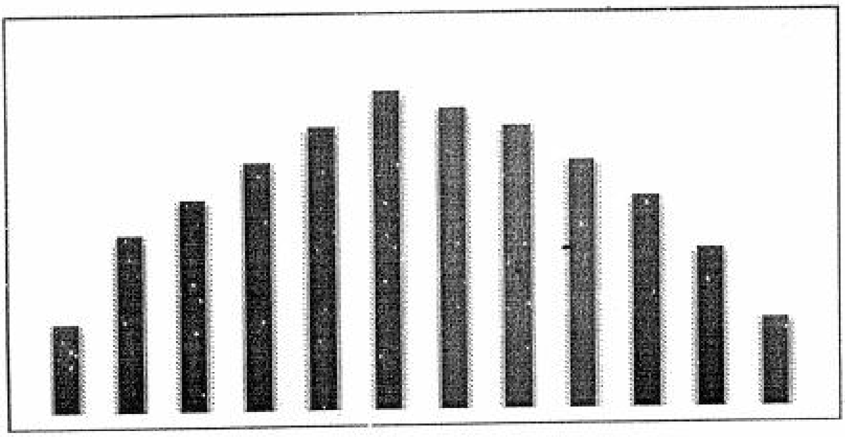
6.8 Area Table

6.9 Further Readings

6.1 Introduction

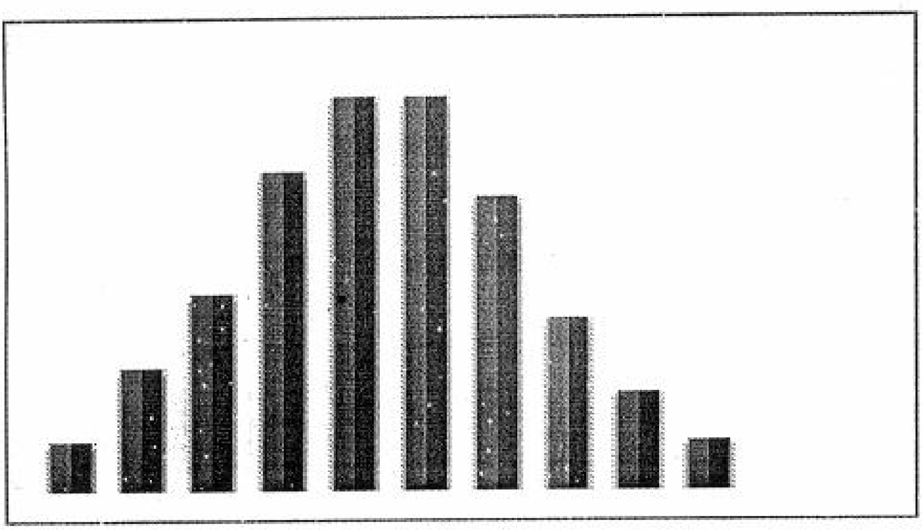
There are many sets of data which we come across in life that follows normal distribution. Some of these are:

- The height of students belonging to class X follow a pattern of distribution which could be represented through the diagram given below:

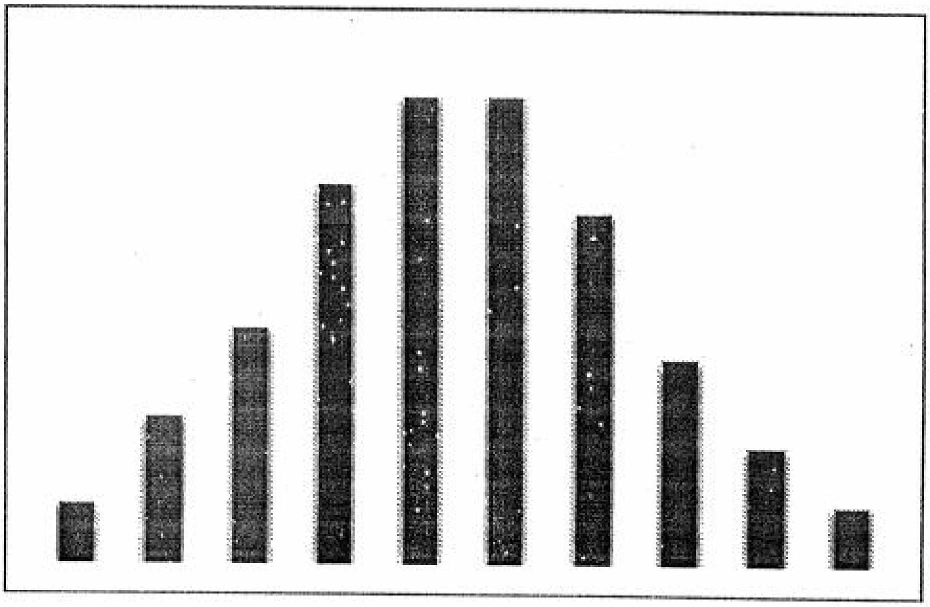


Height of students

- In the mass-production of piston rings with mean inside diameter of 45mm the piston rings with diameters in the range 44.95mm to 45.05mm are acceptable. The distribution of these deviations (errors) (errors are on x-axis and frequency on y-axis) follow a pattern of distribution which could be represented through the diagram given below:



- If we are interested in the average test scores of students belonging to class XII in a particular college. Then these scores follow a pattern of distribution which could be represented through the diagram given below:



Average test scores of students

These frequency histograms that are approximately symmetric and bell-shaped are said to have the shape of a normal probability density function (or normal probability curve). The mathematical function used for representing normal distribution, the normal probability density function is given as follows:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{\left(-\frac{1}{2}\right)\left(\frac{x-\mu}{\sigma}\right)^2}$$;-" < x < "

where

'x' is any value of continuous variable

'μ' is the mean of the variable

'σ' is the standard deviation of the variable

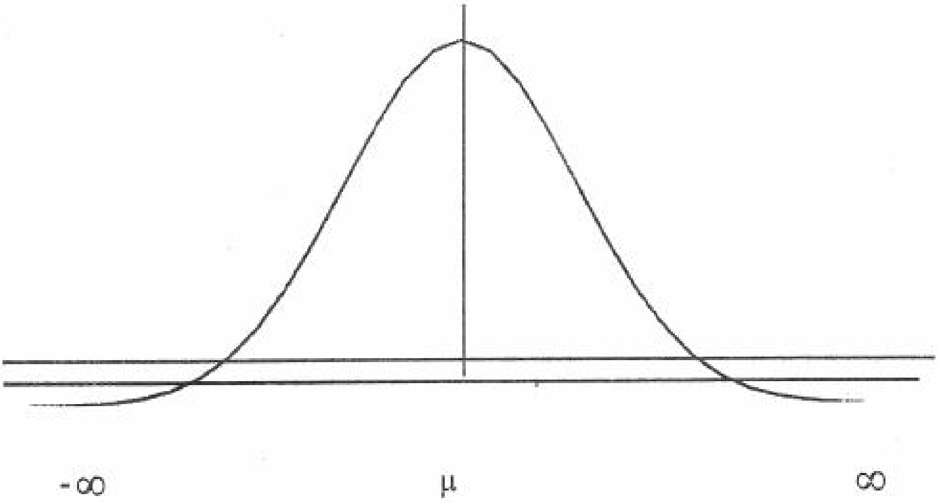
'π' is mathematical constant approximated by 3.14159

'e' is mathematical constant approximated by 2.71828

If 'X' follows a normal distribution with mean μ and variance σ^2 then this could be symbolically expressed as:

$$X \sim N(\mu, \sigma^2)$$

The graph of this normal distribution is a bell shaped curve with peak or top occurring above the mean value μ. The value of standard deviation σ decides whether the normal curve will be flat or steep. If standard deviation is large then the normal curve will be more flat and if standard deviation value is small then the normal curve will be steeper. The general shape of normal curve will be as given below:



The variable 'X' which follows a normal distribution with mean μ and standard deviation σ could be transformed into a standard normal variate 'Z' which has mean 0 and variance 1 by using the relation:

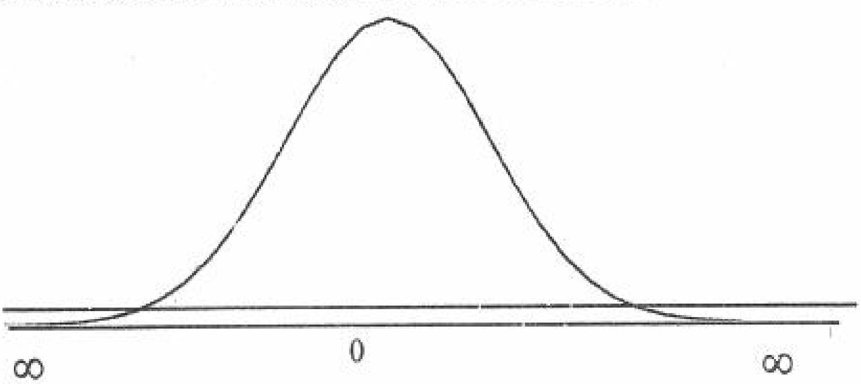
$$Z = \frac{(x - \mu)}{\sigma}$$

Symbolically we can write:

If $X \; N(\mu, \sigma^2)$

Then $Z \; N(0, 1)$

The shape of standard normal variate curve is as follows:



The shape of this curve is same as the shape of a normal variable curve with peak occurring at z=0 and the curve extending from $-\infty$ to $+\infty$.

6.2 Relationship among Binomial, Poisson and Normal Distribution

- Binomial distribution tends towards Poisson distribution when n – the number of trials is large and p – the probability of success is small, so that np is finite.
- Binomial distribution tends towards Normal distribution when n is very large i.e. $n \to \infty$; and neither p nor q is very small.

In this condition $Z = \frac{(x-np)}{\sqrt{npq}}$ follows normal distribution with mean 0 and variance 1

i.e. $Z = \frac{(x-np)}{\sqrt{npq}} \sim N(0, 1)$

- Similarly, Poisson distribution also tends towards Normal distribution and standard normal variate associated with Poisson distribution is Z, which is defined as:

$$Z = \frac{(x-m)}{\sqrt{m}} \sim N(0, 1)$$

6.2 Properties of Normal Distribution

- The normal distribution curve is a bell shaped curve which is symmetrical about its mean value. It means that one half of the curve lies on the left side of mean value and the other half lies on the right side; and the portion of curve which lies on the left side of mean value is exact replica of the portion which lies on the right side. Therefore median value of the distribution lies at the point where mean value occurs i.e. mean and median coincide in normal distribution.

- The peak or the highest point of the curve occurs at the mean value, so this value is also the modal value. In other words value of mode is equal to value of mean i.e. mean and mode also coincide in normal distribution.
- The area under the normal curve is equal to one i.e. unity. The area under the curve to the right of mean equals the area under the curve to the left of mean which is equal to half i.e. 0.5.
- As x increases (gets larger and larger in the positive direction), the graph approaches, but never reaches, the horizontal axis. As x decreases (gets larger and larger in the negative direction), the graph approaches, but never reaches, the horizontal axis. In other words we can say that the range of normal variable is from $-\infty$ to $+\infty$.
- The points of inflection occur at $\mu \pm \sigma$.
- In a normal curve $\mu \pm \sigma$ covers approximately 68.27% of area (or observations), $\mu \pm 2\sigma$ covers approximately 95.45% of area (or observations) and $\mu \pm 3\sigma$ covers approximately 99.73% of area (or observations)
- As the normal curve is symmetrical about median therefore Q_1 (first quartile) and Q_3 (third quartile) are equidistant from median.
- Since the normal curve is symmetrical therefore all the odd order central moments are zero i.e. $\mu_{2n+1} = 0$.
- All the even order central moments can be calculated by the formula given below:

$$\mu_{2n} = 1*3*\ldots\ldots\ldots(2n-1)\,\sigma^{2n}$$

- Linear combination of independent normal variates is also a normal variate i.e. if X N (μ_x, σ_x^2) and Y N (μ_y, σ_y^2) then X+ Y N $(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

6.4 Area Property of Normal Curve

If 'X' follows a normal distribution with mean μ and standard deviation ó then the probability that X is less than equal to 'Q' is given by

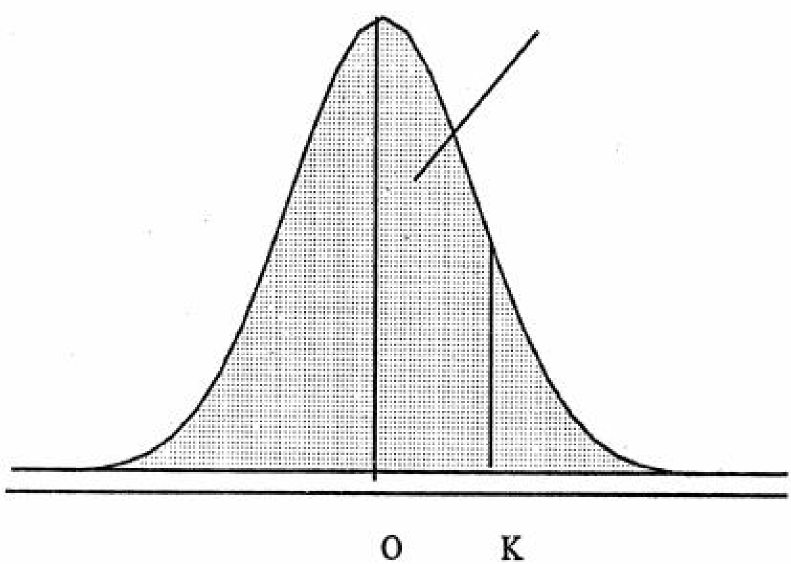
$$P(x \le Q) = P\left(\frac{x-\mu}{\sigma} \le \frac{Q-\mu}{\sigma}\right)$$

$$= P(Z \le \frac{Q-\mu}{\sigma})$$

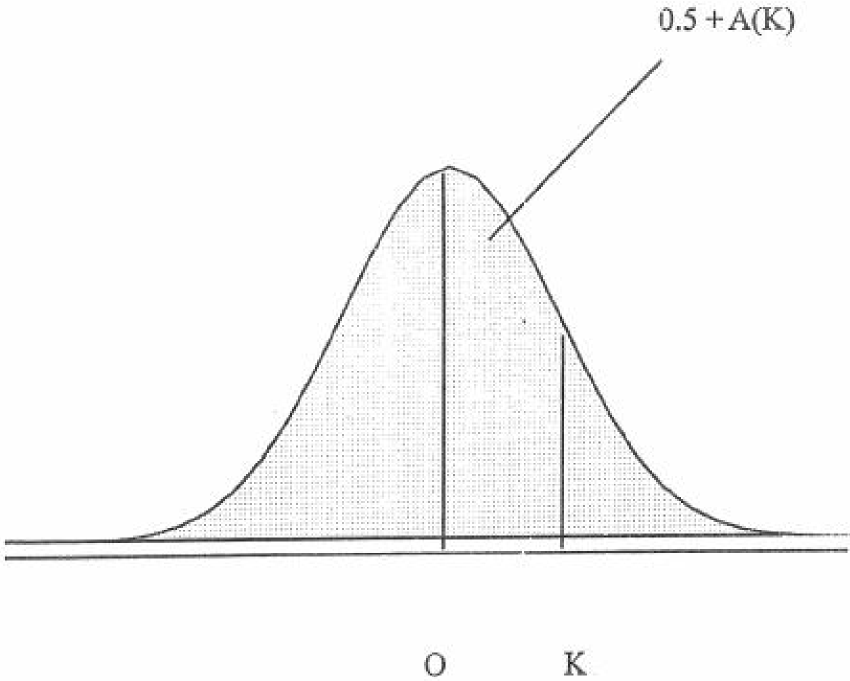
Z is a standard normal variate with mean 0 and standard deviation 1. Area table or probabilities table for standard normal variate Z is available which can be used for finding different probabilities. In this table value of probabilities is given for P(0 d● Z d● K) where K is any positive term. Probabilities are given in the table only for positive K i.e. for the right hand side of mean. As the curve is symmetrical therefore the probabilities for negative vaues of K could be found from this after necessary changes.

Suppose if we denote the probability of P(0 d● Z d● K) by A(K) for any positive value of K, then we could use the following relations for finding different probabilities:

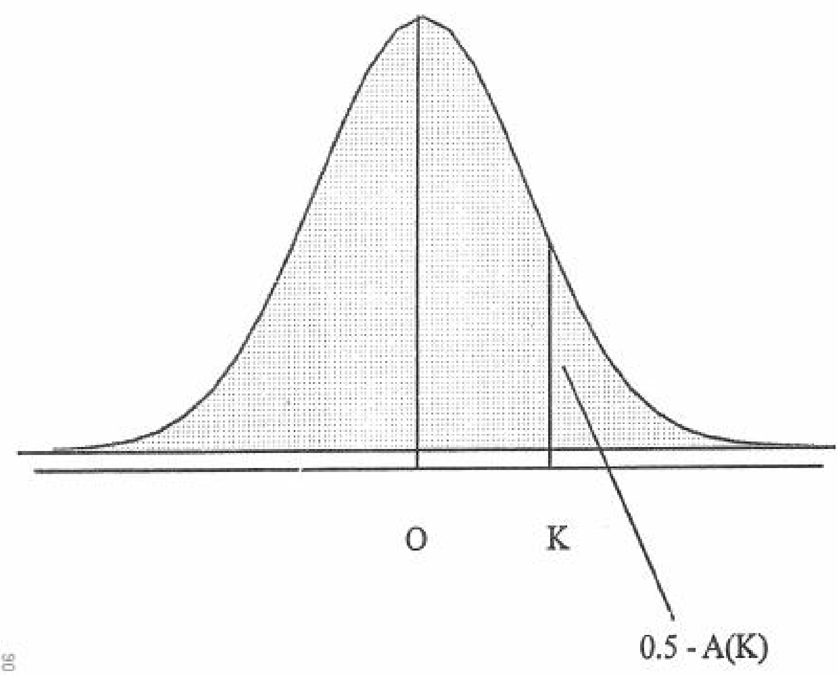
1. P(0 d" Z d" K) = A(K)



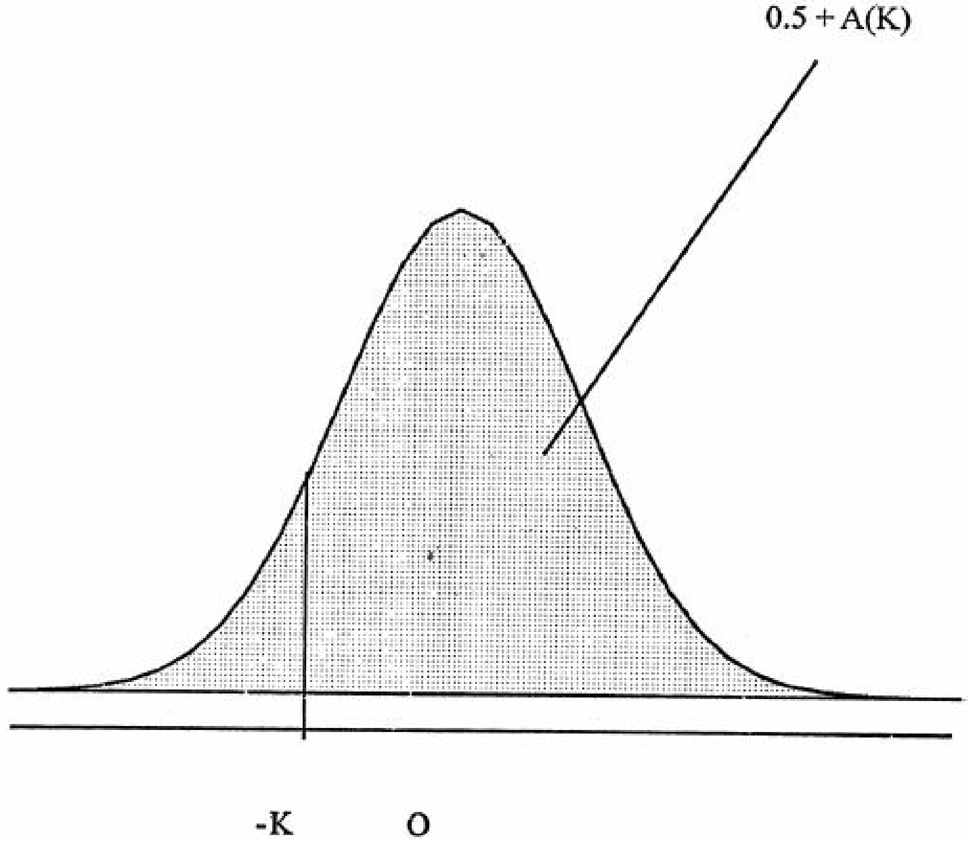
2. $P(Z \leq K) = 0.5 + A(K)$



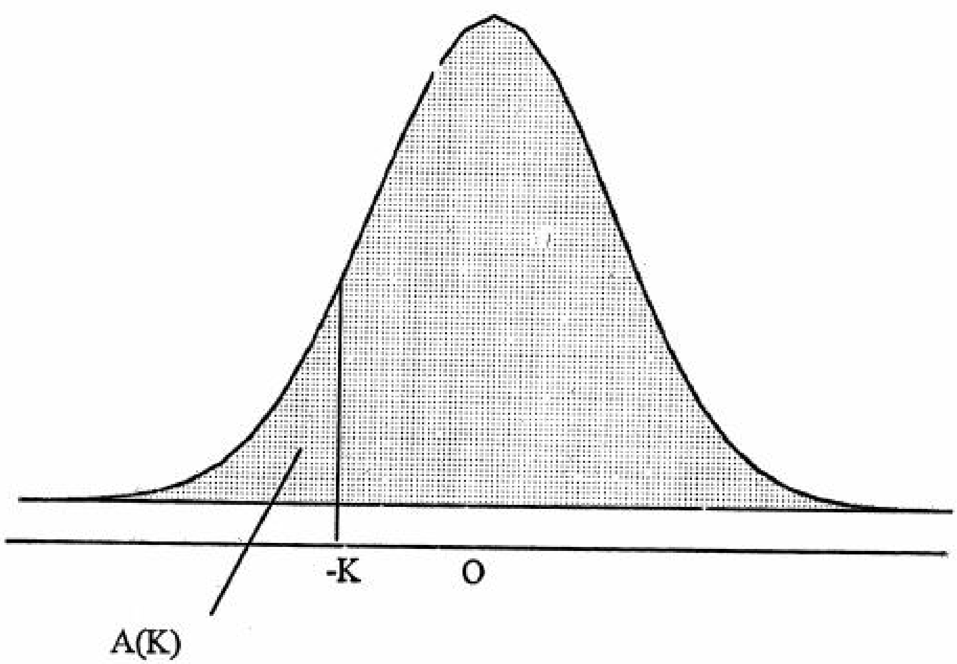
3. P(Z e” K) = 0.5 - A(K)



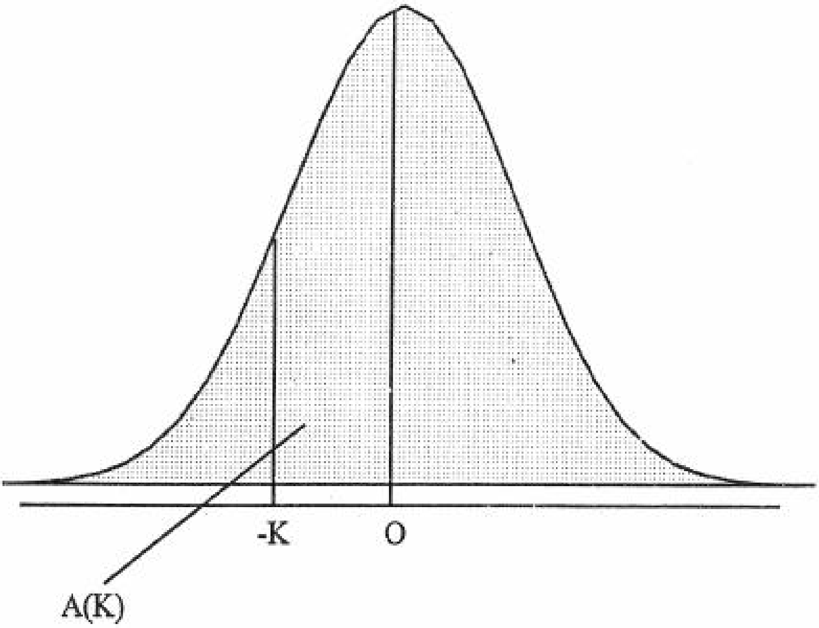
4. P(Z e" - K) = 0.5 + A(K)



5. P(Z d" - K) = 0.5 - A(K)

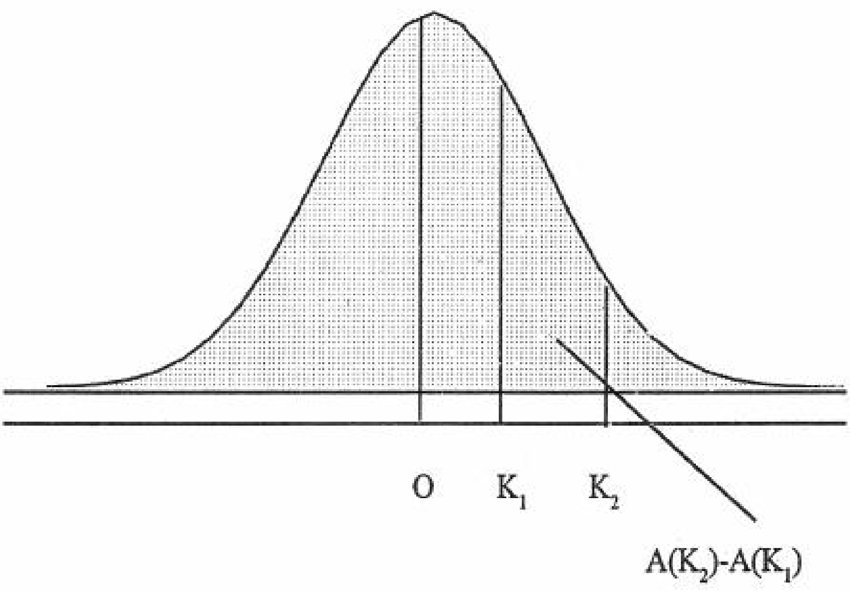


6. P(-K d" Z d" 0) = A(K)

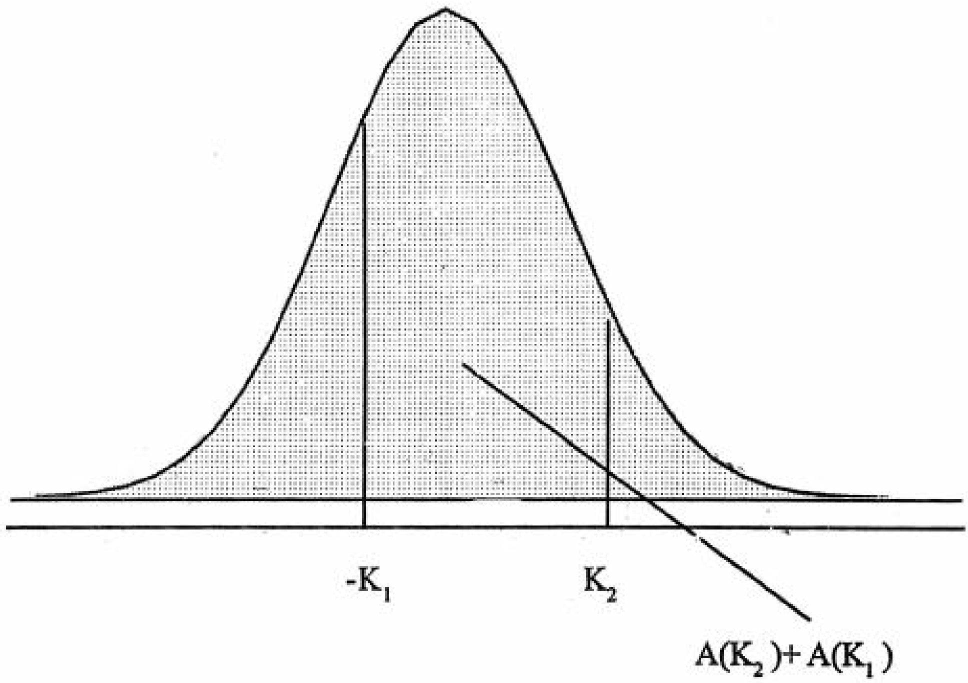


Further if we have $K_1 < K_2$ and $K_1 > 0, K_2 > 0$, then

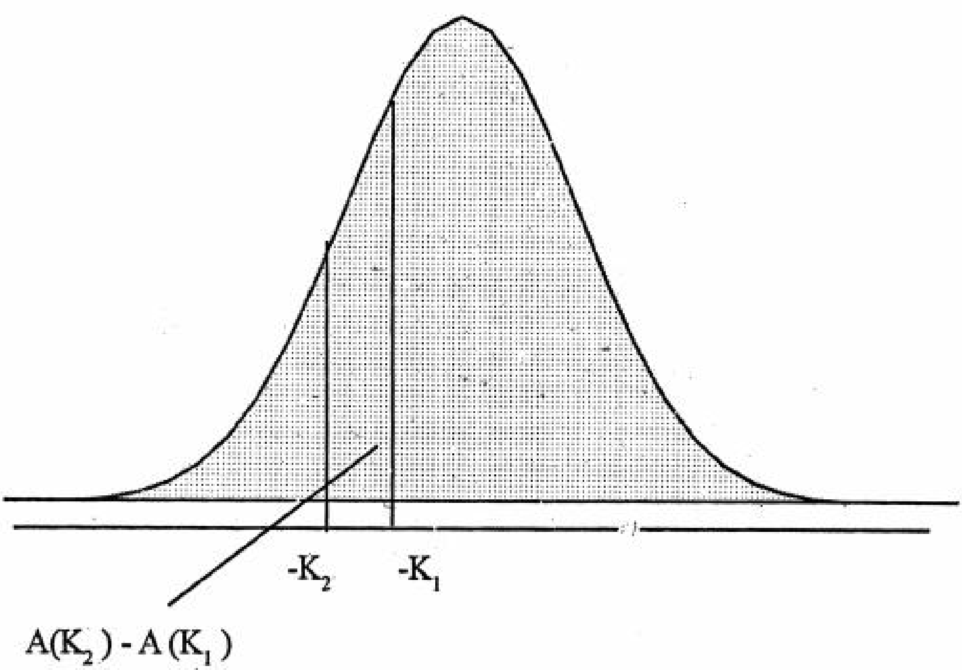
7. P(K_1 d" Z d" K_2) = A(K_2) - A(K_1)



8. $P(-K_1 \text{ d}'' Z \text{ d}'' K_2) = A(K_2) + A(K_1)$



9. $P(-K_2 \text{ d}'' Z \text{ d}'' -K_1) = A(K_2) - A(K_1)$



6.5 Examples of Normal Distribution

6.5.1 Example A machine is designed to produce parts having an average diameter of 3 cm and standard deviation of 0.05 cm. If the distribution of diameters of manufactured product is normal then

(i) What is the probability that a randomly selected part will have a diameter exceeding 3.10 cm?

(ii) What is the probability that a randomly selected part will have a diameter between 2.90 cm and 3.10 cm?

Solution

If 'X' follows a normal distribution with mean μ and standard deviation ó then the probability that X is less than or equal to 'Q' is given by

$$P(x \leq Q) = P\left(\frac{x-\mu}{\sigma} \leq \frac{Q-\mu}{\sigma}\right)$$

$$= P(Z \leq \frac{Q-\mu}{\sigma})$$

(i) Therefore the probability of x (diameter of the part) exceeding 3.10 cm is

$$P(x > 3.10)$$
$$= P(\frac{x-3}{0.05} > \frac{3.10-3}{0.05})$$
$$= P(Z > 2)$$
$$= 0.5 - A(2)$$
$$= 0.5 - 0.4772$$
$$= 0.0228$$

(ii) Therefore the probability of x falling between 2.90 cm and 3.10 cm is

$$P(2.90 < x < 3.10)$$
$$= P(\frac{2.90-3}{0.05} < \frac{x-3}{0.05} < \frac{3.10-3}{0.05})$$
$$= P(-2 < Z < 2)$$

$= 2 * A(2)$

$= 2 * 0.4772$

$= 0.9544$

6.5.2 Example The marks obtained by students in an examination are known to be normally distributed. If 10 percent of the students got less than 40 marks while 15 percent got over 80, what are the mean and standard deviation of the marks?

Solution

If 'X' follows a normal distribution with mean μ and standard deviation ó then the probability that X is less than or equal to 'Q' is given by

$$P(x \leq Q) = P(\frac{x-\mu}{\sigma} \leq \frac{Q-\mu}{\sigma})$$

$$= P(Z \leq \frac{Q-\mu}{\sigma})$$

The probability of x (marks) being less than 40 is 0.10 (ten percent).

i.e. $P(x < 40) = 0.10$

or $P(\frac{x-\mu}{\sigma} < \frac{40-\mu}{\sigma}) = 0.10$

or $P(Z < \frac{40-\mu}{\sigma}) = 0.10$

or $0.5 - A(\frac{40-\mu}{\sigma}) - 0.10$

or $A(\frac{40-\mu}{\sigma}) = 0.40$

or $(\frac{40-\mu}{\sigma}) = 1.28$ ---------(i)

The probability of x (marks) being more than 80 is 0.15 (fifteen percent).

i.e. $P(x > 80) = 0.15$

or $\quad P\left(\frac{x-\mu}{\sigma} > \frac{80-\mu}{\sigma}\right) = 0.15$

or $\quad P\left(Z > \frac{80-\mu}{\sigma}\right) = 0.15$

or $\quad 0.5 - A\left(\frac{80-\mu}{\sigma}\right) = 0.15$

or $\quad A\left(\frac{80-\mu}{\sigma}\right) = 0.35$

or $\quad \left(\frac{80-\mu}{\sigma}\right) = 1.04 \qquad$ ---------------(ii)

Solving (i) and (ii) simultaneously we get $\mu = 61.38$ and $\sigma = 17.24$

6.5.3 Example In a plant automatic machine fills on an average 'm' ml of refined oil in packets. If the amount filled in packets follows a normal distribution with a standard deviation of 4 ml, then what should be the setting for 'm' so that a 400 ml packet will contain an excess amount in only 2% of the cases?

Solution $\qquad \le \frac{Q-\mu}{\sigma})$

If 'X' follows a normal distribution with mean μ and standard deviation σ then the probability that X is less than or equal to 'Q' is given by

$$P(x \le Q) = P\left(\frac{x-\mu}{\sigma} \le \frac{Q-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{Q-\mu}{\sigma}\right)$$

The probability of x (amount of refined oil filled in a packet) exceeding 400 ml is 0.02.

i.e. $\quad P(x > 400) = 0.02$

or $\quad P\left(\frac{x-m}{4} > \frac{400-m}{4}\right) = 0.02$

or $\quad P\left(Z > \frac{400-m}{4}\right) = 0.02$

or $\quad 0.5 - A\left(\frac{400-m}{4}\right) = 0.02$

or $A(\frac{400-m}{4}) = 0.48$

or $\frac{400-m}{4} = 2.56$

or $m = 389.76$

6.5.4 Example A wholesale distributor of fertilizer product finds that the annual demand for one type of fertilizer is normally distributed with mean of 120 tons and standard deviation of 16 tons. If he orders only once a year, what quantity should be ordered to ensure that there is only a 5 percent chance of running short?

Solution

If 'X' follows a normal distribution with mean μ and standard deviation ό then the probability that X is less than or equal to 'Q' is given by

$$P(x \le Q) = P(\frac{x-\mu}{\sigma} \le \frac{Q-\mu}{\sigma})$$

$$= P(Z \le \frac{Q-\mu}{\sigma})$$

In this case μ = 120 tons, ό = 16 tons and let quantity to be ordered is Q. Therefore the probability of x (demand) being less than Q is 0.95 (ninety five percent).

i.e. $P(x < Q) = 0.95$

or $P(\frac{x-120}{16} \le \frac{Q-120}{16}) = 0.95$

or $P(Z \le \left(\frac{Q-120}{16}\right) = 0.95$

or $0.5 + A(\frac{Q-120}{16}) = 0.95$

or $A(\frac{Q-120}{16}) = 0.45$

Therefore $\left(\frac{Q-120}{16}\right) = 1.645$

or $Q = 146.32$

6.5.5 Example The marks obtained by students in an examination are known to be normally distributed with mean marks 80 and standard deviation 20. If 200 students appeared in the examination then find the number of students who will be scoring marks (i) above 90 and (ii) below 60?

Solution

If 'X' follows a normal distribution with mean μ and standard deviation ó then the probability that X is less than or equal to 'Q' is given by

$$P(x \leq Q) = P(\frac{x-\mu}{\sigma} \leq \frac{Q-\mu}{\sigma})$$

$$= P(Z \leq \frac{Q-\mu}{\sigma})$$

The probability of x (marks) being above 90 is equal to

$$P(x > 90)$$

$$= P\left(\frac{x-80}{20} < \frac{60-80}{20}\right)$$

$$= P(Z > 0.5)$$

$$= 0.5 - A(0.5)$$

$$= 0.5 - 0.1915$$

$$= 0.3085$$

Therefore the number of students out of 200 who will be securing marks above 90 is equal to

$$200 * 0.3085$$

$$= 61.7 \text{ or } 62$$

The probability of x (marks) being below 60 is equal to

$$P(x < 60)$$

$$P(\frac{x-80}{20} > \frac{90-80}{20})$$

$$= P(Z < 1)$$

$$= 0.5 - A(1)$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$

Therefore the number of students out of 200 who will be securing marks below 60 is equal to

$$200 * 0.1587$$

$$= 31.74 \text{ or } 32$$

6.5.6 Example The life of an electronic component is normally distributed with a mean of 60 hours and a standard deviation of 4 hours. Three electronic components are used in a circuit. The circuit fails when anyone of these components fails. Find what is the probability that the circuit will last for 50 hours?

Solution

If 'X' follows a normal distribution with mean μ and standard deviation ó then the probability that X is less than or equal to 'Q' is given by

$$P(x \leq Q) = P(\frac{x-\mu}{\sigma} \leq \frac{Q-\mu}{\sigma})$$

$$= P(Z \leq \frac{Q-\mu}{\sigma})$$

The probability of x (life of circuit) exceeding 50 hours is equal to $P(x > 50)^3$

Now we know that $P(x > 50)$

$$= p(\frac{x-60}{4} > \frac{50-60}{4})$$

$$= P(Z > -2.5)$$

$$= 0.5 + A(2.5)$$

$$= 0.5 + 0.4938$$

$$= 0.9938$$

Therefore the probability of x exceeding 50 hours is

$$= P(x > 50)^3$$

$$= (0.9938)^3$$

$$= 0.98152$$

6.5.7 Example The monthly income of a group of 5000 people belonging to a rural habitat in Western Uttar Pradesh is known to be normally distributed with mean income Rs 12000 and standard deviation Rs 1500. What will be the lowest income among the richest 200 people?

Solution

If 'X' follows a normal distribution with mean μ and standard deviation ó then the probability that X is less than or equal to 'Q' is given by

$$P(x \le Q) = P(\frac{x-\mu}{\sigma} \le \frac{Q-\mu}{\sigma})$$

$$= P(Z \le \frac{Q-\mu}{\sigma})$$

Suppose 'I' is the income above which 200 people have income. Then the proportion of people having income greater than or equal to 'I' is 200/5000 i.e. 0.04.

Therefore the probability that P (x < I) = 0.96

$$\text{or} \quad P\left(\frac{x-12000}{1500} < \frac{1-12000}{1500}\right) = 0.96$$

$$\text{or} \quad P\left(Z < \frac{1-12000}{1500}\right) = 0.96$$

$$\text{or} \quad P\left(\left(Z < \frac{1-12000}{1500}\right) = 0.5 + 0.46\right.$$

$$\text{or} \quad A\left(\frac{1-12000}{1500}\right) = 0.46$$

$$\text{or} \quad \frac{1-12000}{1500} = 1.75$$

$$\text{or} \quad 1 = 14625$$

Thus 200 people will have income more than Rs 14625.

6.5.8 Example The income of a group of 10000 persons was found to be normally distributed with mean Rs 1750 per month and standard deviation Rs 50. Show that of this group 95% had income exceeding Rs 1668 and only 5% had income exceeding Rs 1832. What was the lowest income among the richest 100?

Solution

If 'X' follows a normal distribution with mean μ and standard deviation ó then the probability that X is less than or equal to 'Q' is given by

$$P(x \le Q) = P\left(\frac{x-\mu}{\sigma} \le \frac{Q-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{Q-\mu}{\sigma}\right)$$

The probability of x (income) above Rs 1668 is equal to

$$P(x > 1668) = P\left(\frac{x-1750}{50} > \frac{1668-1750}{50}\right)$$

$$= P(Z > -1.64)$$

$$= 0.5 + A(1.64)$$

$= 0.5 + 0.4495$

$= 0.9495$

Therefore the number of people having income above Rs 1668 = 10000 * 0.9495

$= 9495$

This is approximately 95% of the total 10000.

The probability of x (income) above Rs 1832 is equal to

$$P(x > 1832) = P(\frac{x-1750}{50} > \frac{1832-1750}{50})$$

$$= P(Z > 1.64)$$

$$= 0.5 + A(1.64)$$

$$= 0.5 - 0.4495$$

$$= 0.0505$$

Therefore the number of people having income above Rs 1832 = 10000 * 0.0505

$= 505$

This is approximately 5% of the total 10000.

Suppose 'I' is the income above which 100 people have income. Then the proportion of people having income greater than or equal to 'I' is 100/10000 i.e. 0.01.

Therefore the probability that $P(x < I) = 0.99$

or $\quad P(\frac{x-1750}{50} < \frac{1-1750}{50}) = 0.99$

or $\quad P(Z < \frac{1-1750}{50}) = 0.99$

or $\quad P(Z < \frac{1-1750}{50}) 0.5 + 0.49$

or $\quad A(\frac{1-1750}{50}) = 0.49$

or $$\frac{1-1750}{50}=2.33$$

or I = Rs 1866 approximately

Thus the lowest income among the richest 100 people is Rs 1866.

6.6 Key Words

1. Normal Distribution –The normal distribution is the most widely known and commonly used continuous probability distribution. It is also referred to as Gaussian distribution.

2. Mean and Variance – The mean of normal distribution is 'μ' and standard deviation is 'σ.'

3. Standard Normal Variate – The standard normal variate is denoted by Z and Z is equal to $\frac{(x-\mu)}{\sigma}$

4. The mean of standard normal variate is 0 and standard deviation is 1.

5. The total area below the standard normal variate distribution is 1.

6.7 Terminal Questions

6.7.1 Define Normal distribution. What are the main characteristics of normal distribution?

6.7.2 What is the probability that Z takes values between 0 and 1.96?

6.7.3 What are the probabilities that Z takes values (i) between 0 and 2 (ii) less than 2.3 (iii) more than 2.33 (iv) between 1.2 and 2.333?

6.7.4 On an IQ test with a mean of 90 and a standard deviation of 14, at approximately what percentile will an IQ of 104 fall?

6.7.5 Which z score corresponds to a higher value within a distribution of values, -1.04 or 1.00? Explain.

6.7.6 A doctor records the pulse rate of her 200 female patients. The pulse rates are approximately normally distributed, with mean 80 and standard deviation 10. Determine (i) the proportion of the females that have a pulse rate less than 75. (ii) the pulse rate of female at the 20th percentile i.e. find the pulse rate of the female that separates the bottom 20% from the top 80%.

6.7.7 Steel rods are manufactured having a mean length of 50 centimeter. The lengths of the rods are approximately normally distributed, with a standard deviation of 2.0 cm.

(a) What proportion of rods has a length less than 47 cm?

(b) Any rods that are shorter than 48 cm or longer than 53 cm are discarded. What proportion of rods will be discarded?

(c) If 10,000 rods are manufactured in a day, how many should the plant manager expect to discard?

6.7.8 The monthly income of a group of 5000 persons was found to be normally distributed with mean Rs 40,000 per month and standard deviation Rs 4000. Find

(i) the number of persons having income more than Rs 45,000.

(ii) the number of persons having income less than Rs 36,000.

(iii) the highest monthly income among the lowest paid 200 persons.

6.7.9 The daily sales of a grocer's shop is assumed to be normally distributed with mean Rs 20,000 and a standard deviation Rs 4000. Find

(i) the percentage of days on which sales will be more than Rs 22,000.

(ii) the percentage of days on which sales will be within 15% of mean.

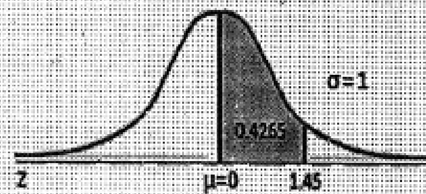
(iii) the limits within which middle 50% of sales will lie.

6.7.10 The wage distribution of workers in a factory is normally distributed with mean Rs 10,000 and standard deviation Rs 1000. If the wages of 50 workers are less than Rs 9,000 then what is the total number of workers?

6.8 Area Table

Areas Under the One-Tailed Standard Normal Curve

This table provides the area between the mean and some Z score.
For example, when Z score = 1.45 the area = 0.4265.



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4964	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000

6.8 Further Readings

1 Levin , R.I. : Statistics for Management(PHI)

2 Gupta , S.P. & Gupta , M.P. : Business Statistics

3 Aczel & Sounderpandian : Complete Business Statistics (Tata McGraw Hill)

4 Lapin , Lawrance : Statistics for Modern Business Decisions (HBJ)

5 Shenoy , G.V. & Pant, M : Statistical Methods in Business and Social Sciencess

UNIT 7 PROBABILITY AND DECISION MAKING

Objectives

After Studying this units, you would be able to understand the:

- States of nature, strategies and payoffs;
- Payoff table and its construction;
- Decision making under certainty; and
- Decision making under uncertainty.

Structure

7.1 Introduction

Decision making is an all pervasive activity in an organization. Executives at all levels and all positions are involved in decision making. The quality of decisions taken separates effective executives from ineffective executives, successful executives from unsuccessful executives and finally it separates successful organizations from unsuccessful organizations. In today's business organizations each managerial activity is concerned with decision making whether it relates to planning, organizing, staffing, directing or controlling. It's an inescapable fact that what an organization gets, good or bad, depends, in part, on what decisions are taken by the executives and, in part, on the circumstances beyond the control of the executives.

Managerial decision making is the process of making a conscious choice between two or more rational alternatives to get a desired result. It is a process which involves collection of information regarding the different alternatives, evaluation of alternatives and selection of an alternative in order to achieve certain stated goals.

7.2 Basic Concepts

The decision model used for decision making makes use of the following concepts:

Decision Maker: The decision maker refers to the person or group of persons or executive or manager or organization which takes decision.

Courses of Action: The different alternatives or actions or strategies that are available to a decision maker from which he/she picks one or more to achieve the desired goals.

States of Nature: The states of nature refer to the events that are beyond the control of the decision maker and which in turn determine the level of success of different acts chosen by decision maker.

Payoffs: It refers to the outcomes associated with different combinations of courses of action and states of nature, which are expressed in terms of benefit that accrues to the decision maker.

Payoff Table: The conditional payoffs in terms of gains associated with different combinations of courses of action and states of nature can be represented in a tabular form, known as payoff table, in which the row designations are the states of nature and column designations are the strategies/actions available to the decision maker.

Suppose there are 'm' states of nature represented by N_1, N_2,........, N_m and 'n' courses of action (strategies) represented by S_1, S_2,........, S_n. Let the payoff corresponding to i^{th} state of nature and j^{th} strategy be represented by p_{ij} where i=1,2,.............,m and j=1,2,...........,n. Then the payoff table corresponding to this is as follows:

States of Nature	*Courses of Action* S_1			
	S_2	S_2	....	S_n
N_1	p_{11}	p_{12}	....	p_{1n}
N_2	p_{21}	p_{22}	....	p_{2n}
....	....	....	....	....
....	....	.....	....	....
N_m	p_{m1}	p_{m2}	....	p_{mn}

7.2.1 Example There is a bouquet seller who makes and sells bouquets in prime market of the city. For preparing bouquets he has to place the order for flowers one day in advance and also he has to make advance payment for the same. He knows from past experience that the demand for number of bouquets could be either 20 or 30 or 40 or 50. The cost of a bouquet for him is Rs 35/- and he sells it at Rs 50/-. Any bouquet left unsold at the end of the day is valueless for the seller. Construct the payoff table for this problem.

Solution

For this problem we can develop the payoff table in the following way:

The various states of nature are -

N_1 – Number of bouquets demanded is 20

N_2 – Number of bouquets demanded is 30

N_3 – Number of bouquets demanded is 40

N_4 – Number of bouquets demanded is 50

The different courses of action (strategies) available are

S_1 – Number of bouquets prepared is 20

S_2 – Number of bouquets prepared is 30

S_3 – Number of bouquets prepared is 40

S_4 – Number of bouquets prepared is 50

Payoff = Sales Revenue – Cost.

The payoffs for different combinations of states of nature and strategies are as follows:

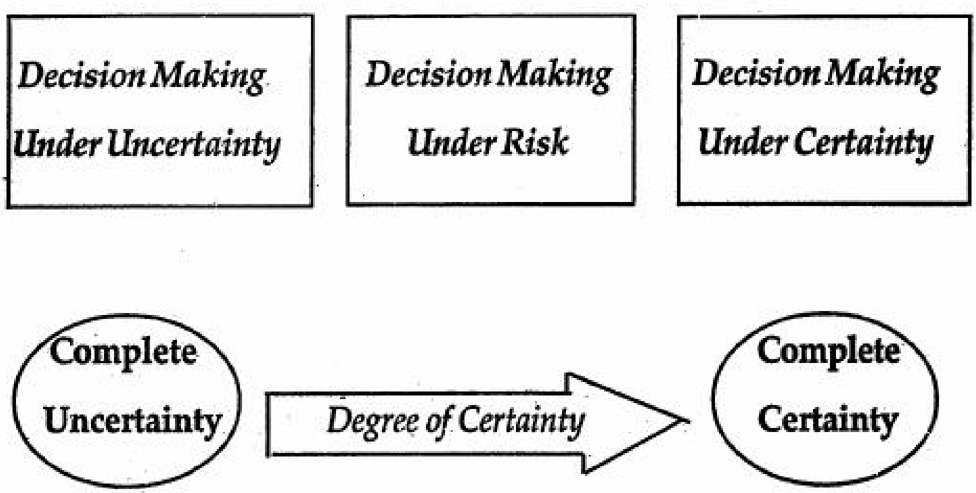
Combination	Payoff in Rs
N_1S_1	20*50 – 20*35 = 1000 – 700 = 300
N_1S_2	20*50 – 30*35 = 1000 – 1050 = -50
N_1S_3	20*50 – 40*35 = 1000 – 1400 = -400
N_1S_4	20*50 – 50*35 = 1000 – 1750 = -750
N_2S_1	20*50 – 20*35 = 1000 – 700 = 300
N_2S_2	30*50 – 30*35 = 1500 – 1050 = 450
N_2S_3	30*50 – 40*35 = 1500 – 1400 = 100
N_2S_4	30*50 – 50*35 = 1500 – 1750 = -250
N_3S_1	20*50 – 20*35 = 1000 – 700 = 300
N_3S_2	30*50 – 30*35 = 1500 – 1050 = 450
N_3S_3	40*50 – 40*35 = 2000 – 1400 = 600
N_3S_4	40*50 – 50*35 = 2000 – 1750 = 250
N_4S_1	20*50 – 20*35 = 1000 – 700 = 300
N_4S_2	30*50 – 30*35 = 1500 – 1050 = 450
N_4S_3	40*50 – 40*35 = 2000 – 1400 = 600
N_4S_4	50*50 – 50*35 = 2500 – 1750 = 750

Now the Payoff table for this problem is as follows:

	Conditional Payoffs (Rs.) *Courses of Action*			
States of Nature	S_1	S_2	S_3	S_4
N_1	300	-50	-400	-750
N_2	300	450	100	-250
N_3	300	450	600	250
N_4	300	450	600	750

7.3 Decision Making Environment

The various decision making situations which a manger faces could be placed in three categories, namely, decision making under uncertainty, decision making under risk and decision making under certainty. This classification depends on the degree to which the future environment determining the outcome of these decisions is known.



7.4 Decision Making Under Certainty

In Decision Making Under Certainty the decision maker is certain about the future state of nature that will occur. This means that the decision maker knows in advance what state of nature will occur and keeping that in mind the decision maker chooses the alternative that gives the most favorable outcome.

7.4.1 Example Let 'A' has Rs 50,000/- to invest and there are three options available – i) Fixed Deposit in bank giving an annual interest of 8%, ii) Government bonds giving an annual interest of 12%, and iii) Investment in a government scheme giving an annual interest of 9%.

Suppose the return is guaranteed in all these three options then 'A' will choose option (ii) as return is 12%, which is highest among the three alternatives.

7.5 Examples of Decision Making Under Certainty

7.5.1 Examples of Production Problem in Decision Making Under Certainty Environment

7.5.1.1 Example A publishing house publishes three weekly magazines – Daily Life, Agriculture Today, and Surf's Up. Publication of one issue of each of the magazines requires the production time and paper indicated in table given below:

	Production (hour)	Paper (Kg)
Daily Life	0.01	0.2
Agriculture Today	0.03	0.5
Surf's Up	0.02	0.3

Each week, the publisher has 120 hours of production time available and 3000 Kg of paper. The total circulation for all three magazines must exceed 5,000 issues per week if the company has to keep its advertisers. The selling price per issue is Rs. 22.50 for Daily Life, Rs. 40 for Agriculture Today and Rs. 15 for Surf's Up. Based on past sales, the publisher knows that the maximum weekly demand for Daily Life is 3000 issues, for Agriculture Today 2000 issues and for Surf's Up 6000 issues. The production manager wants to know the number of issues of each magazine to produce weekiy in order to maximize total sales revenue.

Comment This is a problem related with decision making under certainty which could be formulated as a linear programming problem. It is a **production problem** in which all the values – selling price per magazine; production time and paper requirement per magazine; total production time and paper availability; condition regarding minimum value figure as far as total circulation of three magazines is concerned; and maximum weekly demand of three magazines; are known with certainty. So this problem could be solved using the typical procedure which is normally followed for solving linear programming problems.

7.5.1.2 Example The Omega Data Processing Company performs three types of activity: payrolls, account receivables, and inventories. The profit and time requirements for keypunch computation and office printing for a 'standard job' are shown in the following table:

Job	Profit/Standard job (Rs)	Time requirements (min.)		
		Keypunch	Computation	Print
Payroll	275	1,200	20	100
A/C Receivable	125	1,400	15	60
Inventory	225	800	35	80

Omega guarantees overnight completion of the job. Any job scheduled during the day can be completed during the day or night. Any job scheduled during the night, however, must be completed during the night. The capacities for both day and night are shown in the following table:

Capacity (min.)	Keypunch	Computation	Print
Day	4,200	150	400
Night	9,200	250	650

Formulate the linear programming problem in order to determine the 'mixture' of standard jobs that should be accepted during the day and night.

Comment This is a problem related with decision making under certainty which could be formulated as a linear programming problem. It is a **production problem** in which all the values – profit, time requirement and capacities are known with certainty. So this problem could be solved using the typical procedure which is followed for solving linear programming problems.

7.5.2 Examples of Finance Problem in Decision Making Under Certainty Environment

7.5.2.1 Example An investor has money making activities A_1, A_2, A_3 and A_4. He has only one lakh rupees to invest. In order to avoid excessive investment, no more than 50% of the total investment can be placed in activity A_2 and/or activity A_3. Activity A_1 is very conservative, while activity A_4 is speculative. To avoid excessive speculation, at least Re 1 must be invested in activity A_1 for every Rs 3 invested in activity A_4. The data on the return on investment are as follows:

Activity	Anticipated return on investment(%)
A_1	10
A_2	12
A_3	14
A_4	16

The investor wishes to know how much to invest in each activity to maximize the total return on the investment.

Comment This is a problem related with decision making under certainty which could be formulated as a linear programming problem. It is a **finance problem** in which all the values – amount and interest rates are known with certainty. So this problem could be solved using the typical procedure which is followed for solving linear programming problems.

7.5.2.2 Example A leading Chartered Accountant is attempting to determine a 'best' investment portfolio and is considering six alternative investment proposals. The following table indicates point estimates for the price per share, the annual growth rate in the price per share, the annual dividend per share and a measure of the risk associated with each investment.

	Shares under consideration					
	A	B	C	D	E	F
Current price per share (Rs)	80	100	160	120	150	200
Projected annual growth rate	0.08	0.07	0.10	0.12	0.09	0.15
Projected annual dividend per share	4.00	4.50	7.50	5.50	5.75	0.00
Projected risk in return	0.05	0.03	0.10	0.20	0.06	0.08

The total amount available for investment is Rs 25 lakhs and the following conditions are required to be satisfied.

(i) The maximum rupee amount to be invested in alternative F is Rs 250,000.

(ii) No more than Rs 500,000 should be invested in alternatives A and B combined.

(iii) Total weighted risk should not be greater than 0.10

Total Weighted Risk =

{(Amount invested in alternative j) (Risk of alternative j)} /

(Total amount invested in all the alternatives)

(iv) For the sake of diversity at least 100 shares of each stock should be purchased.

(v) Atleast 10% of the total investment should be in alternatives A and B combined.

(vi) Dividends for the year should be atleast Rs 10,000.

Rupee return per share of the stock is defined as price per share one year hence *less* current price per share *plus* dividend per share. Assuming that the time horizon for investment is one year determine the optimal number of shares to be purchased in each of the shares so as to maximize total rupee return.

Comment This is a problem related with decision making under certainty which could be formulated as a linear programming problem. It is a **finance problem** in which all the values – the price per share, the annual growth rate in the price per share, the annual dividend per share, measure of the risk associated with each investment and different conditions are known with certainty. So this problem could be solved using the typical procedure which is followed for solving linear programming problems.

7.5.3 Examples of Marketing Problem in Decision Making Under Certainty Environment

7.5.3.1 Example An advertisement campaign is to be planned for a new product introduced in the market. It is decided to place advertisements in three newspapers. The company has collected the following set of relevant data:

Reader characteristics	Newspaper A	Newspaper B	Newspaper C
Age: 25-35 yrs	65%	60%	55%
Education level: Graduation & above	70%	50%	60%
Income: Rs 1,5000 and above	60%	65%	45%
Minimum number of advertisements	15	12	8
Maximum number of advertisements	45	50	40
Cost per advertisement (Rs)	5,000	7000	5000
Readership	6,00,000	8,00,000	1,00,000

Additionally, the relative importance of the reader characteristic should be weighted as follows:

Reader characteristics	*Weightage*
Age: 25-35 yrs	0.5
Graduate and above	0.3
Income >= Rs 1,5000	0.2

The advertising budget is Rs 5,00,000. How many advertisements should be placed in different newspapers so as to maximize the effective exposure level.

<u>Comment</u> This is a problem related with decision making under certainty which could be formulated as a linear programming problem. It is a **marketing problem** in which all the values – budget amount, cost of ads, reader characteristics and weightage to be assigned are known with certainty. So this problem could be solved using the typical procedure which is followed for solving linear programming problems.

7.5.3.2 Example A company is planning an advertising campaign for one of its leading brands 'Tezz.' It has decided to insert advertisements in three magazines. The company has collected the following set of relevant data:

	Magazine		
	Zolo	**Y - King**	**Spark**
Readers over 30 yrs	55%	40%	45%
Readers - Graduate & above	60%	50%	65%
Readers(Income): Rs 20,000 and above	40%	55%	65%
Minimum number of advertisements	8	9	6
Cost per advertisement (Rs)	8,000	6000	7000
Readership	5,00,000	6,00,000	4,00,000

Additionally, the relative importance of the reader characteristic should be weighted as follows:

Reader characteristics	*Weightage*
Age: 25-35 yrs	0.4
Graduate and above	0.3
Income >= Rs 1,5000	0.3

The advertising budget is Rs 7,00,000. How many advertisements should be placed in different newspapers so as to maximize the effective exposure level.

<u>Comment</u> This is a problem related with decision making under certainty which could be formulated as a linear programming problem. It is a **marketing problem** in which all the values – budget amount, cost per ad, reader characteristics and weightage to be assigned are known with certainty. So this problem could be solved using the typical procedure which is followed for solving linear programming problems.

7.5.4 Examples of <u>Manpower Scheduling Problem</u> in Decision Making Under Certainty Environment

Example 3.5.4.1 A city hospital has the following minimal daily requirements for nurses.

Period	Clock Time (24-Hour Day)	Minimum No. of Nurses Required
1	6 a.m. – 10 a.m.	2
2	10 a.m. – 2 p.m.	7
3	2 p.m. – 6 p.m.	15
4	6 p.m. – 10 p.m.	8
5	10 p.m. – 2 a.m.	20
6	2 a.m.. - 6 a.m.	6

Nurse report at the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimum number of nurses to be employed so that there will be a sufficient number of nurses available for each period.

Comment This is a problem related with decision making under certainty which could be formulated as a linear programming problem. It is a **manpower scheduling problem** in which all the values – the different periods; number of nurses required in different periods; and number of hours for which nurses work; are known with certainty. So this problem could be solved using the typical procedure which is normally followed for solving linear programming problems.

7.5.4.2 Example A 24 hour supermarket has the following minimal daily requirements for security personnel.

Period	Clock Time (24-Hour Day)	Minimum No. of Security Personnel Required
1	8 a.m. – 12 noon	8
2	12 noon – 4 p.m.	10
3	4 p.m. – 8 p.m.	10
4	8 p.m. – 12 midnight	7
5	12 midnight – 4 a.m.	9
6	4 a.m. – 8 a.m.	6

A security personnel work for 8 consecutive hours. The Personnel Manager wants to determine the minimum number of security personnel to be employed so that there will be a sufficient number of security personnel available for each period.

Comment This is a problem related with decision making under certainty which could be formulated as a linear programming problem. It is again a **manpower scheduling problem** in which all the values – the different periods; number of security personnel required in different periods; and number of hours for which security personnel work; are known with certainty. So this problem could be solved using the typical procedure which is normally followed for solving linear programming problems.

7.5.5 Example of Waiting Line Problem in Decision Making Under Certainty Environment

7.5.5.1 Example Arrivals at a telephone booth are considered to follow Poisson distribution with an average time of 10 minutes between one arrival and next. The length of phone call is assumed to be distributed exponentially with mean 3 minutes.

(i) What is the probability that a person arriving at the booth will have to wait?

(ii) The telephone department will install a second booth when convinced that an arrival would wait for atleast 3 minutes for a phone call. By how much should the flow of arrivals increase in order to justify a second booth?

(iii) What is the average length of the queue that gets formed from time to time?

(iv) What is the probability that it will take him more than 10 minutes to wait for the phone and complete his call?

Comment This is a **waiting line problem** related with decision making under certainty. In this all the values – mean arrival rate, mean phone call time and the distributions which they follow are known with certainty. So this problem could be solved using the typical procedure which is followed for solving waiting line problems.

7.5.6 Example of Assignment Problem in Decision Making Under Certainty Environment

7.5.6.1 Example Suggest an optimum solution to the assignment problem shown below:

Salesman	Markets (Sales in million rupees)			
	I	II	III	IV
A	44	80	52	60
B	60	56	40	72
C	36	60	48	48
D	52	76	36	40

Comment This is a problem related with decision making under certainty which could be formulated as a linear programming problem. It is an **assignment problem** in which salesmen are to be assigned to different markets (only one salesman could be assigned to one market). All the values – number of salesmen/markets and the sales figure for different salesman – market combinations are known with certainty. So this problem could be solved using the typical procedure which is followed for solving assignment problems.

7.5.7 Example of Blending Problem in Decision Making Under Certainty Environment

7.5.7.1 Example The manager of an oil refinery wants to decide on the optimal mix of two blending processes, the inputs and outputs per production run for which are given below:

Process	Input		Output	
	Grade A	Grade B	Gasoline X	Gasoline Y
1	10	12	8	7
2	15	8	5	10

The maximum amount of grade A and grade B crudes available are 500 units and 400 units respectively. The past data regarding market reveals that at least 180 units of gasoline X and 150 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs 1000 and Rs 1200, respectively.

Comment This is a problem related with decision making under certainty which could be formulated as a linear programming problem. It is a **blending problem** in which all the values – the different processes; the inputs and outputs per process run; total availability of crudes A and B; and minimum market demand of two outputs X and Y; is known with certainty. So this problem could be solved using the typical procedure which is normally followed for solving linear programming problems.

7.5.8 Example of Travelling Salesman Problem in Decision Making Under Certainty Environment

Example 3.5.8.1 A salesman has to visit five cities A, B, C, D and E. The distances (in hundred km) between the five cities are given below:

City	A	B	C	D	E
A	-	17	16	18	14
B	17	-	18	15	16
C	16	18	-	19	17
D	18	15	19	-	18
E	14	16	17	18	-

Comment This is a problem related with decision making under certainty which could be formulated as a linear programming problem. It is a **travelling salesman problem** in which salesman has to start from one city and cover all other cities in such a manner so that the total distance travelled is minimised. All the values – distance in hundred kilometres between different cities are known with certainty. So this problem could be solved using the typical procedure which is followed for solving travelling salesman problems.

7.5.9 Example of Transportation Problem in Decision Making Under Certainty Environment

7.5.9.1 Example XYZ tobacco company purchases tobacco and stores in warehouses located in the following four cities:

Warehouse location	Capacity (Tonnes)
City A	90
City B	50
City C	30
City D	60

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The warehouses supply tobacco to cigarette companies in three cities that have the following demand:

Cigarette company	Demand (Tonnes)
Bharat	120
Janata	100
Red Lamp	110

The following railroad shipping costs per tonne (in hundred rupees) have been determined:

	Bharat	Janata	Red lamp
A	7	10	5
B	12	9	4
C	7	3	11
D	9	5	7

Because of railroad construction, shipments are temporarily prohibited from warehouse at city A to Bharat Cigarette company.

(a) Find the optimum distribution for XYZ tobacco company.

(b) Are there multiple optimum solutions? If there are alternative optimum solutions, identity them

Comment This is an example of a problem related with decision making under certainty. It is a **transportation problem** in which all the values – shipping costs per tonne between different warehouses and different cigarette companies situated in different cities; the availabilities at different warehouses; and the demands of different cigarette companies; are known with certainty. So this problem could be solved using the typical procedure which is followed for solving transportation problems.

7.6 Decision Making Under Uncertainty

In Decision Making Under Uncertainty there exist several states of nature but the decision maker is uncertain about which future state of nature will occur. The decision maker has no idea as to which state of nature will occur as the decision maker has no data/no information regarding the chances of occurrence of various states of nature. In this environment the decision maker uses the following different criteria for taking decisions:

- Criterion of Optimism or Maximax/Minimin
- Criterion of Pessimism or Maximin/Minimax
- Criterion of Realism or Hurwicz criterion
- Criterion of Equally Likelihood or Laplace criterion
- Minimax Regret criterion or Savage criterion

These different criteria which are used for decision making under uncertainty have been discussed in detail in next unit.

7.7 Key Words

1. Decision Maker - The decision maker refers to the person or group of persons or executive or manager or organization which takes decision.

2. Courses of Action - The different alternatives or actions or strategies that are available to a decision maker from which he/ she picks one or more to achieve the desired goals

3. States of Nature - The states of nature refer to the events that are beyond the control of the decision maker and which in turn determine the level of success of different acts chosen by decision maker.

4. Payoffs - It refers to the outcomes associated with different combinations of courses of action and states of nature, which are expressed in terms of benefit that accrues to the decision maker.

5. Payoff Table - The conditional payoffs in terms of gains associated with different combinations of courses of action and states of nature can be represented in a tabular form, known as payoff table, in which the row designations are the states of nature and column designations are the strategies/actions available to the decision maker.

6. Decision Making Under Certainty - In Decision Making Under Certainty the decision maker is certain about the future state of nature that will occur. This means that the decision maker knows in advance what state of nature will occur and keeping that in mind the decision maker chooses the alternative that gives the most favorable outcome.

7. Decision Making Under Uncertainty - In Decision Making Under Uncertainty there exist several states of nature but the decision maker is uncertain about which future state of nature will occur. The decision maker has no idea as to which state of nature will occur as the decision maker has no data/information regarding the chances of occurrence of various states of nature.

7.8 Terminal Questions

7.8.1 What do you understand by states of nature? Explain with examples.

7.8.2 What do you understand by courses of action? Explain with real examples from business field.

7.8.3 What is a payoff table? How does it helps in decision making?

7.8.4 What do you understand by decision making under certainty? Explain with suitable examples.

7.8.5 What do you understand by decision making under uncertainty? Explain with suitable example.

7.9 Further Readings

1. N D Vohra : Quantitative Techniques in Management(Tata McGraw Hill)

2. V K Kapoor : Operations Research

3. Taha : Operations Research (Pearson)

4. Sharma , J.K : Operations Research : Theory and Applications (Macmillan India Ltd)

UNIT 8 DECISION MAKING UNDER UNCERTAINTY

Objectives

After studying this unit, you would be able to understand the:

- decision making under uncertainty;
- opportunity loss table;
- criterion of optimism;
- criterion of pessimism;
- criterion of realism;
- criterion of equally likelihood; and
- minimax regret criterion.

Structure

8.1 Introduction

Decision Making Under Uncertainty is the exact opposite of Decision Making Under Certainty. In this the decision maker knows about the possible states of nature, but cannot assess the probability of occurrence for the various states of nature. The decision maker has no idea as to which state of nature will occur as the decision maker has no data/ information regarding the chances of occurrence of various states of nature.

8.1.1 Example - Suppose a manufacturing company is thinking of introducing a totally new type of product. The company is trying to decide whether to introduce it on a large scale, medium scale or small scale. There is no past data available regarding demand for the product. The company feels that there can be only three levels of product acceptance and on the basis of this it has estimated the different payoffs for various possible strategy/state of nature combinations. The management has to take its decision on the basis of expected profit from the first year of production. The relevant data are shown in the following table:

Anticipated First Year Profit (in lakhs of Rs)

Product Acceptance	Large Scale	Medium Scale	Small Scale
Good	60	50	35
Fair	40	35	25
Poor	-20	-15	0

8.2 Different Criteria used for Decision Making Under Uncertainty

The different criteria that are used for decision making under uncertainty are:

- Criterion of Optimism or Maximax/Minimin
- Criterion of Pessimism or Maximin/Minimax

- Criterion of Realism or Hurwicz criterion
- Criterion of Equally Likelihood or Laplace criterion
- Minimax Regret criterion or Savage criterion

8.3 Criterion of Optimism or Maximax/Minimin

In this criterion the decision maker is highly optimistic in his/her approach. He/She believes that whatever can go right will go right. If he/she selects a strategy/course of action/alternative then he/she believes that the state of nature that will occur will be the one which will result in the highest payoff (profit/sales/increase, etc). So his/her approach is to find out the maximum payoffs pertaining to different strategies/ courses of action/alternatives and then from them select the strategy for which payoff is the highest i.e. this is a decision making criterion in which we select the strategy/course of action/alternative related to the maximum payoff among the maximum payoffs. That's why this criterion is also called maximax criterion i.e. a criterion in which we select the maximum value among the maximum values.

If instead of profits, losses are given then under this criterion the decision maker will select the strategy/course of action/alternative which could result in minimum possible loss i.e. the decision maker selects the strategy/course of action/alternative related to the minimum loss among the minimum losses. That's why this criterion is also called minimin criterion i.e. a criterion in which we select the minimum value among the minimum values.

8.4 Examples of Criterion of Optimism or Maximum/ Minimum

8.4.1 Example Suppose a manufacturing company is thinking of introducing a totally new type of product. The company is trying to decide whether to introduce it on a large scale, medium scale or small scale. There is no past data available regarding demand for the product. The company feels that there can be only three levels of product acceptance and on the basis of this it has estimated the different payoffs for various possible strategy/state of nature combinations. The management has to take its decision on the basis of expected profit from the first year of production. The relevant data are shown in the following table:

Anticipated First Year Profit (in lakhs of Rs)

Product Acceptance	Large Scale	Medium Scale	Small Scale
Good	60	50	35
Fair	40	35	25
Poor	-20	-15	0

What strategy should be adopted by the company? Answer the question using Criterion of Optimism or Maximax.

Solution

First of all we write down the maximum payoffs corresponding to different strategies/alternatives as shown below:

Anticipated First Year Profit (in lakhs of Rs)

Product Acceptance	Large Scale	Medium Scale	Small Scale
Good	60	50	35
Fair	40	35	25
Poor	-20	-15	0
Maximum Payoffs	*60*	*50*	*35*

Now from these maximum payoffs (60, 50 and 35) we select the highest, which is 60. The strategy corresponding to 60 is selected i.e. the company should introduce the product on a large scale.

So as per criterion of optimism the company should introduce the product on a large scale and its payoff will be Rs 60 lakh.

8.4.2 Example A manufacturing company has estimated the costs involved in the first month in introducing a totally new type of product. The company has to decide whether to introduce the product on a large scale, medium scale or small scale. The company feels that there can be only three levels of product acceptance and on the basis of this it has estimated the different costs, in the first month, for various possible strategy/state of nature combinations. The relevant data are shown in the following table:

Anticipated First Month Cost (in thousands of Rs)

Product Acceptance	Large Scale	Medium Scale	Small Scale
Good	15	20	30
Fair	25	35	34
Poor	60	40	25

What strategy should be adopted by the company? Answer the question using Criterion of Optimism or Minimin

Solution

First of all we write down the minimum costs corresponding to different strategies/alternatives as shown below:

Anticipated First Month Cost (in thousands of Rs)

Product Acceptance	Large Scale	Medium Scale	Small Scale
Good	15	20	30
Fair	25	35	34
Poor	60	40	25
Minimum costs	*15*	*20*	*25*

Now from these minimum costs (15, 20 and 25) we select the smallest, which is 15. The strategy corresponding to 15 is selected i.e. the company should introduce the product on a large scale.

Loss per report unsold is Rs 40

So as per criterion of optimism the company should introduce the product on a large scale and its cost will be Rs 15000.

8.5 Criterion of Pessimism or Maximin/Minimax

In this criterion the decision maker is highly pessimistic in his/her approach. He/She believes that whatever can go wrong will go wrong. If he/she selects a strategy/course of action/alternative then he/she believes that the state of nature that will occur will be the one which will result in least payoff (profit/sales/increase, etc). So his/her approach is to find out the minimum payoffs pertaining to different strategies/courses of action/ alternatives and then from them select the strategy for which payoff is highest i.e. this is a decision making criterion in which we select the strategy/course of action/alternative related to the maximum payoff from the minimum payoffs. That's why this criterion is also called maximin criterion i.e. a criterion in which we select the maximum value from the minimum values.

If instead of profits losses are given then under this criterion the decision maker will select the strategy/course of action/alternative related to the minimum loss among the maximum losses. That's why this criterion is also called minimax criterion i.e. a criterion in which we select the minimum value from the maximum values.

8.6 Examples of Criterion of Pessimism or Maximin/ Minimax

8.6.1 Example Times Corporation prepares information reports and prints them on daily basis, which are purchased daily by mutual funds, banks and insurance companies. This information is very expensive and the demand for the reports is limited to a maximum of 50 units. The possible demands are 20, 30, 40 and 50 reports per day. The profit per report sold is Rs 70 and the loss per report unsold is Rs 40. No production of extra reports during a day is possible. Further, there is a penalty cost of Rs 2500 for not meeting the demand. Unsold reports cannot be carried on to the next day. Find out the number of reports to be produced if Criterion of Pessimism or Maximin is used.

Solution

For this problem we can develop the payoff table in the following way:

Profit per report sold is Rs 70

Loss per report unsold is Rs 40

Penalty for not meeting the demand is Rs 2500

The various states of nature are -

N_1 – Number of reports demanded is 20

N_2 – Number of reports demanded is 30

N_3 – Number of reports demanded is 40

N_4 – Number of reports demanded is 50

The different courses of action (strategies) available are

S_1 – Number of reports produced is 20

S_2 – Number of reports produced is 30

S_3 – Number of reports produced is 40

S_4 – Number of reports produced is 50

The payoffs for different combinations of strategies and states of nature are as follows:

Combination	Payoff in Rs	Combination	Payoff in Rs
N_1S_1	20*70 = 1400,	N_2S_1	20*70 – 2500 = -1100
N_3S_1	20*70 – 2500 = -1100	N_4S_1	20*70 – 2500 = -1100
N_1S_2	20*70 –10*40 = 1000	N_2S_2	30*70 = 2100
N_3S_2	30*70 – 2500 = - 400	N_4S_2	30*70 – 2500 = - 400
N_1S_3	20*70 – 20*40 = 600	N_2S_3	30*70 –10*40 = 1700
N_3S_3	40*70 = 2800	N_4S_3	40*70 -- 2500 = 300
N_1S_4	20*70 – 30*40 = 200	N_2S_4	30*70 –20*40 = 1300
N_3S_4	40*70 – 10*40 = 2400	N_4S_4	50*70 = 3500

Now the Payoff table for this problem is as follows:

Number of Reports Demanded	*Conditional Payoffs (Rs.) Number of Reports Produced*			
	S_1	S_2	S_3	S_4
N_1	1400	1000	600	200
N_2	-1100	2100	1700	1300
N_3	- 1100	- 400	2800	2400
N_4	- 1100	- 400	300	3500

First of all we write down the minimum payoffs corresponding to different strategies/alternatives as shown below:

Number of Reports Demanded	*Conditional Payoffs (Rs.) Number of Reports Produced*			
	S_1	S_2	S_3	S_4
N_1	1400	1000	600	200
N_2	-1100	2100	1700	1300
N_3	- 1100	- 400	2800	2400
N_4	- 1100	- 400	300	3500
Minimum Payoffs	*-1100*	*-400*	*300*	*200*

Now from these minimum payoffs (*-1100, -400, 300 and 200*) we select the highest, which is 300. The strategy corresponding to 300 is selected i.e. Times Corporation should produce 40 reports per day.

So as per criterion of pessimism or maximin Times Corporation should produce 40 reports per day and its payoff will be Rs 300.

8.6.2 Example Again take Example 4.4.2 and find the solution using criterion of pessimism or minimax.

Solution

First of all we write down the maximum costs corresponding to different strategies/alternatives as shown below:

Anticipated First Month Cost (in thousands of Rs)

Product Acceptance	Large Scale	Medium Scale	Small Scale
Good	15	20	30
Fair	25	35	34
Poor	60	40	25
Maximum Costs	*60*	*40*	*34*

Now from these maximum costs (60, 40 and 34) we select the smallest, which is 34. The strategy corresponding to 34 is selected i.e. the company should introduce the product on a small scale.

So as per criterion of optimism or minimax the company should introduce the product on a small scale and its cost will be Rs 34000.

8.7 Criterion of Realism or Hurwicz criterion

In this criterion the decision maker is neither extremely optimistic nor extremely pessimistic but realistic in his/her approach. This criterion given by Hurwicz realizes the fact that the decision maker's view could fall between extreme optimism and extreme pessimism. In this criterion á known as coefficient of optimism is defined; and the value of á lies between 0 and 1. When á is 1 it signifies extreme optimism and the criterion becomes same as the criterion of optimism whereas when á is 0 it signifies extreme pessimism and the criterion becomes same as the criterion of pessimism. 1 – á is known as coefficient of pessimism.

In this criterion, in order to take a decision regarding selection of strategy/course of action/alternative, the decision maker multiplies the highest payoff pertaining to each alternative with á and multiplies the smallest payoff pertaining to each alternative with 1 – á. The decision maker then finds the sum of these two for all the alternatives and selects the alternative for which this sum is the highest.

If instead of profits, losses/costs are given then under this criterion the decision maker multiplies the lowest cost pertaining to each alternative with á and multiplies the highest cost pertaining to each alternative with 1 – á. The decision maker then finds the sum of these two for all the alternatives and selects the alternative for which this sum is the lowest.

8.8 Example of Criterion of Realism or Hurwicz Criterion

8.8.1 Example A manufacturer makes a product, of which the principal ingredient is a chemical X. At the moment, the manufacturer spends Rs 1,000 per year on supply of X, but there is a possibility that the price may soon increase to four times its present figure because of a worldwide shortage of the chemical. There is another chemical Y, which the manufacturer could use in conjunction with a third chemical Z, in order to give the same effect as chemical X. Chemicals Y and Z would together cost the manufacturer Rs 3,000 per year; but their prices are unlikely to rise. What action should the manufacturer take? Apply criterion of realism or Hurwicz criterion for decision making (coefficient of optimism is 0.4).

Solution

For this problem we can develop the payoff table in the following way:

Present Payoff on supply of X is minus 1000 rupees i.e. Rs – 1000 (minus sign expresses costs in terms of profit)

Payoff on supply of X after increase in price is minus 4000 rupees i.e. Rs – 4000

Payoff on supply of Y and Z together is minus 3000 rupees i.e. Rs – 3000

The various states of nature are -

N_1 – Price of X increases

N_2 – Price of X does not increases

The different courses of action (strategies) available are

S_1 – Use X

S_2 – Use Y and Z

The payoffs for different combinations of strategies and states of nature are as follows:

Combination	Payoff in Rs
N_1S_1	- 4000
N_2S_1	- 1000
N_1S_2	- 3000
N_2S_2	- 3000

Now the Payoff table for this problem is as follows:

	Strategies	
States of Nature	S_1 (Use X)	S_2 (Use Y & Z)
N_1 (Price of X increases)	-4000	-3000
N_2 (Price of X does not increases)	- 1000	-3000

Now we calculate the value of H = á (highest payoff pertaining to strategy) + (1 – á) (smallest payoff pertaining to strategy) for both the strategies.

For S_1 the value of H = 0.4 (-1000) + 0.6 (-4000)

= -2800

And for S_2 the value of H = 0.4 (-3000) + 0.6 (-3000)

= -3000

Now since the highest payoff among the two strategies is -2800. Therefore the manufacturer should continue using chemical X.

Alternate Approach

This problem could also be solved directly without converting costs into payoffs. If we directly take the costs then the relevant data could be represented through the following table:

Costs in Rupees

States of Nature	*Strategies*	
	S_1 (Use X)	S_2 (Use Y & Z)
N_1 (Price of X increases)	4000	3000
N_2 (Price of X does not increases)	1000	3000

Now we calculate the value of H = á (smallest cost pertaining to strategy) + (1 – á) (highest cost pertaining to strategy) for both the strategies.

For S_1 the value of H = 0.4 (1000) + 0.6 (4000)

= 2800

And for S_2 the value of H = 0.4 (3000) + 0.6 (3000)

= 3000

Now since the smallest cost among the two strategies is Rs 2800. Therefore the manufacturer should continue using chemical X.

8.9 Criterion of Equally Likelihood or Laplace Criterion

In this criterion it is assumed that all the states of nature have an equal chance of occurrence (equally likely to occur) since the decision maker does not have any idea regarding the chances of occurrence of different states of nature. In this criterion the expected payoff for each strategy/course of action/alternative is calculated (which is sum of all payoffs pertaining to an alternative divided by the number of states of nature) and then the alternative with the highest expected payoff is selected.

If instead of profits, losses/costs are given then under this criterion the decision maker should find the expected loss/cost for each strategy/course of action/alternative and then select the alternative for which the expected loss/cost is the least.

8.10 Example of Criterion of Equally Likelihood or Laplace Criterion

8.10.1 Example Scarlett Ltd. has installed a machine costing Rs 10 lakhs and is in the process of placing order for appropriate number of spare parts required for repairs. The spare parts cost Rs 8000 each and are available only if they are ordered in advance. In case of machine failure if no spares are available then cost to company comes out to be Rs 30,000 per unavailable spare part. The management feels that the number of spare parts which may be required could be 1, 2, 3, 4 and 5. Find how many spare parts should the company order using criterion of equally likelihood or Laplace criterion.

Solution

For this problem we can develop the cost table in the following way:

Cost per spare part is Rs 8000

Cost per unavailable spare part is Rs 30,000

The various states of nature are -

N_1 – Number of spare parts required is 1

N_2 – Number of spare parts required is 2

N_3 – Number of spare parts required is 3

N_4 – Number of spare parts required is 4

N_5 – Number of spare parts required is 5

The different courses of action (strategies) available are

S_1 – Number of spare parts ordered is 1

S_2 – Number of spare parts ordered is 2

S_3 – Number of spare parts ordered is 3

S_4 – Number of spare parts ordered is 4

S_5 – Number of spare parts ordered is 5

If 'R' stands for number of spares ordered and 'D' stands for number of spares demanded, then

Cost, C, = 8,000 R if D d≤R

or

= 8,000 R + 30,000 (D – R), if D e ≥ R

The costs for different combinations of strategies and states of nature are as follows:

Combination	Cost in Rs	Combination	Cost in Rs
N_1S_1	8000	N_2S_1	38000
N_3S_1	68000	N_4S_1	98000
N_5S_1	128000	N_1S_2	16000
N_2S_2	16000	N_3S_2	46000
N_4S_2	76000	N_5S_2	106000
N_1S_3	24000	N_2S_3	24000
N_3S_3	24000	N_4S_3	54000
N_5S_3	84000	N_1S_4	32000
N_2S_4	32000	N_3S_4	32000
N_4S_4	32000	N_5S_4	62000
N_1S_5	40000	N_2S_5	40000
N_3S_5	40000	N_4S_5	40000
N_5S_5	40000		

Now the cost table for this problem is as follows:

Costs (in thousand of Rs.)

Number of Spare Parts Required	Number of Spare Parts Ordered				
	S_1	S_2	S_3	S_4	S_5
N_1	8	16	24	32	40
N_2	38	16	24	32	40
N_3	68	46	24	32	40
N_4	98	76	54	32	40
N_5	128	106	84	62	40

Under Laplace criterion we find the expected cost (average cost) for each strategy and then select the strategy for which the expected cost is the least.

Expected (average) cost (in '000 of Rs) associated with S_1 = (8+38+68+98+128)/5 = 68

Expected (average) cost (in '000 of Rs) associated with S_2 = (16+16+46+76+106)/5 = 52

Expected (average) cost (in '000 of Rs) associated with S_3 = (24+24+24+54+84)/5 = 42

Expected (average) cost (in '000 of Rs) associated with S_4 = (32+32+32+32+62)/5 = 38

Expected (average) cost (in '000 of Rs) associated with S_5 = (40+40+40+40+40)/5 = 40

Since the least expected cost is Rs 38,000 and is associated with strategy S_4, this implies that the company should place an order for 4 spare parts.

8.11 Minimax Regret criterion or Savage criterion

This criterion is based on the opportunity losses (regrets) associated with different alternatives that are adopted by the decision maker. So in order to use this criterion first of all Opportunity Loss Table (Regret Table) is to be developed.

Opportunity loss or regret is the difference between the highest payoff/profit associated with a given state of nature and the actual payoff/ profit that accrues due to adopting a particular strategy i.e. it is the loss that accrues to the decision maker for not adopting the best strategy in a particular state of nature.

8.11.1 Example There is a bouquet seller who makes and sells bouquets in prime market of the city. For preparing bouquets he has to place the order for flowers one day in advance and also he has to make advance payment for the same. He knows from past experience that the demand for number of bouquets could be either 20 or 30 or 40 or 50. The cost of a bouquet for him is Rs 35/- and he sells it at Rs 50/-. Any bouquet left unsold at the end of the day is valueless for the seller. Construct the opportunity loss table for this problem.

Solution

The Payoff table for this problem is as follows:

States of Nature	*Conditional Payoffs (Rs.)* *Courses of Action*			
	S_1	S_2	S_3	S_4
N_1	300	-50	-400	-750
N_2	300	450	100	-250
N_3	300	450	600	250
N_4	300	450	600	750

The opportunity losses for different combinations of states of nature and strategies are as follows:

Combination	Payoff in Rs
N_1S_1	300 – 300 = 0
N_1S_2	300 + 50 = 350
N_1S_3	300 + 400 = 700
N_1S_4	300 + 750 = 1050
N_2S_1	450 - 300 = 150
N_2S_2	450 - 450 = 0
N_2S_3	450 - 100 = 350
N_2S_4	450 + 250 = 700
N_3S_1	600 - 300 = 300
N_3S_2	600 - 450 = 150
N_3S_3	600 - 600 = 0
N_3S_4	600 - 250 = 350
N_4S_1	750 - 300 = 450
N_4S_2	750 - 450 = 300
N_4S_3	750 - 600 = 150
N_4S_4	750 - 750 = 0

Thus the opportunity loss table for this problem is as follows:

States of Nature	*Opportunity Losses (Rs.)* *Courses of Action*			
	S_1	S_2	S_3	S_4
N_1	0	350	700	1050
N_2	150	0	350	700
N_3	300	150	0	350
N_4	450	300	150	0

Minimax Regret criterion or Savage criterion is based on the opportunity losses (regrets) associated with different alternatives that are adopted by the decision maker. In this criterion the decision maker finds out the maximum regret associated with different strategies/courses of action/alternatives and then select the strategy/course of action/ alternative for which among these values regret is minimum i.e. this is a decision making criterion in which we select the strategy/course of action/ alternative related to the minimum regret from the maximum regrets. That's why this criterion is also called minimax regret criterion i.e. a criterion in which we select the minimum regret value from the maximum regret values.

8.12 Examples of Minimax Regret Criterion or Savage Criterion

8.12.1 Example An Electronics Company is thinking of manufacturing special modern components. The General Manager Marketing has developed the following table concerning profits.

	Profit (in '000 of Rs.)		
	Large Scale Prodn	Medium Scale Prodn	Small Scale. Prodn
Strong Market	500	250	150
Fair Market	200	350	200
Poor Market	- 80	100	120

Which strategy should be adopted by the company if the General Manager applies minimax regret criterion or Savage criterion for taking decision?

Solution

The opportunity loss table along with maximum regret for this problem is as follows:

	Profit (in '000 of Rs.)		
	Large Scale Prodn.	Medium Scale Prodn.	Small Scale Prodn.
Strong Market	0	250	350
Fair Market	150	0	150
Poor Market	200	20	0
Maximum Regret	*200*	*250*	*350*

The maximum regret associated with different strategies are 200, 250 and 350. The minimum value among these three is 200. Hence strategy **Large Scale Production** should be adopted by the company when General Manager uses minimax regret criterion or Savage criterion for taking decision.

8.13 Key Words

1. **Payoff Table -** The conditional payoffs in terms of gains associated with different combinations of courses of action and states of nature can be represented in a tabular form, known as payoff table, in which the row designations are the states of nature and column designations are the strategies/actions available to the decision maker.

2. **Opportunity loss or regret -** It is the difference between the highest payoff/profit associated with a given state of nature and the actual payoff/profit that accrues due to adopting a particular strategy

3. **Opportunity Loss Table (Regret Table)** – It is the table that represents the opportunity losses (regrets) associated with different combinations of courses of action and states of nature.

4. **Decision Making Under Uncertainty -** In Decision Making Under Uncertainty there exist several states of nature but the decision maker is uncertain about which future state of nature will occur. The decision maker has no idea as to which state of nature will

occur as the decision maker has no data/no information regarding the chances of occurrence of various states of nature.

5. **Criterion of Optimism or Maximax/Minimin** – It is a decision making criterion under uncertainty in which the decision maker is highly optimistic and selects a strategy that will result in highest payoff (profit/sales/increase, etc). In this criterion the decision maker finds out the maximum payoffs pertaining to different strategies and then from them selects the strategy for which payoff is highest.

6. **Criterion of Pessimism or Maximin/Minimax** – It is a decision making criterion under uncertainty in which the decision maker is highly pessimistic and selects a strategy that will result in least payoff (profit/sales/increase, etc). In this criterion the decision maker finds out the minimum payoffs pertaining to different strategies and then from them selects the strategy for which payoff is highest.

7. **Criterion of Realism or Hurwicz criterion** - It is a decision making criterion under uncertainty in which the decision maker is realistic in his/her approach. In this criterion in order to take a decision regarding selection of strategy the decision maker multiplies the highest payoff pertaining to each alternative with á and multiplies the smallest payoff pertaining to each alternative with 1 – á; and then finds the sum of these two for all the alternatives and selects the alternative for which this sum is highest.

8. **Criterion of Equally Likelihood or Laplace Criterion** - It is a decision making criterion under uncertainty in which the decision maker assumes that all the states of nature have an equal chance of occurrence and then selects the strategy for which the expected payoff is highest.

8.14 Terminal Questions

8.14.1 What do you understand by opportunity loss? How opportunity loss table could be developed from payoff table?

8.14.2 Explain with real examples the different criteria that are used for decision making under uncertainty.

8.14.3 The following payoff table represents the payoffs for different combinations of strategies and states of nature:

States of Nature	*Conditional Payoffs (Rs.'000)* *Strategies*			
	S_1	S_2	S_3	S_4
N_1	40	55	70	65
N_2	55	75	90	20
N_3	65	35	20	-10

Find the strategy to be adopted using (i) Maximax criterion (ii) Maximin criterion (iii)Hurwicz criterion (iv) Laplace criterion and (v) Savage criterion.

8.14.4 The Parker Flower Shop promises its customers delivery within four hours on all flower orders. All flowers are purchased on the previous day and delivered to Parker by 8:00 a.m. the next morning. Parker's daily demand for dozens of roses could be 7, 8, 9 or 10. Parker purchases roses for Rs. 10.00 per dozen and sells them for Rs 30.00. All unsold roses are donated to a local hospital. How many dozens of roses should Parker order each evening to maximize its profits? Use Maximax criterion, Maximin criterion and Hurwicz criterion for finding the answer.

8.14.5 ABC Company needs to increase its production beyond its existing capacity. It has narrowed the alternatives to two approaches to increase the production capacity:

- expansion at a cost of Rs 8 million or
- modernization at a cost of Rs 5 million.

Both approaches would require the same amount of the time for implementation. Management believes that over the required payback period, demand will either be high or moderate. If the demand is high, expansion would gross an additional Rs 12 million but modernisation only an additional amount of Rs 6 million, due to lower maximum production capacity. On the other hand, if the demand is moderate, the comparable figures would be Rs 7 million for expansion and Rs 5 million for modernization.

(i) Calculate the conditional profit in relation to various action and states of nature combination.

(ii) Find the strategy to be adopted if Minimax regret criterion is used for decision making.

8.15 Further Readings

1. N D Vohra : Quantitative Techniques in Management(Tata McGraw Hill)
2. V K Kapoor : Operations Research
3. Taha : Operations Research (Pearson)
4. Sharma , J.K : Operations Research : Theory and Applications (Macmillan India Ltd)

Block
3

विशेषज्ञ -समिति

1. Dr. Omji Gupta, Director SoMS UPRTOU, Allahabad
2. Prof. Arvind Kumar, Prof., Deptt. of Commerce, Lucknow University, Lucknow
3. Prof. Geetika, HOD, SoMS, MNNIT, Allahabad
4. Prof. H.K. Singh, Prof., Deptt. of Commerce, BHU, Varanasi

लेखक

Dr. Sanjay Mishra, Asso. Prof. MPJ Rohilkhand University, Bareilly.

सम्पादक

Prof. S.A.Ansari, Ex-Dean, Director and Head MONIRBA, University of Allahabad

परिमापक

अनुवाद की स्थिति में

मूल लेखक	अनुवाद
मूल सम्पादक	भाषा सम्पादक
मूल परिमापक	परिमापक

सहयोगी टीम

संयोजक Dr. Gaurav Sankalp, SoMS, UPRTOU

प्रकाशन --उत्तर प्रदेश राजर्षि टण्डन मुक्त विश्वविद्यालय, इलाहाबाद

प्रकाशन- उत्तर प्रदेश राजर्षि टण्डन मुक्त विश्वविद्यालय, प्रयागराज की ओर से डॉ. अरूण कुमार गुप्ता, कुलसचिव द्वारा पुनः मुद्रित एवं प्रकाशित वर्ष-2020।

मुद्रक- चन्द्रकला यूनिवर्सल प्राइवेट लिमिटेड 42/7 जवाहर लाल नेहरू रोड, प्रयागराज-211002

Block 3 : Quantitative Techniques for Business Decisions

Block Introduction

Block three comprises of four units. Unit nine deals with decision making. Unit ten deals with probability theory while unit eleven deals with operations with matrix and introduction to vector. Unit twelve highlights decision three analysis.

UNIT 9 DECISION MAKING

Learning Objectives

After completing this unit you will be able to understand the:

- Role of manager as decision-maker;
- Importance of decision-making through mathematical model;
- Decision making process; and
- Various decision making environment.

Structure

9.0 Learning Objectives

After completing this unit you will be able to understand the:

- Role of manager as decision-maker;
- Importance of decision-making through mathematical model;
- Decision making process; and
- Various decision making environment.

9.1 INTRODUCTION

Decision making is an essential part of human life at every stage. To a great extent, the success or failure that a person experience, in life depends on the decision that he or she makes. Similarly, the success or failure of organization also depends upon its ability of making decision. The decision maker selects one strategy (course of action) over others depending on some criteria, like: utility, sales, cost or rate of return. Thus, decision theory or decision analysis is a rational method of decision making and developing method for improving decisions in a variety of circumstances. Decision theory may be defined as:

"A process of selecting one best course of action out of all availabl alternatives, which is considered to meet the objective of the decision problem, more satisfactorily than others as judged by the decision maker."

"Decision making is a process evaluating all possible alternatives and selecting the best one out of all that satisfies the objective of the decision-maker under various state-of-natures."

9.2 TERMINOLOGY USED IN DECISION MAKING PROBLEMS

Irrespective of the type of decision you want to make there are some common terms, which are same in all the problems. These are:

9.2.1 Courses of action

A decision is made out of various alternatives known and available to decision-maker. These are called courses of action, acts or strategies and are under the control of the decision-maker.

9.2.2 State of nature

Courses of action that depend upon various situations or factors which are not under the control of decision maker, these situations or factors are known as state of nature (future).

9.2.3 The preference or value system

Decision maker generally keeps certain criteria in mind while making decision, like: profit, income, utility etc. These are known as the preferences or value system of a decision maker.

9.2.4 Payoff

A quantitative value of each possible alternative resulting as a combination of course of action and state of nature is called pay-off. The pay-off values are always conditional values because of unknown state of nature.

9.3 PAYOFF TABLE

A tabular arrangement of these conditional outcome (payoff) values is known as payoff matrix.

Suppose the problem under consideration has n possible events (state of nature) denoted by $S_1, S_2, \ldots\ldots\ldots\ldots S_n$ and m alternative acts (strategies) denoted by $A_1, A_2 \ldots\ldots\ldots\ldots A_m$. then the payoff corresponding to strategy A_j of the decision maker under the event (state

of nature) S_i will be denoted by p_{ij} where, i= 1, 2, ...n & j= 1, 2
.................m.

9.3.1 Construction of payoff table

State of Nature	Decision Alternatives (courses of Action)			
	A_1	A_2	A_3.........	A_m
S_1	p_{11}	p_{21}	p_{31}...............	p_{n1}
S_2	p_{12}	p_{22}	p_{32}...............	P_{n2}
S_3	p_{13}	p_{23}	p_{33}............	p_{n3}
S_n	p_{1m}	p_{2m}	p_{3m}..................	p_{nm}

9.4 OPPORTUNITY LOSS

The difference between highest possible profit and actual outcom (profit) for that state of nature and course of action taken is known a opportunity loss. In other words opportunity loss is incurred due to failur of not adopting most favorable courses of action or strategy.

9.4.1 Construction of Opportunity loss table

Suppose in the previous payoff table if Mi is the maximum pay-o possible corresponding to n strategies and if A_1 is used by decision make then opportunity loss will be M1-pi1 and so on, then a table showin opportunity loss can be computed as follows:

State of Nature	Decision Alternatives (courses of Action) A_1	A_2	A_3........	A_m
S_1	M_1-p_{11}	M_1- p_{21}	M_1-p_{31}..............	M_1- p_{n1}
S_2	M_2-p_{12}	M_2-p_{22}	M_2-p_{32}..............	M_2-P_{n2}
S_3	M_3-p_{13}	M_3- p_{23}	M_3-p_{33}..........	M_3-p_{n3}
	.			
	.			
	.			
S_n	M_n-p_{m1}	M_n-p_{m2}	M_n-p_{m3}................	M_n-p_{nm}

9.5 DECISION – MAKING PROCESS

The decision making process involves the following steps:

Step 1- Clearly identify & define the problem.

Step 2- Search all possible courses of actions out of which decision is to be taken and state of nature under which decision is to be taken.

Step 3- Calculate payoff corresponding to all state of nature and courses of action.

Step 4- Select and apply an appropriate mathematical decision theory model to select the best course of action from the payoff table that results in the optimum (desired) payoff.

Check your progress A

1. What do you understand by decision making? Why is it essential for a manager?

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2. Explain the following terms:

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a) course of action

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b) state of nature

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c) pay-off

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....

3. What do you understand by expected opportunity loss?

....

....

9.6 DECISION MAKING ENVIRONMENT

The ultimate objective of decision theory is to provide the best course of action to the decision maker amongst the available courses of action. The type of decisions people make depends on how much knowledge or information they have about the environment under which they need to take decision. There are three decision making environments:

- Decision making under certainty
- Decision making under uncertainty
- Decision making under risk

9.6.1 Decision making under certainty

When a decision maker knows about the possible outcomes of every course of action with certainty i.e. with 100 % surety, such situation is called decision making under certainty. In this situation decision maker adopts the most favorable action i.e. the action that leads to maximum satisfaction.

Suppose you want to invest Rs. 1,00,000 to gain future benefits. You have three courses of action like investing in saving bank account at a rate of 3.5 % per annum or investing in fixed deposit at an interest rate of 8.5 % p.a. or investing in a Govt. treasury note paying 10 % interest to you per annum. If all three investment options are secure and guaranteed, then there is a certainty that you will like to invest in a Govt. treasury note because it will give you the maximum gain.

The various techniques for solving problems under certainty are:

i) Linear programming,

ii) Integer programming,

iii) Queue models,

iv) Inventory models,

v) Capital budgeting or

vi) Break even analysis etc.

9.6.2 Decision making under uncertainty

This refers to situations where more than one outcome can result from any single decision. The decision maker in this type of situation cannot determine the outcome with 100 % surety.

The various techniques for solving problems under uncertainty are:

i) Laplace criterion,

ii) Maximin or Minimax criterion,

iii) Maximax or Minimin criterion,

iv) Savage criterion,

v) Hurwicz criterion or criteria of realism.

9.6.3 Decision making under risk

When a decision maker selects from several possible options whose probabilities of occurrence can be stated, he is said to take decisions under risk. In this situation, the decision maker faces various states of nature but he selects on the basis of his knowledge, experience, information available or judgments to select the best course of action.

In this case, most widely used criterion are:

i) Expected Monetary Value (EMV) Or Expected Utility (EU)

ii) Expected Opportunity Loss (EOL) criterion.

9.7 EXPECTED MONETARY VALUE (EMV)

The EMV, or the mean value, is the long-run average value of that decision. In this criterion, the expected monetary value (EMV) for each alternative is calculated by taking the sum of possible payoffs of each alternative, each weighted by the probability of occurrence for that pay off. Then the alternative corresponding to highest EMV (lowest EMV in case of loss) is selected as a decision by the decision maker.

Suppose there is a prior knowledge either on the basis of past experience or on a subjective basis that the state of nature S_i has a probability of occurrence $p(s_i)$ where i = 1,2,3,.... n. then the expected monetary value corresponding to a course of action A_j of the decision maker is given by:

$$EMV(A_j) = P_{1j} * p(s_1) + P_{2j} * p(s_2) + P_{3j} * p(s_3) + \ldots\ldots\ldots + P_{nj} * p(s_n)$$

9.7.1 Steps for calculating EMV

Step 1: Construct pay off matrix related to all possible courses of action and state of natures.

Step 2: Calculate payoff value for all combination of courses of action and states of nature.

Step 3: Calculate probability of corresponding state of natures.

Step 4: Find out EMV for each course of action by multiplying the conditional payoff with the respective probability and then summing these weighted values for each course of action.

Step 5: select the course of action that yields the optimal EMV.

Check your knowledge B

1. Explain the following term

a) Decision making under certainty

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b) Decision making under uncertainty

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c) Decision making risk

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9.7.2 Illustrations based on EMV

Example- 1

The payoff (in Rs.) of three courses of actions A1, A2 and A3 and possible states of nature S1, S2 and S3 are given in the table.

The probabilities of states of nature are 0.40, 0.35 and 0.25 respectively. Determine the optimal act using EMV model.

	Course of Actions		
States of nature	A1	A2	A3
S1	-50	100	200
S2	250	-150	-50
S3	300	400	-200

Solution

State of Nature	Probability	Conditional pay off (Rs.)				Expected Payoff (Rs.)		
S	P (S)	A1	A2	A3		A1	A2	A3
(1)	(2)	(3)	(4)	(5)		(2)*(3)	(2)*(4)	(2)*(5)
S1	0.40	-50	100	200		-20	40	80
S2	0.35	250	-150	-50		87.5	-52.5	-17.5
S3	0.25	300	400	-200		75	100	-50
EMV						**142.5**	87.5	12.5

The maximum value of EMV is corresponding to action A1. Hence according to the EMV criterion, the optimal action is A1.

Example 2

A book seller sells weekly magazine, which he purchases at a wholesale price of Rs. 35 each; and sells it a retail price of Rs. 40 each. An unsold copy after a week is a clear loss to the seller. The seller estimated the following probabilities for the number of copies demanded.

No. of copies:	50	51	52	53	54	55
Probabilities:	0.07	0.1	0.28	0.30	0.22	0.03

Prepare a payoff table and find out how many copies should he order so that his expected profits will be maximum. Loss of sale due to stock out situation is negligible.

Solution

Given that:

Cost per magazine = Rs. 35

Selling price per magazine = Rs. 40

Profit per magazine = Rs.40-35 = Rs. 5

Loss per magazine in case it is not sold in one week = Cost price of magazine = Rs. 35 per magazine.

Thus, the conditional payoff matrix with all alternative decisions (stock levels) and the states of nature (demands) and corresponding probabilities are computed in the following table:

Table: 1 Conditional Payoff Table (In Rs.)

D Possible demand (S_i)	Probability P (S_i)	Payoff for various possible course of actions i.e. for different Stock levels (A_j)					
		A_1 (50)	A_2 (51)	A_3 (52)	A_4 (53)	A_5 (54)	A_6 (55)
S_1 (50)	0.07	250	215	180	145	110	75
S_2 (51)	0.10	250	255	220	185	150	115
S_3 (52)	0.28	250	255	260	225	190	155
S_4 (53)	0.30	250	255	260	265	230	195
S_5 (54)	0.22	250	255	260	265	270	235
S_6 (55)	0.03	250	255	260	265	270	275

Working note: *To calculate payoff at each cell possible demand and stock level is to be considered simultaneously. Like in first cell (i.e. S1A1) stock level is 50 and possible demand is also 50, so total profit will be 50** Rs.5= Rs.250. *In cell S2A1 stock level is 50 where as demand is 51, seller can sell 50 because he is having this much of stock only so total profit will be 50** Rs.5= Rs.250. *In second column first row i.e. S1A2 stock level is 51 and demand is 50 so one unit will remain unsold and that will be a loss for seller thus total profit in this case will be no. of units sold * profit per unit-cost price of unsold unit i.e. 50** Rs.5- Rs.35= Rs.215. *Similarly other payoff can be calculated.*

Expected Monetary Values for each course of action are calculated by taking a summation of weighted values for each course of action. Weighted values are calculated by multiplying conditional payoffs of each course of action with the corresponding probabilities of states of nature as shown in the following table:

Table: 2 EMV Table (In Rs.)

D Possible demand (S_i)	Probability $P(S_i)$	EMV for various possible course of actions i.e. for different Stock levels (A_j)					
		A_1 (50)	A_2 (51)	A_3 (52)	A_4 (53)	A_5 (54)	A_6 (55)
S_1 (50)	0.07	17.5	15.05	12.6	10.15	7.7	5.25
S_2 (51)	0.10	25	25.5	22	18.5	15	11.5
S_3 (52)	0.28	70	71.4	72.8	63	53.2	43.4
S_4 (53)	0.30	75	76.5	78	79.5	69	58.5
S_5 (54)	0.22	55	56.1	57.2	58.3	59.4	51.7
S_6 (55)	0.03	7.5	7.65	7.8	7.95	8.1	8.25
EMV (in Rs.)		250	**252.2**	250.4	237.4	212.4	178.6

Working note: *To calculate EMV for each course of action each payoff will be multiplied with its corresponding probability and then summed up column wise.*

Maximum EMV is Rs. 252.2, which corresponds to the stock level of 51 magazines per week. Hence 51 magazines per week should be stocked to maximize EMV or Net profit.

Example 3

A baker manufactures bakery items. The items must be consumed within a month from its date of manufacturing, otherwise it becomes worthless. The cost of one item is Rs. 100 whereas the baker charges Rs. 150 per item. In the past 12 months the baker experiences the sale of items in the following way:

No of items:	20	30	40	50
No. of weeks	4	3	3	2

Using EMV criterion, suggest the baker, how many bakery items he must produce to maximize his profits.

Solution:

Here no. of bakery items he produces is an act and monthly demand of that is an event. As per information given, baker must not produce less than 20 or more than 50 items per month. It is also given that sale of one item yields a profit of Rs. 50 or otherwise it is a dead loss of Rs. 100. Now, the various conditional payoff (profit) values for each event combination are given by:

Conditional profit = marginal profit * unit sold – marginal loss*unit unsold

= (150-100) * unit sold – 100* unit unsold

The resulting conditional payoffs and corresponding expected pay offs are computed in the following table:-

Event (Dem- and per month)	Proba bility	Conditional Pay off (Rs.) for each action (production level per month)				Expected pay off (Rs.) for each action (production level per month)			
		20	30	40	50	20	30	40	50
20	4/12= 0.34	1000	0	-1000	-2000	340	0	-340	-680
30	3/12= 0.25	1000	1500	-500	-500	250	375	-125	-125
40	3/12= 0.25	1000	1500	2000	1000	250	375	500	250
50	2/12= 0.16	1000	1500	2000	2500	160	240	320	400
EMV						1000	990	355	-155

Since the action production of 20 items yields the highest EMV of Rs. 1000, the optimum act for the baker would be to produce 20 items per month.

9.8 EXPECTED OPPORTUNITY LOSS

Expected opportunity loss (EOL) is an alternative approach to maximize EMV by minimizing Expected Opportunity Loss (EOL). It is also known as expected value of regret. EOL is calculated in the same manner as EMV criterion discussed earlier.

If P(S1), P(S2)... P(Sn) are the prior probabilities corresponding to the state of nature S1, S2, ... Sn then the expected opportunity loss corresponding to action Aj is

$$EOL\ (A_j) = (M_1-P_{1j}) * p\ (s_1) + (M_2 -P_{2j}) * p\ (s_2) + (M_3-P_{3j}) * p\ (s_3) +... +(M_n- P_{nj})*p\ (s_n)$$

Where $(Mj-P_{ij})$, refers to the possible regret of an alternative with reference to the highest payoff of an alternative.

9.8.1 Illustrations based on Expected Opportunity Loss

Example 4

A book seller sells weekly magazine, which he purchases at a wholesale price of Rs. 35 each; and sells it a retail price of Rs. 40 each. An unsold copy after a week is a clear loss to the seller. On the other hand, if a customer desires a magazine and all of them has been sold, the customer will buy elsewhere and the seller will be in a loss of expected profit out of that. The seller estimated the following probabilities for the number of copies demanded.

No. of copies:	50	51	52	53	54	55
Probabilities:	0.07	0.1	0.28	0.30	0.22	0.03

Prepare a payoff table and find out how many copies should be orders so that his expected profits will be maximum.

Solution

Given that:

Cost per magazine	= Rs. 35
Selling price per magazine	= Rs. 40
Profit per magazine = 40-35	= Rs. 5

Loss per magazine in case it is not sold in one week = Cost price of magazine = Rs. 35 per magazine.

Loss per magazine in case of out of stock condition= profit per magazine=Rs. 5

The first step here is to calculate opportunity loss through opportunity loss table. This can be done through determining the opportunity loss for not choosing the best alternative for each state of nature, or any row, is calculated by subtracting each outcome in the row from the best outcome in the same row.

Table: 1 Conditional Payoff Loss Table (In Rs.)

D Possible demand (S_i)	Probability $P(S_i)$	Payoff for various possible course of actions i.e. for different Stock levels (A_j)					
		A_1 (50)	A_2 (51)	A_3 (52)	A_4 (53)	A_5 (54)	A_6 (55)
S_1 (50)	0.07	0	35	70	105	140	175
S_2 (51)	0.10	5	0	35	70	105	140
S_3 (52)	0.28	10	5	0	35	70	105
S_4 (53)	0.30	15	10	5	0	35	70
S_5 (54)	0.22	20	15	10	5	0	35
S_6 (55)	0.03	25	20	15	10	5	0

Working note: *To calculate loss at different cell two type of loss must be taken care off. First loss is due to unsold units & second loss is due to out of stock condition (opportunity loss). In first cell (i.e. S1A1) stock level is 50 and demand is also 50 so vendor will not face any loss. Then at A1S2 stock is 50 where as total demand is 51, so one customer will remain unsatisfied and that will be a loss for vendor equal to Rs 5. In second column i.e. S1A2, stock is of 51 units whereas total demand is 50 only, so one unit will remain unsold and that will be a loss for vendor and will be equal to its cost price i.e. Rs. 35. Similarly other loss can be calculated.*

Expected Opportunity Loss for each course of action can be calculated by taking a summation of weighted values for each course of action. Weighted values are calculated by multiplying conditional loss of each course of action with the corresponding probabilities of states of nature as shown in following table:

D Possible demand (S_i)	Probability $P(S_i)$	EOL for various possible course of actions i.e. for different Stock levels (A_j)					
		A_1 (50)	A_2 (51)	A_3 (52)	A_4 (53)	A_5 (54)	A_6 (55)
S_1 (50)	0.07	0	2.45	4.9	7.35	9.8	12.25
S_2 (51)	0.10	0.5	0	3.5	7.0	10.5	14.0
S_3 (52)	0.28	2.8	1.4	0	9.8	19.6	29.4
S_4 (53)	0.30	4.5	3.0	1.5	0	10.5	21
S_5 (54)	0.22	4.4	3.3	2.2	1.1	0	7.7
S_6 (55)	0.03	0.75	0.6	0.45	0.3	0.15	0
EOL (in Rs.)		12.95	**10.75**	12.55	25.55	50.55	84.35

Working note: *To calculate EOL at each cell, each payoff will be multiplied by its corresponding probability and then summed up column wise.*

Minimum EOL is Rs. 10.75, which corresponds to the stock level of 51 magazines per week. Hence 51 magazines per week should be stocked to minimize EOL or Net loss.

Alternative method:

The same problem can be solved through an alternative approach;

Table: 1 Conditional Payoff Table (In Rs.)

D Possible demand	Probability P (S_i)	Payoff for various possible course of actions i.e for different Stock levels (A_j)						Maximum payoff M
		A_1	A_2	A_3	A_4	A_5	A_6	
(S_i)		(50)	(51)	(52)	(53)	(54)	(55)	
S_1 (50)	0.07	250	215	180	145	110	75	**250**
S_2 (51)	0.10	250	255	220	185	150	115	**255**
S_3 (52)	0.28	250	255	260	225	190	155	**260**
S_4 (53)	0.30	250	255	260	265	230	195	**265**
S_5 (54)	0.22	250	255	260	265	270	235	**270**
S_6 (55)	0.03	250	255	260	265	270	275	**275**

Working note: *Payoff of the table is calculated same as in Example 2. Then the maximum value of each row is written in the last column.*

To calculate conditional opportunity loss each payoff wil be subtracted from maximum payoff of the respective row.

Table: 2 Conditional opportunity loss Table (In Rs.)

D Possible (A_j) demand	Probability P (S_i)	Opportunity loss for various possible course of actions i.e. for different Stock levels					
		A_1	A_2	A_3	A_4	A_5	A_6
(S_i)		(50)	(51)	(52)	(53)	(54)	(55)
S_1 (50)	0.07	0	35	70	105	140	175
S_2 (51)	0.10	5	0	35	70	105	140
S_3 (52)	0.28	10	5	0	35	70	105
S_4 (53)	0.30	15	10	5	0	35	70
S_5 (54)	0.22	20	15	10	5	0	35
S_6 (55)	0.03	25	20	15	10	5	0

Working note: *Now the expected opportunity loss for each alternative is calculated by adding the product of the probability of each state of nature with the corresponding conditional opportunity loss value for the alternative.*

Table: 3 Conditional opportunity loss Table (In Rs.)

D Possible (A_j) demand	Probability $P(S_i)$	Conditional opportunity loss of possible course of actions i.e. for different Stock levels					
		A_1	A_2	A_3	A_4	A_5	A_6
(S_i)		(50)	(51)	(52)	(53)	(54)	(55)
S_1 (50)	0.07	0	2.45	4.9	7.35	9.8	12.25
S_2 (51)	0.10	0.5	0	3.5	7.0	10.5	14.0
S_3 (52)	0.28	2.8	1.4	0	9.8	19.6	29.4
S_4 (53)	0.30	4.5	3.0	1.5	0	10.5	21
S_5 (54)	0.22	4.4	3.3	2.2	1.1	0	7.7
S_6 (55)	0.03	0.75	0.6	0.45	0.3	0.15	0
EOL		12.95	10.75	12.55	25.55	50.55	84.35

Minimum EOL is Rs. 10.75, which corresponds to the stock level of 51 magazines per week. Hence 51 magazines per week should be stocked to minimize EOL or Net loss.

9.9 EXPECTED VALUE OF PERFECT INFORMATION

The expected value of perfect information is the expected or average return, in the long run, if we have perfect information before a decision has to be made. To find this value, we select the best alternative for each state of nature and multiply it by corresponding probability of occurrence.

Mathematically,

EPPI = Ó(best payoff in state of nature i) * (probability of state of nature i)

In perfect market situations, there is no control over the occurrence of any given state of nature. However, in some situations like oligopoly market the decision maker knows with certainty that any increase in his price will lead to either increase in price by the competitor or the latter remains with unchanged price. Suppose the decision maker knows that the competitor will not raise the price even if there is price rise by him, the decision maker will have completely reliable information. Now he would be able to choose the optimal course of action with certainty. Thus, Expected Value of Perfect Information (EVPI) is the maximum amount one would be willing to pay to obtain perfect information about the state of nature that would occur. Hence,

$$EVPI = EPPI - EMV$$

EPPI is Expected Profit with Perfect Information

9.9.1 Illustrations based on EVPI

Example 5:-

Based on data provided in example 2 calculate EPPI & EVPI.

Solution-

Given that:

Cost per magazine = Rs. 35

Selling price per magazine = Rs. 40

Profit per magazine = 40-35 = Rs. 5

Loss per magazine in case it is not sold in one week = Cost price of magazine = Rs. 35 per magazine.

Thus, the conditional payoff matrix with all alternative decisions (stock levels) and the states of nature (demands) and corresponding probabilities are computed in the following table:

Table: 1 Conditional Payoff Table (in Rs.)

D actions levels (A_j) demand	Probability Possible	Payoff for various possible course of P (S_i) i.e. for different Stock					
		A_1	A_2	A_3	A_4	A_5	A_6
(S_i)		(50)	(51)	(52)	(53)	(54)	(55)
S_1 (50)	0.07	250	215	180	145	110	75
S_2 (51)	0.10	250	255	220	185	150	115
S_3 (52)	0.28	250	255	260	225	190	155
S_4 (53)	0.30	250	255	260	265	230	195
S_5 (54)	0.22	250	255	260	265	270	235
S_6 (55)	0.03	250	255	260	265	270	275

Table: 2 Computation of EPPI (in Rs.)

Event demand	Probability	Payoff under perfect information	Payoff under expected perfect information
50	0.07	250	17.5
51	0.10	255	25.5
52	0.28	260	72.8
53	0.30	265	79.5
54	0.22	270	59.4
55	0.03	275	8.25
		Total	262.95

Working note: *To calculate Expected Value of Perfect Information first we calculate Expected Profit with Perfect Information (EPPI). For EPPI values from table corresponding to equal demand and stock will be taken up. These values then will be multiplied by corresponding probabilities and then summed up to get EPPI. For EVPI maximum value of EMV will be subtracted from this EPPI.*

Thus,

EVPI=EPPI-Maximum EMV

= Rs.262.95 – Rs.252.20

= Rs.10.75

9.10 KEY WORDS

Decision making environment: Circumstances under which one need to take decisions.

Decision process: A process which results in the selection from a set of alternative courses of action.

Opportunity loss: Difference between highest possible profit and actual outcome for a particular course of action.

Payoff: Quantitative value of all alternative courses of action.

Payoff table: Tabular form of quantitative values of all alternative courses of action.

State of nature: Situations which are not under the control of decision maker.

Value system: Preconceived mind set of decision maker.

9.11 SELF ASSESSMENT QUESTIONS

1. Explain in detail various ingredients of a decision problem.
2. What do you understand by decision making process? Also explain steps involved in this process.
3. What are different emvironmentS in which decisionS are made?
4. Write a short note on decision making under risk.
5. Indicate the difference between decision making under risk and decision making under uncertainty.
6. What is EMV? How is it computed to use it as a criterion of decision making and when?

7. Define EVPI. How is it calculated?

8. Write a short notes on the following terms:

   a. Decision making

   b. Pay-off table

   c. EOL

   d. Decision making under certainty

   e. Decisoion making environment.

9. A physicion purchases a particular vaccine on Monday of each week. The vaccine must be used within the week following, otherwise it becomes worthless. The vaccine costs Rs.20 per dose and the physician charges Rs.40 per dose. In the past 50 weeks, the physicion has administered it in the following quantities:

Doses per week:	20	25	30	35
No. of weeks:	5	15	25	5

Determine how many doses the physician should buy every week.

10. A retailer has the following probabilities of selling an item:

No. of items:	10	11	12	13	14
Probabilities:	0.10	0.15	0.20	0.25	0.30

Cost of one item is Rs.30 and sale price is Rs.50. He can not return unsold item.

   a. How many items should he order by EMV criterion?

   b. How many items should he order under EOL criterion?

   c. Calculate EPPI and EVPI.

11. A news paper distributor assigns probabilities to the demand for a magazine as follows:

Copies demanded:	10	20	30	40
Probabilities:	0.4	0.3	0.2	0.1

A copy of magazine which sells for Rs. 25 costs Rs.20 What can be the maximum posssible Expected Monetary Value if the distributor can return unsold copies for Rs.10.

12. Calculate the expected loss from the following pay-off table.

Action	Events			
	E1	E2	E3	E4
A1	50	300	-150	50
A2	400	0	100	0
A3	-50	200	0	100
A4	0	300	300	0

Suppose that the probability of the events in this table are:

P(E1)=0.15, P(E2)=0.45, P(E3)=0.25, P(E4)=0.15

Calculate the expected payoff and expected loss of each action.

13. An electronic goods seller finds that the cost of holding LED in stock for a week is Rs 50. Customer who can not obtain a new LED immediately tend to go to other dealers and he estimates that every customer who cannot get immediate delivery he losses an average of Rs.200. For one particular model of LED the probabilities of a demand of 0,1,2,3,4 & 5 in a week are 0.05, 0.10, 0.20, 0.30, 0.20, & 0.15 respectively.

a. How many LED per week should he order? Assume that there is no time lag between ordering and delivery.

b. Compute EVPI.

c. The seller is thinking of spending on a small market survey to obtain additional information regarding the demand levels. How much should he be willing to spend on such a survey?

14. A dealer has to decide on the optimal number of units to be stocked in respect of a certain item under the following circumstances:

a. Cost price in season: Rs. 20

b. Selling price in season: Rs.22

c. Bargain price after season: Rs.10

d. Cost of holding an item in inventory beyond the season is Rs. 2

The distribution of demand based on past data is shown below:

Demand:	10	12	14	16	18	20
Probabilities:	0.10	0.20	0.25	0.15	0.20	0.10

Determine the optimal act based on the expected monetary value criterion.

9.12 Further Readings

1. N D Vohra : Quantitative Techniques in Management(Tata McGraw Hill)

2. V K Kapoor : Operations Research

3. Taha : Operations Research (Pearson)

4. Sharma , J.K : Operations Research : Theory and Applications (Macmillan India Ltd)

UNIT 10 PROBABILITY THEORY

Structure

10.0 Objectives

After studying this unit, you will be able to:

- Understand the amount of uncertainty that is involved in decision making.
- Understand the fundamentals of probability and various probability rules that help in calculating probability.
- Perform several analyses that help in decision making involving uncertainty.

10.1 INTRODUCTION

In our day to day life, we usually listen and use statements like: "Probably it will rain today", "It is likely that Mr. X will not come for this party", "Team A seems to have more chances to win this game", "It is possible that he will join us at 2'o'clock." All these words probably, likely, chances, possible sound same and have similar meaning. In all these statements an element of uncertainty is associated. In the layman's language the word 'Probability' thus connotes that there is uncertainty about the happening of the event.

10.2 PROBABILITY DEFINED

The probability of a given event is an expression of likelihood or chance of occurrence of that event. A probability ranges from zero to one. Zero means event will not occur whereas one means event will definitely occur. Thus, the theory of probability provides a numerical measure of the element of uncertainty. It enables us to take decision under conditions of uncertainty with a calculated risk.

10.3 TERMINOLOGY USED IN PROBABILITY THEORY

Before starting the calculation of probability, it is important to understand few terms:

10.3.1 Event and Experiment

The term experiment means an act which can be repeated under some given conditions. The term Random Experiment means, an experiment if conducted repeatedly under homogeneous conditions, does not give the same result. If an unbiased dice is thrown, any of six numbers on the dice can come up. Here, throwing the dice is an experiment.

The outcome of any random experiment is called an event. For example, in throwing a dice, appearance of a number i.e. 1, 2, 3, 4, 5, or 6 is an event.

An event could be either simple or compound. If the possible outcome corresponding to an event is simple, it is known as simple event otherwise it is known as compound event. For example, in throwing a dice, probability of getting 4 is a simple event as there is only one outcome i.e. 4 is on the dice, but getting an even number as an outcome is a compound event as there are three possible even numbers (2, 4, 6) on a dice that can be the outcome.

10.3.2 Exhaustive Cases

In a random experiment, all possible outcomes are known as exhaustive cases. For example, in tossing a coin, total number of possible outcomes is two (Head or Tail). Thus exhaustive cases are two. Similarly in throwing a dice, total number of possible outcomes is six, so the exhaustive cases are six. However, if two dices are thrown simultaneously, total number of possible outcomes will be 6 *6 = 36.

10.3.3 Favorable event

In any random experiment, the total number of desired outcomes is known as favorable event. For example, in a game of cards, if one card is chosen at random from a deck of cards, the probability of getting a diamond card will be 13/52 as there are 13 diamond cards in one set. Hence total number of favorable events will be 13.

10.3.4 Equally likely event

Events are said to be equally likely if the possibility of their happening is equal. For example, if we toss a coin, we can get either head or tail, occurrence of these two events is equally likely.

10.3.5 Independent and Dependent events

If the happening of one event is totally free from the happening of other event, the events are called as independent events. For example, in tossing a coin, possibility of getting a head is independent of getting a tail. Similarly in playing cards, possibility of getting a king is equal in every experiment (i.e.4/52 as there are 4 kings in one set), if the cards are replaced (total cards are 52 every time) after any experiment. But if the card is not replaced after every experiment, then the probability of getting a king is 3/51 as one king is dropped after an experiment so remaining kings in the set are three and total cards in the set are 51 only. This event of getting a king in the second experiment is known as dependent event.

CHECK YOUR KNOWLEDGE- A

1. Define probability. How is it useful for a manager?

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2. Define the following terms

a. Event

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b. Random Experiment

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c. Favorable event

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d. Equally likely event

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e. Independent and dependent event

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10.4 TYPES OF PROBABILITY

Probability can be defined in three ways-

10.4.1 Classical or a priori Probability

10.4.2 Relative frequency theory of probability

10.4.3 Subjective theory of probability

10.4.1 Classical or a priori Probability

This is the oldest and simplest way of defining probability. This approach was given by French Mathematician 'Laplace' in the Eighteenth century. According to this theory, probability is the ratio of favorable events to the possible events. Thus probability of the occurrence of an event A is given as:

$$p(A) = a/n$$

Where 'a' is the number of favorable events and n is the total number of possible events. Therefore,

$$\text{Probability of an event} = \frac{\text{No. of favorable outcomes}}{\text{Total no. of possible outcomes}}$$

For example :- In a random throw of a dice, the probability of getting a number 2 on the face of dice is - $p(2) = 1/6 = 0.166$, as 1 is the number of favorable outcome in one throw and total number of possible outcomes in one throw is 6.

Thus, for calculating probability under this approach we must calculate:

(i) No. of favorable outcomes

(ii) Total number of possible outcomes

Another example of tossing a coin can be taken, in this case the probability of getting a Head is

$$p(H) = 1/2 = 0.5$$

Similarly, in a random selection of a card from a deck of playing cards, the probability of getting a specific card (Ace of spade) is p (A) = 1/52

Classical probability is also called as priori probability because it assumes that various outcomes of an event are equally likely and the device with which the experiment is performed is a fair device, and so the probability of their happening is also equal. Thus, the theory determines the probability of an event in advance i.e. before or without an actual experiment is being performed.

Since this theory assumes all event are equally likely, hence this approach is applicable in situations like throwing a dice, tossing a coin, playing card selection etc. but in real life there are many conditions where occurrence of events are not equally likely. For example, if a person is suffering with a very serious disease, his probability of survival will not be 50 % since survival and death, i.e. the two mutually exclusive and exhaustive events are not equally likely.

Mathematically, it can be said that if there are 'a' possible outcomes which are favorable and 'b' number of events are unfavorable, then probability of occurring of favorable events will be:

$$p = a/(a + b)$$

And, probability of non- occurrence will be:

$$q = b/(a + b)$$

Here it should be noted that,

$$p + q = 1$$

$$\text{or,} \quad p = 1 - q$$

10.4.2 Relative Frequency Theory of Probability

In Eighteenth century, British statisticians tried to calculate risk of losses in life insurance and other commercial insurances. They defined probability on the basis of collection of statistical data of births and deaths. This theoretical foundation of probability is known as relative frequency theory of probability.

This approach of probability is impossible to implement until you once actually perform the experiment. Secondly, this approach may not be able to explain certain cases.

For example; if a coin is tossed 10 times, it is possible that you get 7 heads and 3 tails. The probability of head is thus 0.7 and that of a tail is 0.3. However, if the same experiment is continued up to 100 or 1000 times, we should expect approximately equal number of heads and tails. Thus as n (total trials) increases or tends towards ", we find that probability of all the events is equal. The probability of an event can thus be defined as the relative frequency with which it occurs in indefinitely large number of trials.

Thus, the probability of an event can be defined as the relative frequency with which it occurs in an infinitely large number of trials.

Thus for an event A,

$$p(A) = \text{Lim}_{n \to \infty} (a/n)$$

Where 'a' is the number of time an event occurred out of n trials, where n is large or approaches to infinity.

Hence,

$$\text{Probability} = \frac{\text{Relative frequency}}{\text{Total no. of trials}}$$

For example; If XYZ firm produces 1000 items per month and 10 items out of that are defective, then the probability of being an item defective would be 10/1000 i.e. 0.01.

10.4.3 Subjective Theory of Probability

This theory of probability is subjective in nature as it depends on the belief of the decision maker. The subjective probability is defined as the probability assigned to an event by any decision maker depending upon his/her knowledge or information with him.

For example; We have data related to share prices of XYZ Company for the last 1 year. Out of the 100 values, share price of this company rises 40 times. According to classical theory the probability of price rise of this Company will be 40/100 = 0.4. Now this is an individual decision whether he/she will purchase share of this company or not. Many investors may think that a rise in price of the shares is on the card, hence they will purchase the shares where as others may like to play safer by not purchasing shares of this company. Since the decision taken by any individual reflects his/her personality, hence this theory is also known as Personality theory of probability.

Since most of the times, top-level management decisions are concerned with specific conditions, rather than what happened in the past, therefore, the decision makers at this level make considerable use of subjective probability.

10.4.4 Illustration based on Probability theory

Example 1:

From a bag containing 20 green and 50 yellow balls, a ball is drawn at random. What is the probability that it is a green ball?

Solution:

Total number of balls in the bag = 20 + 50 =70

Number of green ball in the bag = 20

Probability of getting a green ball

$$P(G) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

$$= 20/70$$

$$= 2/7$$

Example 2:

One card is drawn from a standard pack of 52. What is the probability that it is a King?

Solution:

Total number if cards in a standard pack = 52

Number of King in a standard pack = 4

Probability that the card drawn is a King

$$P(K) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

= 4/52

= 1/13

Example 3:

What is the probability of getting a head in tossing a coin?

Solution:

Total number of possible events = 2 (either a head or tail)

Number of favorable event = 1 (getting a head)

Probability of getting a head

$$P(H) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

= 1/2

Example 4:

A dice is thrown, what is the probability of getting Number Six?

Solution:

Total number of possible events = 6 (since, a dice is having six faces numbered 1, 2, 3, 4, 5 and 6 coded on it)

Number of favorable event = 1(getting the number 6 only)

Probability of getting six

$$P(6) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

= 1/6

Example 5:

One card is drawn from a standard pack of 52. What is the probability that it is a card of Diamond?

Solution:

Total number of cards in a standard pack = 52

Number of Diamond cards in a standard pack = 13

Probability that the card drawn is a Diamond card

$$P(D) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

= 13/52

= 1/4

CHECK YOUR KNOWLEDGE B

1. Why is classical theory of probability also known as priori theory?

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2. Briefly explain the Subjective theory of probability.

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3. One dice is thrown. What is the probability of getting number 5?

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10.5 THEOREMS OF PROBABILITIES

There are two important theorems of probabilities, namely:

- The addition theorem, and
- The multiplication theorem

10.5.1 The addition theorem

According to this theorem if two events A and B are mutually exclusive the probability of the occurrence of either A or B will be the sum of the individual probability of A and B.

Mathematically,

$$P\,(A \text{ or } B) = P(A) + P(B)$$

Justification of theorem- Suppose in an experiment of n trials, A can happen in a_1 ways and B can happen in a_2 ways, then total number of ways in which either of the event can happen can be represented as $a_1 + a_2$. Then by definition, Probability of A or B can be calculated as

$$P\,(A \text{ or } B) = \frac{a_1+a_2}{n}$$

$$= \frac{a_1}{n} + \frac{a_2}{n}$$

Thus,

$$\mathbf{P(A \text{ or } B) = P(A) + P(B)}$$

The theorem can be extended to three or more mutually exclusive events.

Thus,

$$\mathbf{P\,(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)}$$

10.5.2 Illustration based on addition theorem

Example 6:

One card is drawn from a standard pack of 52. What is the probability that it is either a queen or a king?

Solution-

Total number of cards in a standard pack = 52

Number of King in a standard pack = 4

Number of Queen in a standard pack = 4

Probability that the card drawn is a King

P(K) = Number of favorable cases / Total number of equally likely cases

= 4/52

= 1/13

Probability that the card drawn is a Queen

P(Q) = Number of favorable cases / Total number of equally likely cases

= 4/52

= 1/13

Since, the event are mutually exclusive, the probability that the card drawn is either a king or a queen

P(K or Q) = P(K) + P(Q)

P(K or Q) = 1/13 + 1/13

= 2/13

Modification of addition rule:

In many cases it is possible that events are not mutually exclusive. It means that it is possible that both the events will occur. For example if we want to draw a card from a standard pack of cards and that should be a queen or a card of diamond , then it is possible that you will get a queen of diamond , means both the favorable events happened simultaneously. Thus in this case addition theorem can be stated as,

P (A or B) = P(A) + P(B) – P(A and B)

Here P(A and B) means probability of happening A and B togther.

Thus in the above example,

P (Queen or Diamond) = P(Queen) + P(Diamond) – P(Queen and Diamond)

= 4/52+ 13/52- (4/52 * 13/52)

= 4/52+ 13/52- 1/52

= 16/52

= 4/13

Similarly, in case of three events, the theorem can be stated as,

P(A or B or C)= P(A)+P(B)+P(C)-P(A and B)-P(B and C)-P(C and A)+P(ABC)

Example 7-: A student is called for GD/PI for admission against one vacant seat by three Business Schools, X, Y and Z. The call letters for GD/PI issued by X are 10, by Y 14 and by Z 15. Find the chance of the student getting finally selected to any one of the three Business Schools.

Solution:

Let P(A) = probability of the student getting finally selected to business school X,

and P(B) and P(C), the probability of the student getting finally selected to business schools and Z respectively.

P(A or B or C) = P(A)+P(B)+P(C) - P(A and B) - P(B and C) - P(C and A) +P(A and B and C)

= 1/10 + 1/14 + 1/15 – (1/10*1/14)–(1/14*1/15)–(1/15 * 1/10) + (1/10*1/14* 1/15)

= (210/2100 + 150/2100 + 140/2100) – (1/140) – (1/210) – (1/150) + 1/2100

= 500/2100 – (1/140) – (1/210) – (1/150) + 1/2100

= 501/2100 – 15/2100 – 10/2100 – 14/2100

= 462/2100

= 11/50

Thus, the chance of the student getting finally selected to any one of the business schools is 11/50.

Example 8:

Find the probability of getting

a) A King or jack.

b) Any card of spade, and

c) The Jack of diamond, from a standard pack of playing cards.

Solution:

a) No. of Kings in a standard pack of 52 cards = 4

No. of Jacks in a standard pack of 52 cards = 4

Thus, Probability of getting a King or Jack = (4+4)/52 = 8/52= 2/13

b) No. of Spade cards in a standard pack of 52 cards = 13

Thus, probability of getting a Spade card = 13/52 = ¼

c) No of Jacks in a standard pack of 52 cards = 4

No. of Diamond cards in a standard pack of 52 cards = 13

In a standard pack of 52, there is also a Jack of diamond,

Thus, P(A Jack or Diamond) = P(Jack)+P(Diamond)-P(Jack and Diamond)

= 4/52 + 13/52 – (4/52 * 13/52)

= 4/52+ 13/52- 1/52

= 16/52 = 4/13

Example 9:

From a bag containing 50 balls numbering 1 to 50, one ball is picked randomly. What is the probability that the ball picked is a multiple of

a) 7 or 9

b) 6 or 7

Solution:

a) Probability of getting a number multiple of 7 is

P (7, 14, 21, 28, 35, 42, 49) = 7/50

Probability of getting a number multiple of 9 is

P (9, 18, 27, 36, 45) = 5/50

Since events are mutually exclusive,

Hence P (multiple of 7 or 9) = 7/50 + 5/50

= 12/50 = 6/25

b) Probability of getting a number multiple of 6 is

P (6, 12, 18, 24, 30, 36, 42, 48) = 8/50

Probability of getting a number multiple of 7 is

P (7, 14, 21, 28, 35, 42, 49) = 7/50

Since 42 is a multiple of 6 as well as 7, thus getting a ball of no. 42 is common for the events hence the probability of getting a multiple of 6 or 7 is

P(6 or 7) = 8/50 + 7/50 – 1/50

= 14/50

= 7/25

10.5.3 The Multiplication Theorem

According to this theorem, multiplication of probabilities of two independent events A and B is equal to the product of their marginal probabilities. Symbolically:

P (A and B) = P(A) x P(B)

Justification of the theorem- Suppose in an experiment event A can happen successfully a_1 times out of n_1 ways and event B can happen successfully a_2 times out of n_2 ways. Then the total number of successful events in this case is $a_1 \times a_2$. Similarly, the total number of possible cases is $n_1 \times n_2$.

Then by definition the probability of the occurrence of both events is,

$$\frac{a_1 \times a_2}{n_1 \times n_2} = \frac{a_1}{n_1} \times \frac{a_2}{n_2}$$

P (A and B) = P(A) x P(B)

In the similar way the theorem can be extended to three or more events.

P (A, B and C) = P(A) x P(B) x P(C)

10.5.4 Illustration based on Multiplication Theorem

Example 10-

A bag contains 5white and 7black balls. A ball is drawn out, its colour is noted and then replaced in the bag. Then the ball is drawn again. What is the probability that (i) both the balls drawn were white, (ii) both were black, (iii) the first ball was white and the second black, (iv) the first balls was black and the second white, (v) both were of the same colour, (vi) both were of different colours.

Solution-

(i) Probability of first ball being white P(A) = 5/12

Probability of second ball also being white P(B) = 5/12

Since, both the events are independent. Hence the probability of both balls being white:

$$P(A \& B) = P(A) \times P(B)$$
$$= 5/12 \times 5/12$$
$$= 25/144$$

(ii) Probability of first ball being black P(A) = 7/12

Probability of second ball also being black P(B) = 7/12

Since, both the events are independent. Hence the probability of both balls being black:

$$P(A \& B) = P(A) \times P(B)$$
$$= 7/12 \times 7/12$$
$$= 49/144$$

(iii) Probability of first ball being white P(A) = 5/12

Probability of second ball being black P(B) = 7/12

Since, both the events are independent. Hence the probability of first ball being white and second being black:

$$P(A \& B) = P(A) \times P(B)$$
$$= 5/12 \times 7/12$$
$$= 35/144$$

(iv) Probability of first ball being black P(A) = 7/12

Probability of second ball being white P(B) = 5/12

Since, both the events are independent. Hence the probability of first ball being black and second being white:

$$P(A \& B) = P(A) \times P(B)$$

$$= 7/12 \times 5/12$$

$$= 35/144$$

Both are of same colour means either both are white or both are black:

Probability of both being white = 25/144

Probability of both being black = 49/144

Both the events are mutually exclusive. Hence, the probability of both balls of same colour = 25/144 + 49/144

$$= 74/144$$

$$= 37/72$$

Both balls of different colours means :

Either first ball is black and second white or first ball is white and second black:

$$= (7/12 \times 5/12) + (5/12 \times 7/12)$$

$$= 35/144 + 35/144$$

$$= 70/144$$

$$= 35/72$$

ple 11:

Find the probability in the following cases:

Two cards are drawn from a pack of cards in succession with replacement. Find the probability that both are aces.

From a pack of cards four cards are drawn with replacement. What is the probability that they all will be (a) Aces, (b) Club, (c) Red colour.

ion-

Probability of first card being an ace P(A) = 4/52

Probability of second card also being an ace P(B) = 4/52

Since, both the events are independent, i.e.

$$P(A \times B) = P(A) \times P(B)$$

$$= 4/52 \times 4/52$$

$$= 16/2701 = 1/169$$

(2) Total number of events in each draw = 52 (since the cards are being replaced)

(a) Probability of getting an ace = 4/52 = 1/13

Hence the probability of getting all the four cards of ace

$$= 1/13 \times 1/13 \times 1/13 \times 1/13$$

$$= 1/28,561$$

(b) Probability of getting the card of Club in each draw = 13/52 = ¼

Hence, probability of getting all the four cards of Club:

$$= 1/4 \times 1/4 \times 1/4 \times 1/4 \times 1/4$$

$$= 1/256$$

(c) Probability of getting the cards of red colour in each draw =

$$26/52 = 1/2$$

Hence, probability of getting all the four cards of red colour

$$= 1/2 \times 1/2 \times 1/2 \times 1/2 = 1/16$$

10.6 RANDOM VARIABLE FUNCTION

The outcome of any random experiment can be termed in the form of a variable. Such variable is known as random variable. For example- in two dices are thrown simultaneously, the 36 outcomes are possible whose value can vary from 2 to 12, if we represent all these values by a variable X, and then X will be called as random variable. Since the value of such variable totally depends upon chance hence it is also known as chance or stochastic variable.

A random variable can be of two types

- Discrete random variable
- Continuous random variable

If the value of any random variable is not following any pattern or a fixed interval it is called as discrete random variable. For example- number of patients a doctor has treated in a particular month, number of runs per match a batsman is scoring in a particular series.

On the other hand, a variable is called continuous random variable if the value of that variable is within a certain interval.

Symbolically, if $X_1, X_2, X_3, \ldots\ldots\ldots\ldots X_k$ are the set of values of a discrete variable X with their respective probabilities $p_1, p_2, p_3, \ldots\ldots\ldots\ldots p_k$, where $p_1+p_2+p_3+\ldots\ldots\ldots p_k = 1$, we can say it the function P(X) which has respective values $p_1, p_2, p_3, \ldots\ldots\ldots\ldots p_k$ for X= X1, X2, X3,..........X_k and this is called the probability function or frequency function of X.

It should be noted that a probability distribution is analogous to relative frequency distribution ,with probability replacing relative frequencies. Thus we can think of probability distribution as theoretical or ideal limiting forms of relative frequency distribution when the number of observation made is very large. For this reason, we can think of probability distribution as being distribution for population, whereas relative frequency distribution is distribution drawn from this population.

10.6.1 Illustration

Example 12:

A vendor has estimated from his past experience the probability of his selling breads in a day. These are as follows:

No. of breads sold in a day:	0	1	2	3	4	5
Probability:	0.02	0.20	0.30	0.35	0.10	0.03

Find the mean number of breads sold in a day.

Solution:

Mean no of breads sold= 0.02x0+0.20x1+0.30x2+0.35x3+0.10x4+ 0.03x5

= 0+0.20+0.60+1.05+0.40+0.15

=2.4

Hence, mean number of breads sold in a day is 3.

Example 13:

A book seller sells weekly magazine, which he purchases at a wholesale price of Rs. 35 each; and sells it a retail price of Rs. 40 each. An unsold copy after a week is a clear loss to the seller. The seller estimated the following probabilities for the number of copies demanded.

No. of copies:	50	51	52	53	54	55
Probabilities:	0.07	0.1	0.28	0.30	0.22	0.03

How many copies should he order, so that his expected profit will be maximum?

Solution-

Given that:

Cost per magazine = Rs. 35

Selling price per magazine = Rs. 40

Profit per magazine = 40-35 = Rs. 5

Expected profit= no. of copies sold x probability x profit per copy

Computation of expected profit

No. of copies	Probabilities	Profit per copy	Expected profit
50	0.07	5	17.5
51	0.1	5	25.5
52	0.28	5	72.8
53	0.30	5	79.5
54	0.22	5	59.4
55	0.03	5	8.25

53 copies will give the maximum expected profit of Rs. 79.50

10.7 MISCELLANEOUS EXAMPLES

Example 14-

A bag contains balls numbered from 1 to 100. A ball is picked. What is the probability that the picked ball contains:

a. An even number

b. The number 5 or multiple of 5

c. A number which is greater than 75

Solution-

a. Let A be the event of getting an even no.

Total even no. from 1 to 100 = 50

Total no. of events = 100

Thus, probability of getting an even no P(A). = 50/100

= 1/2

b. Let B be the event of getting 5 or multiple of 5

Total no. of favorable events from 1 to 100 = {5, 10, 15, ...100} = 20 events

Total no. of events = 100

Thus, probability of getting 5 or multiple of 5 P(B) = 20/100

= 1/5

c. Let C be the event of getting a number greater than 75

Total no. of favorable events from 1 to 100 = {76, 77, 78, ...100) = 25 events

Total no. of events = 100

Thus, probability of getting a number greater than 75 P(C). = 25/100 = 1/4

Example 15:

What will be the probability of 53 Mondays in a leap year selected randomly.

Solutions:

In a leap year there are 366 days . i.e. 52 weeks and 2 days. It means there will be 52 Monday for sure but remaining two days can be a set as:

1. Monday & Tuesday
2. Tuesday & Wednesday
3. Wednesday & Thursday
4. Thursday & Friday
5. Friday & Saturday
6. Saturday & Sunday
7. Sunday & Monday

Out of these 7 sets there are two such sets where Monday occurs

Thus,

The probability of getting 53 Mondays in a leap year = **2/7**

Example 16:

A coin is tossed two times. Find the probability of having

(a) 2 heads

(b) Exactly one head.

Solution :

(a) Probability of getting 2 heads in two throws of a coin= 1/2 x 1/2

= 1/4

(b) Probability of getting exactly one head means first head and second tail or first tail and then head, i.e.

= (1/2 x 1/2) + (1/2 x 1/2)

= 1/4 + 1/4

= 2/4 = 1/2

10.8 KEY WORDS

Classical approach: The probability of an event A is the ratio of the number of outcomes in favour of A to the number of all possible outcomes, provided experimental outcomes are equally likely to occur.

Events: Any subset of outcomes of an experiment.

Exhaustive events: Events that represent all possible experimental outcomes.

Mutually exclusive events: Events which cannot occur together or simultaneously.

Probability: A numerical measurement of the likelihood of occurrence of an uncertain event.

Random experiment: A process of obtaining information through observation or measurement of a phenomenon whose outcome is subject to change.

Random variable: The outcome of any random experiment can be termed in the form of an variable. Such variable is known as random variable.

Subjective approach: The probability of an event based on the personal beliefs of an individual.

10.9 SELF ASSESSMENT QUESTIONS

1. What do you understand by the term probability? Discuss its importance in business decision making?
2. Critically examine the definition of priori probability showing clearly the importance which the empirical version of probability makes over it.
3. Define independent and mutually exclusive events. Can two events be mutually exclusive and independent simultaneously? Support your answer with suitable example.
4. State the Multiplication Theorem of probability.

5. Three unbiased coins are tossed. What is the probability of obtaining:

   a. All heads
   b. Two heads
   c. One head
   d. At least one head
   e. At least two heads
   f. All tails.

Ans: a. 1/8 b. 3/8 c. 3/8 d. 7/8 e. 1/2 f. 1/8

6. A card is drawn from a pack of cards, find the probability that it is

   a. A black,
   b. A red
   c. A diamond
   d. An ace,
   e. A black ace,
   f. Ace of diamond
   g. A queen or a King
   h. Not a spade card.

Ans: a. 1/2 b. 1/2 c. 1/4 d. 1/13 e.1/26
f.1/52 g. 2/13 h. 3/4

7. What is the probability that a leap year, selected randomly, will contain 53 Sunday?

Ans. 2/7.

8. A bag contains 20 black and 30 white balls. What is the probability of drawing:

   a. A black ball
   b. A white ball

Ans; a. 2/5 b. 3/5

9. Two dice are tossed. What is the probability that the total is divisible by 3 or 4?

Ans: 5/9

10. If two dice are thrown simultaneously. What is the probability of throwing a total of 7?

Ans: 1/6

10.10 Further Readings

1. Levin , R.I. : Statistics for Management(PHI)
2. Gupta , S.P. & Gupta , M.P. : Business Statistics
3. Aczel & Sounderpandian : Complete Business Statistics (Tata McGraw Hill)
4. Lapin , Lawrance : Statistics for Modern Business Decisions (HBJ)
5. Srivastava , T.V. & Rego, S : Statistics for Management

UNIT-11 OPERATIONS WITH MATRIX & INTRODUCTION TO VECTOR

STRUCTURE

11.0 Learning Objectives

After studying this unit you will be able to understand the:

- fundamentals of matrix
- importance of matrix for business problem solving
- basic operations of matrix
- use of matrix in understanding the different tools of quantitative technique.

11.1 INTRODUCTION

Suppose A, B, C & D are the four competitors dealing in three products X, Y & Z. Let the market share of A in all three products be 20, 40 & 70 respectively. Similarly market shares of B, C & D in all three products can be 10, 20, 10; 40, 10, 12 and 30, 30, 08 respectively. The market shares in each product by all businessmen can be shown like this,

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccc} X & \quad Y \quad & Z \end{array} \\ \begin{pmatrix} 20 & 40 & 70 \\ 10 & 20 & 10 \\ 40 & 10 & 12 \\ 30 & 30 & 08 \end{pmatrix} \end{array}$$

In this arrangement row 1 show share of A in all three products X, Y, & Z similarly rows 2, 3, & 4 show shares of B, C, & D respectively. On the other hand, column shows total share distribution of product X by different businessmen. An arrangement of this type is known as matrix.

Matrix algebra is a very useful technique in solving linear equations; particularly the Simplex method is totally based on the concept of matrix. Many tools of quantitative techniques like duality, game theory optimum strategies etc. are totally based on this. Hence, it is very important to understand the concept of matrix.

11.2 DEFINITION

A matrix is a rectangular array of numbers arranged in rows and columns enclosed by a pair of brackets and subject to certain rules of presentation.

For example-

2 3 8 6 7 8 9

$$\begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix} \qquad \begin{pmatrix} 3 & 5 & 8 \\ 6 & 8 & 5 \\ 7 & 9 & 4 \end{pmatrix} \qquad \begin{pmatrix} 6 & 7 & 8 & 9 \\ 9 & 8 & 5 & 4 \\ 5 & 3 & 6 & 9 \\ 7 & 4 & 9 & 5 \end{pmatrix}$$

2x2 matrix 3x3 matrix 4x4 matrix

Followed by two suffixes, the first suffix indicates the row and second the column in which the elements appear.

Matrix A of size m x n is a rectangular array (table) of ordered numbers arranged into m rows and n columns represented by-

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots a_{2n} \\ & \\ a_{m1} & a_{m2} \ldots\ldots\ldots a_{mn} \end{bmatrix} = ((a_{ij}))\ m \times n$$

where, i=1, 2, 3,............m; j=1, 2, 3,................n and a_{ij} is the element in the i^{th} row and j^{th} column and referred as, $(i, j)^{th}$ element.

11.3 TYPES OF MATRIX

11.3.1 Square Matrix

A matrix in which number of rows is equal to number of column, is called square matrix. Thus a m x n matrix will be a square matrix if m = n.

Example:- $\begin{bmatrix} 3 & 8 & 9 \\ 6 & 5 & 2 \\ 8 & 7 & 2 \end{bmatrix}$ $\begin{bmatrix} 3 & 9 \\ 8 & 4 \end{bmatrix}$

11.3.2 Diagonal Matrix

A square matrix is called a diagonal matrix in which all elements except those in the leading diagonal are zero.

Example:- $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ $\begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}$

11.3.3 Scalar Matrix

A diagonal matrix is called scalar matrix if its diagonal elements are equal.

Example:-

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

11.3.4 Unit Matrix

A diagonal matrix is called unit matrix if its diagonal matrix are unity (one).

Example:-

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$$

11.3.5 Null Matrix or Zero Matrix

A matrix is called null or zero matrix if it's all elements are zero.

Example-

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

11.3.6 Row matrix & Column matrix

A matrix consisting of a single row is called a row matrix or a row vector.

Example:- $(3 \quad 7 \quad 6)$ $\qquad$ $(6 \quad 9)$

A matrix consisting of a single column is called column matrix or column vector.

Example:-

$$\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \qquad \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

11.3.7 Sub Matrix

A matrix obtained by deleting some rows or columns or both of a matrix is known as sub matrix of that matrix.

Example:- Let

$$A=\begin{pmatrix} 3 & 5 & 4 \\ 2 & 8 & 6 \\ 4 & 5 & 7 \end{pmatrix}$$

If we delete third row and third column of matrix A, then sub-matrix of A is,

$$\begin{pmatrix} 3 & 5 \\ 2 & 8 \end{pmatrix}$$

11.3.8 Transpose of a Matrix

If we interchange all rows by columns of the matrix A, the new obtained matrix is said to be transpose of matrix A and denoted by A^T or A′.

Example- Let

$$\begin{pmatrix} 3 & 5 & 4 \\ 2 & 8 & 6 \\ 4 & 5 & 7 \end{pmatrix}$$

If we interchange all rows by columns then transpose of A is,

$$A^T=\begin{pmatrix} 3 & 2 & 4 \\ 5 & 8 & 5 \\ 4 & 6 & 7 \end{pmatrix}$$

11.3.9 Symmetric Matrix

A symmetric matrix is a special kind of matrix for which $a_{ij}=a_{ji}$ for all i and j. In other words, in a symmetric matrix elements placed on either side of the leading diagonal are equal.

Example:- $\begin{pmatrix} 2 & 6 & 9 \\ 6 & 3 & 4 \\ 9 & 4 & 7 \end{pmatrix}$ $\begin{pmatrix} a & p & q \\ p & b & r \\ q & r & c \end{pmatrix}$

11.3.10 Skew Symmetric Matrix

A square matrix is called skew symmetric matrix if $a_{ij} = -a_{ji}$

Example:-

$$\begin{pmatrix} 2 & 6 & 9 \\ -6 & 3 & 4 \\ -9 & -4 & 7 \end{pmatrix} \qquad \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix}$$

11.4 MATRIX OPERATIONS

Mathematical operations like addition, subtraction, multiplicaiion etc. can be performed on matrix. Let's understand it.

11.4.1 Addition of matrix

Two matrix A and B can be added together to get the addition in the form of A+B. While doing so, we must be careful that the order of both the matrix must be the same. It means that number of elements in each row and column in both the matrix must be equal.

Example:- Find A+B, if

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 8 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 5 & 7 \\ 6 & 4 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 2+5 & 4+7 \\ 3+6 & 8+4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 11 \\ 9 & 12 \end{pmatrix}$$

This addition is also known as element wise addition.

11.4.2 Subtraction of matrix

Similar to the addition, subtraction operation can also be performed.

Example:-

Find B-A, if $A=\begin{pmatrix}2 & 4\\ 3 & 8\end{pmatrix}$ *and* $B=\begin{pmatrix}5 & 7\\ 6 & 4\end{pmatrix}$

$$B-A=\begin{pmatrix}5-2 & 7-4\\ 6-3 & 4-8\end{pmatrix}$$

$$=\begin{pmatrix}3 & 3\\ 3 & -4\end{pmatrix}$$

11.4.3 Scalar Product

Multiplication of matrix with a scalar quantity is known as scalar product. Suppose matrix A is multiplied by a scalar quantity m then product of both will be represented by mA.

Example:- If

$$A=\begin{pmatrix}2 & 4\\ 3 & 8\end{pmatrix} \; Find 3A$$

$$3A = 3\begin{pmatrix}2 & 4\\ 3 & 8\end{pmatrix}$$

$$=\begin{pmatrix}3\times 2 & 3\times 4\\ 3\times 3 & 3\times 8\end{pmatrix}$$

$$=\begin{pmatrix}6 & 12\\ 9 & 24\end{pmatrix}$$

11.4.4 Multiplication of Matrix

If A is an m x n order matrix and B is an n x p order matrix then the multiplication of A & B can be represented as AB or with a different symbol, say, C of order m x p.

Example- If

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 8 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & 7 \\ 6 & 4 \end{pmatrix} \text{ Find } AB$$

$$AB = \begin{pmatrix} 2\times5+4\times6 & 2\times7+4\times4 \\ 3\times5+8\times6 & 3\times7+8\times4 \end{pmatrix}$$

$$= \begin{pmatrix} 34 & 30 \\ 63 & 53 \end{pmatrix}$$

11.5 PROPERTIES OF MATRIX

Matrix follows some properties that are as follows:

11.5.1 Associative Law

Matrix multiplication is associative in nature, i.e. if A, B, & C are three matrix such that A(BC) and (AB)C are defined, then

$$A(BC) = (AB)C$$

11.5.2 Distributive Law

Matrix multiplication is distributive over addition, i.e.

$$A(B+C) = (AB) + (AC)$$

Provided multiplication and addition of matrix are defined.

11.5.3 Multiplication of matrix is not commutative

Multiplication of matrix is not commutative in general, i.e if A & B are two matrix such that both AB & BA are defined, then it is not compulsory that AB is equal to BA.

11.6 DETERMINANT OF MATRIX

The determinant of a matrix A is a number and is denoted by $|A|$

The determinant of the 2x2 matrix

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ is given by}$$

$$|A| = a_{11} \times a_{22} - a_{12} \times a_{21}$$

The determinant of 3x3 matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$|B| = b_{11}\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} - b_{12}\begin{vmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{vmatrix} + b_{13}\begin{vmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{vmatrix}$$

$|B|=b_{11}(b_{22}xb_{33}-b_{23}xb_{32})-b_{12}(b_{21}xb_{33}-b_{23}xb_{31})+$
$b_{13}(b_{21}xb_{32}-b_{22}b_{31})$

Example-

$$For\ A=\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}\ Find\ |A|$$

$$|A| = 1\begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}$$

$$= 1(5\times9-8\times6) - 4(2\times9-8\times3) + 7(2\times6-5\times3)$$
$$=(-3)-(-24)+(-21)$$
$$=0$$

11.7 MINORS & COFACTOR

In a square matrix $A = |a_{ij}|$, the minor of the element a_{ij} is defined as the determinant of the sub-matrix. In other words minor of a_{ij} is formed by deleting i^{th} row and j^{th} column of that matrix.

Cofactor of an element is defined as negative of minor of that element.

Example :- find minor and cofactor of element 1,5 & 9

$$For\ A=\begin{pmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{pmatrix}$$

Minor of element 1 = $\begin{vmatrix}5 & 8\\ 6 & 9\end{vmatrix}$

$= 5 \times 9 - 8 \times 6$

$= -3$

= -12

Cofactor of element 5 = -(-12)

= 12

Minor of element 9 = 1 4

2 5

= 1x5 – 4x2

= -3

Cofactor of element 9 = -(-3)

= 3

11.8 ADJOINT MATRIX

The adjoint of a matrix is the transpose of the matrix of cofactors. In other words, the adjoint of a square matrix A is obtained by taking the transpose of the cofactor matrix of A and is denoted by Adj A. i.e.

Adj A = (cof A)^T

Example:- Find Ad joint of A, if A= $\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}$

Cofactor of element 1= -4

Cofactor of element -3= 2

Cofactor of element-2= 3

Cofactor of element4= -1

Matrix of cofactors $\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$

Transpose of matrix $\begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}$

Thus, adjoint of matrix A = $\begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}$

11.9 INVERSE OF MATRIX

Definition:- For a non-singular square matrix A, if there exists a matrix B such that AB = BA = I, then B is called the inverse of A and is denoted by A^{-1}.

The inverse of a square matrix A is obtained by taking the product of 1/|A| and Adj A.

i.e.

$$A^{-1} = \frac{1}{|A|} \times \text{Adj } A$$

Example:- Find Inverse of A, if A= $\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$

Determinant of A = |A| = (1x4) – (-2x-3)

= 4-6= -2

Now, for adjoint of A

Cofactor of element 1= -4

Cofactor of element -3= 2

Cofactor of element-2= 3

Cofactor of element4= -1

Matrix of cofactors $\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$

Transpose of matrix $\begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}$

Thus, adjoint of matrix A= $\begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}$

Thus, inverse of A= $A^{-1} = \frac{1}{|A|} \times \text{Adj A}$

$$= \frac{1}{-2} \times \begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4/-2 & 3/-2 \\ 2/-2 & -1/-2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1.5 \\ -1 & 0.5 \end{pmatrix}$$

11.10 RANK OF MATRIX

Definition: A positive integer r is said to be the rank of the matrix A, and denoted by ñ(A) if,

1. Matrix A possesses at least one r-rowed minor which is not zero, and

2. Matrix A does not possess any non zero (r+1) rowed minor.

For example:- Let a matrix A= $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 9 \end{pmatrix}$

Here, matrix A possesses a largest minor of 2x2 order (if at least one 2x2 minor is not equal to zero) which may be obtained by deleting any of the column. Thus minor can be obtained as·

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \qquad \begin{pmatrix} 3 & 4 \\ 6 & 9 \end{pmatrix} \qquad \begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix}$$

$=6-6$ $\qquad$ $=27-24$ $\qquad$ $=9-8$

$=0$ $\qquad$ $=3$ $\qquad$ $=1$

Since, at least one minor is non zero; hence rank of matrix A is 2.

Two matrix A & B is said to be equivalent, if any and only if, ñ(A) = ñ(B) and denoted by A~B.

If the matrix A* is obtained from matrix A by suitable row/column operations, then also A* is known as equivalent matrix of A, and denoted by A*~A.

11.11 ILLUSTRATIONS

Example 1:-

Find A+B and A-B if

$$A=\begin{bmatrix} 6 & 8 & 5 \\ 7 & 6 & 4 \\ 6 & 4 & 9 \end{bmatrix} \quad B=\begin{bmatrix} 7 & 6 & 4 \\ 3 & 4 & 6 \\ 1 & 3 & 2 \end{bmatrix}$$

Solution –

$$A+B \begin{bmatrix} 6+7 & 8+6 & 5+4 \\ 7\;\;3 & 6+4 & 4+6 \\ 6\;\;1 & 4+3 & 9+2 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 14 & 9 \\ 10 & 10 & 10 \\ 7 & 7 & 11 \end{bmatrix}$$

$$AND\;,\; A-B = \begin{bmatrix} 6-7 & 8-6 & 5-4 \\ 7-3 & 6-4 & 4-6 \\ 6-1 & 4-3 & 9-2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 1 \\ 4 & 2 & 2 \\ 5 & 1 & 7 \end{bmatrix}$$

Example 2:-

Find A+2B-C if,

$$A=\begin{pmatrix} 6 & 8 & 5 \\ 7 & 6 & 4 \\ 6 & 4 & 9 \end{pmatrix} \quad B=\begin{pmatrix} 7 & 6 & 4 \\ 3 & 4 & 6 \\ 1 & 3 & 2 \end{pmatrix} \quad C=\begin{pmatrix} 3 & 4 & 3 \\ 1 & 5 & 3 \\ 6 & 4 & 2 \end{pmatrix}$$

Solution –

$$A+2B-C=\begin{pmatrix} 6 & 8 & 5 \\ 7 & 6 & 4 \\ 6 & 4 & 9 \end{pmatrix} +2\begin{pmatrix} 7 & 6 & 4 \\ 3 & 4 & 6 \\ 1 & 3 & 2 \end{pmatrix} -\begin{pmatrix} 3 & 1 & 6 \\ 1 & 5 & 3 \\ 6 & 4 & 2 \end{pmatrix}$$

$$A+2B-C=\begin{pmatrix} 6 & 8 & 5 \\ 7 & 6 & 4 \\ 6 & 4 & 9 \end{pmatrix} +\begin{pmatrix} 14 & 12 & 8 \\ 6 & 8 & 12 \\ 2 & 6 & 4 \end{pmatrix} -\begin{pmatrix} 3 & 1 & 6 \\ 1 & 5 & 3 \\ 6 & 4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 6+14-3 & 8+12-1 & 5+8-6 \\ 7+61 & 6+8-5 & 4+12-3 \\ 6+2-6 & 4+6-4 & 9+4-2 \end{pmatrix}$$

$$=\begin{pmatrix} 17 & 19 & 7 \\ 12 & 9 & 13 \\ 2 & 6 & 11 \end{pmatrix}$$

Example 3:-

If $A=\begin{pmatrix}6 & 8 & 5\\ 7 & 6 & 4\\ 6 & 4 & 9\end{pmatrix}$ $B=\begin{pmatrix}7 & 6 & 4\\ 3 & 4 & 6\\ 1 & 3 & 2\end{pmatrix}$ Find the matrix X so that A+2B-X=0

Solution-

$$A+2B-X=0$$

$$X=A+2B$$

$$A=\begin{pmatrix}6 & 8 & 5\\ 7 & 6 & 4\\ 6 & 4 & 9\end{pmatrix}+2\begin{pmatrix}7 & 6 & 4\\ 3 & 4 & 6\\ 1 & 3 & 2\end{pmatrix}$$

$$=\begin{pmatrix}6 & 8 & 5\\ 7 & 6 & 4\\ 6 & 4 & 9\end{pmatrix}+\begin{pmatrix}14 & 12 & 8\\ 6 & 8 & 12\\ 2 & 6 & 4\end{pmatrix}$$

$$X=\begin{pmatrix}6+14 & 8+12 & 5+8\\ 7+6 & 6+8 & 4+12\\ 6+2 & 4+6 & 9+4\end{pmatrix}=\begin{pmatrix}20 & 20 & 13\\ 13 & 14 & 16\\ 8 & 10 & 13\end{pmatrix}$$

Example 4:-

$$If\ A=\begin{pmatrix}2 & 3\\ 4 & 1\end{pmatrix}\ Find\ A^2\ and\ A^3$$

Solution- $A^2 = A \times A$

$$Thus=\begin{pmatrix}2 & 3\\ 4 & 1\end{pmatrix}x\begin{pmatrix}2 & 3\\ 4 & 1\end{pmatrix}$$

$$= \begin{pmatrix} 2x2+3x4 & 2x3+3x1 \\ 4x2+1x4 & 4x3+1x1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix}$$

Similarly, $A^3 = A^2 \times A$

$$= \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16x2+9x4 & 16x3+9x1 \\ 12x2+13x4 & 12x3+13x1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 68 & 57 \\ 76 & 49 \end{pmatrix}$$

Example 5:-

$$If \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad showthat\ A^2 - 31 = 2A$$

Solution –

$$2A = 2 \times \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$$

$$And,\ A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence,

$$31 = 3 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Now, $A^2 - 31 =$

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5-3 & 4-0 \\ 4-0 & 5-3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$$

$= \quad 2A \quad$ Hence proved

Example 6:- A book seller supplies 100 books of Management Accounting, 120 books of Statistics, and 150 books of Economics to a College. The prices per book of these books are Rs.150, Rs.200 and Rs.300 respectively. Find the total amount of the bill furnished by the book seller.

Solution-

The no. of books can be written in row matrix and price in column matrix as:

Order matrix= $\begin{pmatrix} 100 & 120 & 150 \end{pmatrix}$

Price matrix= $\begin{pmatrix} 150 \\ 200 \\ 300 \end{pmatrix}$

Now total bill= order matrix x price matrix

$$\begin{pmatrix} 100 & 120 & 150 \end{pmatrix} \times \begin{pmatrix} 150 \\ 200 \\ 300 \end{pmatrix}$$

= 100x150 + 120x200 + 150x300

= 15000 + 24000 + 45000

= Rs. 84000

11.12 INTRODUCTION TO VECTORS

The study of vector is required to understand some basics of Mathematics; as these concepts are essential for the study of Operations Research.

In basic mathematics it is explained that a point is the smallest unit of geometry, and it can be explained it terms of an origin and co-ordinate axes. If two dimensional space is represented by a set of real numbers on axes; like (2,3), (-3,5), (4,-8), (-1,-3) etc., here each number is having a unique significance. First number shows distance from origin on X axis whereas the second number shows distance of point from origin on Y axis.

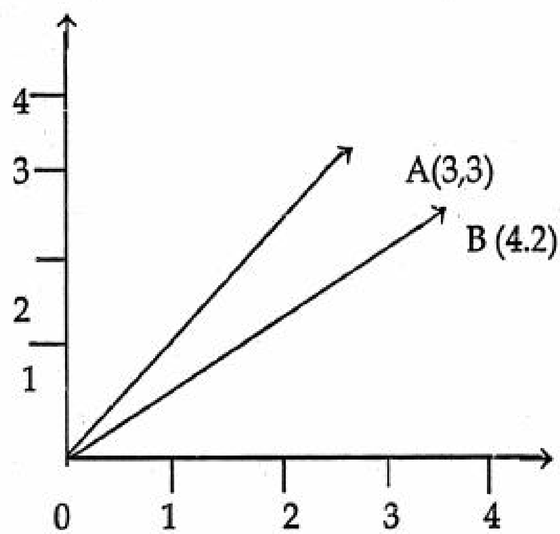


Figure : vector in 2 spaces

From this figure it is clear that a vector is having direction and magnitude. Direction means from origin to the point and magnitude means length of the line.

Similarly concept of vector can be understand in n-dimensional space .

11.12.1 Definition

A vector in n-space is an ordered set of n-real numbers. For example vector A $(a_1, a_2, \ldots\ldots a_n)$ is a vector of n element, where $a_1, a_2, \ldots\ldots a_n$ are known as components or elements of vector.Two vectors are said to be equal if all its components are equal. For example vector $A=(a_1, a_2, \ldots\ldots a_n)$ and vector $B=(b_1, b_2, \ldots\ldots b_n)$ are said to be equal if; $a_1=b_1, a_2=b_2, \ldots\ldots a_n=b_n$

Thus, we can say that,

$$A=B \qquad \text{or} \qquad B=A$$

11.12.2 Addition & Subtraction of Vector

Two vectors $A=(a_1, a_2, \ldots\ldots a_n)$ and $B=(b_1, b_2, \ldots\ldots b_n)$ can be added or subtracted like that,

$$A+B=(a_1+b_1, a_2+b_2, \ldots\ldots a_n+b_n)$$

$$A-B=(a_1-b_1, a_2-b_2, \ldots\ldots a_n-b_n)$$

11.12.3 Multiplication

Multiplication of a vector by scalar

A vector $A=(a_1, a_2, \ldots\ldots a_n)$ can be multiplied by a scalar μ quantity in the following way.

$$\mu A = \mu(a_1, a_2, \ldots\ldots a_n)$$

$$= \mu a_1, \mu a_2, \ldots\ldots \mu a_n$$

Scalar – product

Multiplication of two vectors is known as scalar product. Since the result of this multiplication is always a scalar quantity hence, it is known as scalar product. This can be done in the following way:

If vector A= (2, 4, 6, 8) and vector B=(1, 3, 5, 7)

Then, AB = (2, 4, 6, 8) x (1, 3, 5, 7)

= 2 + 12 + 30 + 56

= 100

11.12.4 Properties of vectors

1. Vectors follow addition law.

i.e., for three vectors A,B & C

i) A + B = B + A

ii) A + (B + C) = (A + B) + C

2. Vectors follow multiplication & distribution law.

i.e, for two vectors A & B and a scalar quantity μ

i) $\mu(A + B) = \mu A + \mu B$

ii) A B = B A

iii) A (B + C) = A B + A C

11.12.5 Distance between two points

Distance between two points is also known as length of vector, and can be calculated as;

Say for two vectors A= $(a_1, a_2, \ldots\ldots\ldots a_n)$ and B= $(b_1, b_2, \ldots\ldots\ldots b_n)$ the distance or length can be;

$|A-B| = " (a_1-b_1)^2 + (a_2-b_2)^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + (a_n-b_n)^2$

11.12.6 Special types of vector

Some special name has been given to the commonly used vectors They are:

1. Null Vector:

A vector whose all elements are zero are called as null vector.

0=(0, 0, 0,.......0)

2. Sum Vector:

A vector, whose all elements are 1 are called as sum vector.

1=(1, 1, 1.........1)

3. Unit Vector:

   A vector whose i^{th} element is 1 and rest all other element are 0 is known as unit vector.

   e_1 = (1, 0, 0,.....0)

   e_2 = (0, 1, 0,.....0), similarly a unit vector whose i^{th} element is 1 will be denoted by e_n.

11.13 KEY WORDS

Adjoint : The adjoint of a matrix is the transpose of the matrix of cofactors.

Cofactor: Cofactor of an element is defined as negative of minor of that element.

Length of vector: Distance between two points is also known as length of vector.

Matrix: A matrix is a rectangular array of numbers arranged in rows and column enclosed by a pair of brackets.

Minor: determinant of the sub-matrix is known as minor

Null Vector: A vector whose all elements are zero and called as null vector: 0=(0, 0, 0,.......0)

Scalar Product of matrix: Multiplication of matrix with a scalar quantity is known as scalar product.

Scalar product of vectors: Multiplication of two vectors is known as scalar product.

Sum Vector: A vector, whose all elements are 1 is called as sum vector : 1=(1, 1, 1.........1)

Unit Vector: A vector whose i^{th} element is 1 and rest all other element are 0 is known as unit vector.

Vector: A vector in n-space is an ordered set of n-real numbers.

11.14 Self Assessment Questions

1. Find P + Q : If

a) P = (2 4 6) and Q = (8 7 2)

b) $P=\begin{pmatrix}6 & 8 & 5\\7 & 5 & 2\end{pmatrix}$ *and* $Q=\begin{pmatrix}4 & 7 & 2\\9 & 6 & 3\end{pmatrix}$

c) $P=\begin{pmatrix}5 & 8 & 3\\7 & 9 & 4\\6 & 4 & 7\end{pmatrix}$ *and* $Q=\begin{pmatrix}7 & 4 & 2\\9 & 6 & 4\\8 & 4 & 2\end{pmatrix}$

2. Find 3A+2B-C : if

a) $A=\begin{pmatrix}2 & 4 & 6\\8 & 6 & 3\\8 & 6 & 4\end{pmatrix}$ $B=\begin{pmatrix}6 & 8 & 4\\9 & 4 & 1\\2 & 4 & 7\end{pmatrix}$ $\&C=\begin{pmatrix}5 & 9 & 6\\8 & 5 & 2\\8 & 5 & 3\end{pmatrix}$

b) $A=\begin{pmatrix}7 & 6 & -8\\-3 & -7 & 9\\0 & -6 & 9\end{pmatrix}$ $B=\begin{pmatrix}-8 & -9 & -5\\-7 & -6 & -5\\-6 & 0 & 1\end{pmatrix}$ $\&C=\begin{pmatrix}5 & 6 & 8\\0 & 0 & 1\\6 & -8 & -9\end{pmatrix}$

3. If $\begin{pmatrix}x & 7\\0 & 9\\2 & 0\end{pmatrix}+\begin{pmatrix}8 & 9\\8 & 6\\7 & y\end{pmatrix}-\begin{pmatrix}5 & 16\\8 & 15\\9 & 5\end{pmatrix}$ Find the value of x & y

Such that, a) A-X=3B And b) A+2Y=4B, then find the value of X & Y.

5. Calculate A^2 and AB

If

$$A=\begin{pmatrix}2 & 3 & 4\\1 & 2 & 3\\1 & -1 & 1\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 3 & 2\\-1 & 0 & 1\\0 & 0 & 1\end{pmatrix}$$

6. Three firms A, B and C supplied 40, 35 and 25 truckloads of stones and 10, 5 and 8 truck loads of sand respectively to a contractor. If the cost of stone and sand are Rs. 1,200 and Rs. 500 per truck load respectively, find the total amount paid by the contractor to each of these firms by using matrix method.

7. The annual sale volumes of three products X, Y, Z whose sales prices per unit are Rs. 3.50, Rs. 2.75 and Rs. 1.50 respectively, in two different markets A and B are shown below:

Market	Product		
	X	Y	Z
A	6000	9000	13000
B	12000	6000	17000

Find the total revenue in each market with the help of matrix.

11.15 Further Readings

1. Sharma, S.D. : Operations Research (Kedarnath Ramnath publishers)

2. Rajagopalan, S.P. & Sattanathan, R. : Business Mathematics (Tata McGraw Hill)

3. Gupta, K.L. & Agarwal, R. : Business Mathematics (Navyug Sahitya Sadan)

UNIT 12 DECISION TREE ANALYSIS

Structure

12.0 Learning Objectives

After studying this unit, you will be able to understand the:

- Amount of uncertainty that is involved in decision making;
- Usefulness of multistage decision making situations; and
- Sequential decision making solutions.

12.1 INTRODUCTION

The problems and situations we discussed in Unit-I refers to single-stage decision making, where the impact of one decision on further decision is supposed to be neutral. But in many situations, the decision taken by the decision maker puts some impact on further decisions, or in other words, one decision is always based upon the outcome of previous decision. Thus the problem becomes a sequence of other decision. Such situations are known as multi-stage decision problem.

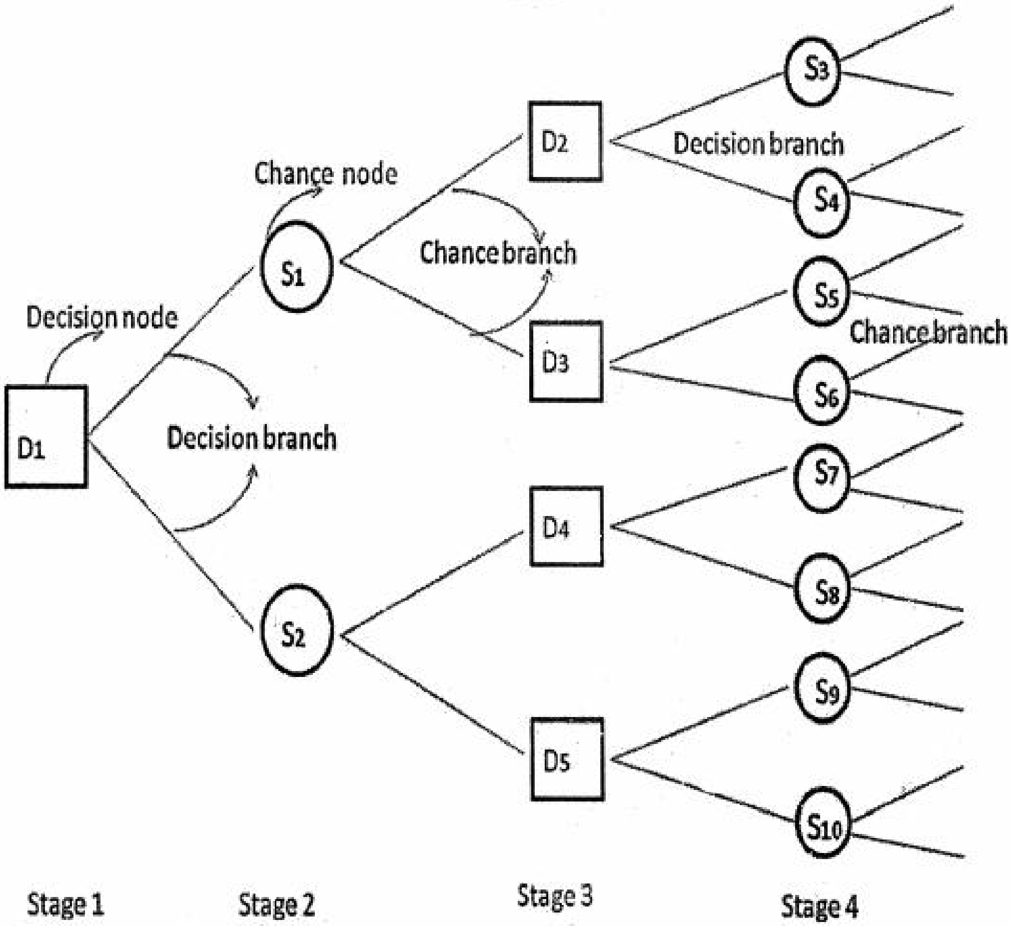
The decision making problems discussed in Unit – I are referred to single-stage decision problems because the states of nature, pay-offs, the courses of action and probability distribution are not subject to change under the assumption that no new information is sought and time also does not change any basic component of decision making environment. But in many situations the decision maker needs to revise his previous decisions on getting information and make a sequence of other decisions. In other words, the outcome of one decision works as an input for the next decision. Hence a chance event occurs which in turn influences the next decision and make a sequence of decisions. Such situations are known as multi-stage decision problems or sequential decision problems.

For example, in the process of new product development an important stage is 'Test Marketing'. For such marketing, one could go for either intensive or normal test marketing. The outcome of any of these can be favorable, fair or poor. On the basis of this result one will take decision to either market the product or drop the idea or rather re-design the product and again go for test marketing. Given the decision, there will be an outcome leading to another decision and so on. For this purpose, the technique we popularly use is Decision Tree or Decision Flow Diagram.

12.2 DECISION TREE

A decision tree is a graphic representation of the sequences of action-event combination available to the decision maker. In this, decision problem, alternative courses of action, states of nature and the likely outcomes of alternatives are diagrammatically or graphically depicted as if they are branches and sub-branches of a tree, and thus it is known as decision tree.

As the name implies, a decision tree consists of network of nodes, branches, probability estimate and payoffs.



12.3 IMPORTANT TERMS

12.3.1 Node

If we compare a decision tree with a natural tree, we can say a node is the trunk of the tree. Nodes are of two types; decision/choice node and chance node. Decision nodes are represented by a square ' ' and it shows an action or a decision whereas chance nodes are represented by a circle 'O' and it shows point of uncertainty. It is important to note that sum of probabilities of occurrence of events at a node is always one, and at any of the chance node, the expected payoffs of different alternatives at that particular tree are always added to obtain single EMV value. At decision node any one decision is chosen out of different options present on the basis of their EMV values.

12.3.2 Branches

In a decision tree various strategies act as branches. Branches connect various nodes. There are two types of branches: decision branch or chance branch. The decision branch indicates a course of action or strategy that can be selected at this decision point, whereas the chance branch indicates the state of nature of a set of chance factors. Any branch that makes the end of decision tree (i.e. no further decision or chance node after that point) is called a terminal branch; though it can indicate either a course of action or a chance outcome.

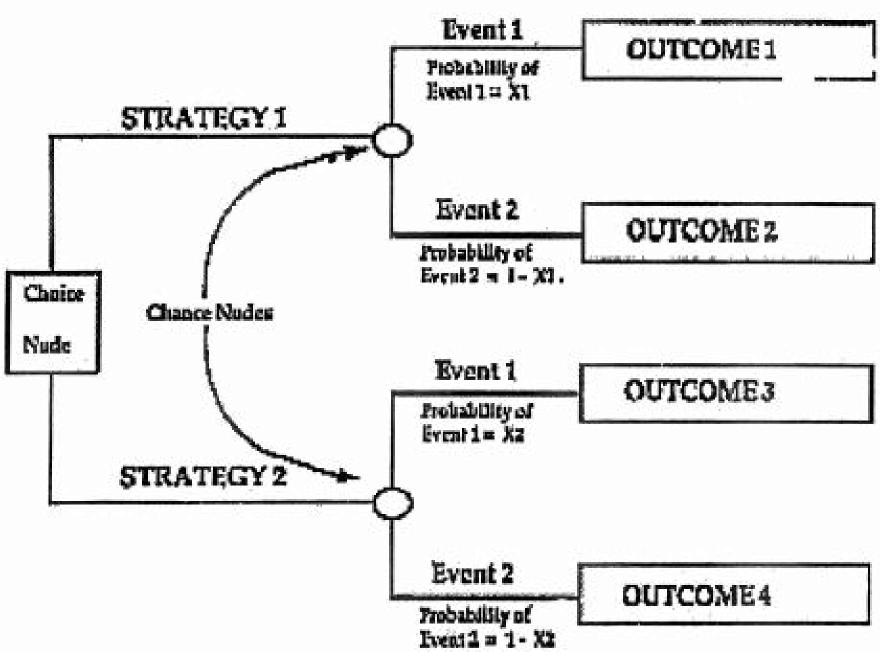
12.3.3 Probability estimate

Probabilities are the likelihood that the chance outcome will assume the value assigned to the given branches.

12.3.4 Payoffs

As explained in unit-I, payoff is the effectiveness associated with specified combination of a course of action and state of nature. The payoff can be positive or negative, and it can be associated either with decision or the chance branches.

Choice Nodes



A decision problem is solved using the roll back technique. In this technique, we start from the right hand side and rolling backward till the first node for each of the possible decisions.

12.5 METHOD OF DRAWING DECISION TREE

To draw a decision tree one must follow the following steps-

Step 1- Clearly identify all decision nodes, chance nodes and alternative course of action before you start drawing tree.

Step 2- Arrange all nodes and branches in an appropriate order. Remember decision node comes first and then followed by alternative actions and chance nodes.

Step 3- Assign probability to each course of action, and find out payoff for each of them.

Step 4- Using Roll-Back Technique calculate maximum EMV, and repeat the method till the first decision point is reached.

Step 5- Mark the sequence of action of courses adopted from beginning to end.

CHECK YOUR KNOWLEDGE A

1. Write a short note on decision tree.

___

___

___

2. Explain the following terms

a) Nodes

___

___

___

b) Branch

___

___

___

c) Probability estimates

12.6 ILLUSTRATIONS

Example 1-

A company XYZ Ltd. has to decide whether to advertise or not to advertise its product. If company advertises the product and customers give the positive response company will gain Rs.5lakhs, but if customers do not accept the product the company will lose Rs.2lakh. Company expects that there is 0.6 probability of getting success if it advertises. No profit or loss is attached to its decision not to advertise.

a) Construct a decision tree to help analyze this problem.

b) What action the marketing manager should take?

Solution:

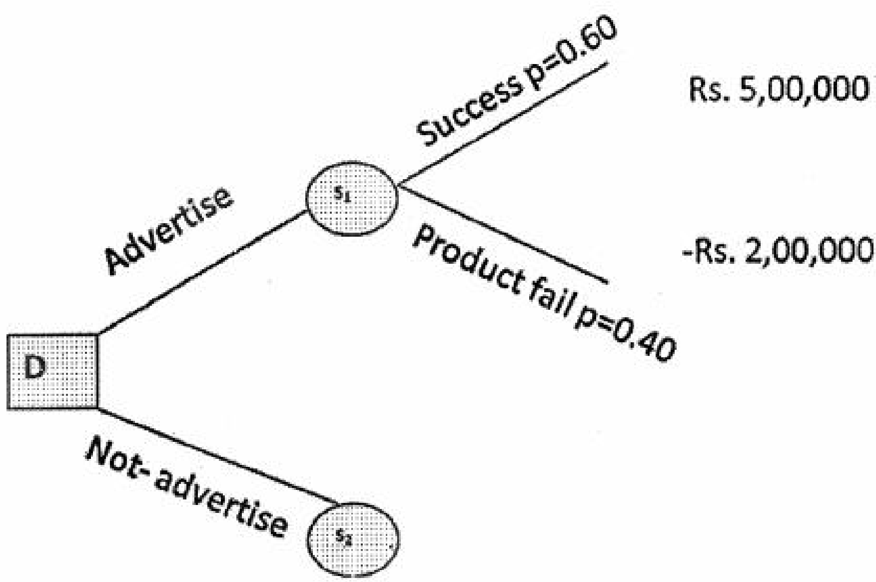


Fig: decision tree for Marketing problem

EMV = Rs.(5lakhs x 0.6) + Rs.(-2lakh x 0.4)

= Rs. 3lakh – Rs.80,000

= Rs. 2,20,000

Conclusion:

This analysis shows EMV under uncertainty is Rs. 2,20,000. Thus company should advertise the product.

Working note:

For the preparation of decision tree

First the company needs to take decision whether to advertise or not advertise the product. Thus a decision node D is formed. Then two decision branches are formed first for 'advertising' and second for 'not advertising'. At first decision branch one chance node is required as at this point company has two uncertainties (either success or failure). Now from this chance node two chance branches are to be constructed, showing probability of success and failure.

To calculate EMV

To calculate EMV roll-back technique has been used. It means that we start from extreme right side of the diagram and start multiplying probability by its corresponding outcome. At every chance node we add all EMV for all chance branches that will become the total EMV for that particular chance node. We continue this process till last node.

Example 2:

A glass factory specialized in crystal is developing a substantial back-log and the firm's management is considering three courses of action: arrange for sub contracting (S1), begin overtime production (S2), and construct new facilities (S3). The correct choice depends largely upon future demand which may be low, medium, or high. By consensus, management ranks the respective probabilities as 0.10, 0.50 and 0.40. A cost analysis reveals the effects upon the profits in Lakh Rs. that is shown in the table below:

Demand (Lakh Rs.)	Proba-bility	Effects on profit of Courses of action		
		Subcontrac-ting S1	Begin overtime S2	Construct new facilities S3
Low (L)	0.10	10	-20	-150
Medium (M)	0.50	50	60	20
High (H)	0.40	50	100	200

Show this decision situation in the form of a decision tree and indicate the most preferred decision and the corresponding expected value.

Solution-

Calculation for EMV

At node S_1 = 0.10x10 + 0.50x50 + 0.40x50 = Rs.46 Lakh

At node S_2 = 0.10x(-20) + 0.50x60 + 0.40x100 = Rs.68 Lakh

At node S_3 = 0.10x(-150) + 0.50x20 + 0.40x200= Rs.75 Lakh

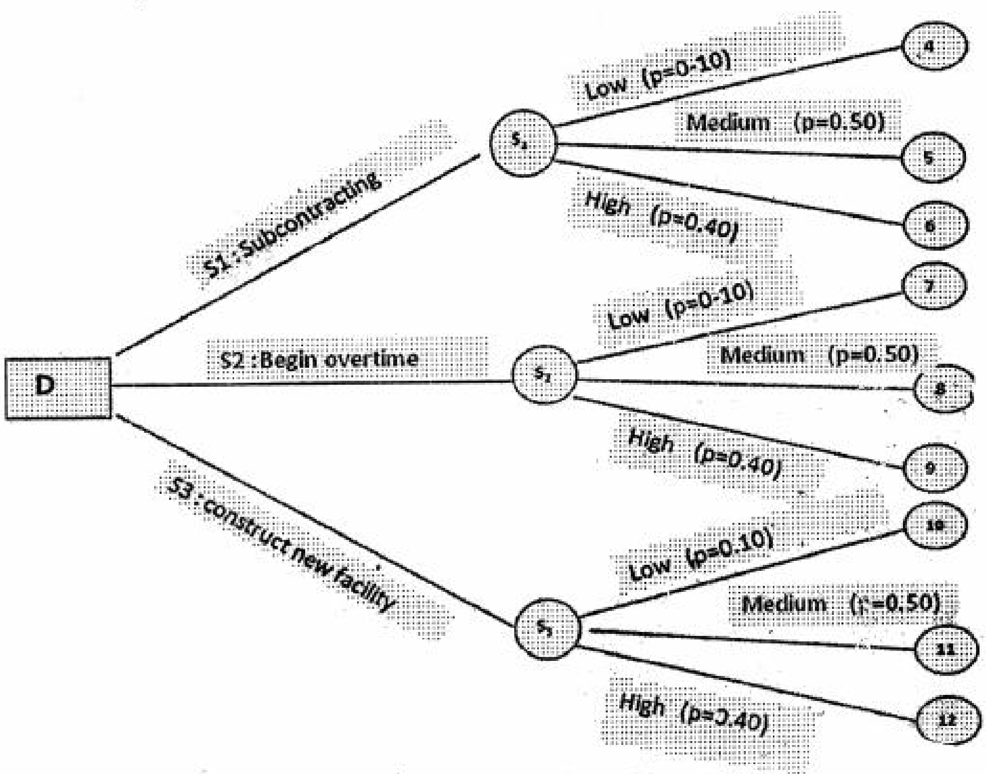


Fig: Decision Tree for construction problem

Conclusion:

Since expected EMV is the highest at node S_3, hence factory should select construct new facility

Working note:

For constructing Decision Tree:

In this problem factory has three options: sub contracting, begin overtime and construct new facility. Firstly, draw a decision node D and three decision branches for three decision alternatives. From these three decision alternatives, draw three chance nodes S_1, S_2 and S_3. Each chance node then be followed be three chance branch as factory has three possible conditions as: low, medium and high.

For calculating highest expected EMV:

For calculating EMV we will use roll-back technique so we start from the extreme right side of the figure and multiply possible outcomes by its corresponding probability. For each chance node we will add all possible outcomes of each chance branch.

Example 3:-

The investment staff of TNC bank is considering four investment proposals for a client; shares, bonds, real estate and saving certificates. These investments will be held for one year. The past data regarding the four proposals are given below;

Shares: There is 25% chance that shares will decline by 10%, a 30% chance that they will remais stable and a 45% chance that they will increase by 15%. Also, the shares under consideration do not pay any dividend.

Bonds: These bonds stand a 40% chance of increase in value by 5% and 60% chance of remaining stable and they yield 12% return.

Real estate: This proposal has a 20% chance of increasing 30% in value, a 25% chance of increasing in 20% value, a 40% chance of increasing in 10% value, a 10% chance of remaining stable amd a 5% chance of losing 5% of its value.

Saving certificates: These certificates yield 8.5% with certainty.

Solution-

Calculation for EMV

At node $S_1 = 0.25 \times 0.90 + 0.30 \times 1.00 + 0.45 \times 1.15 = 1.0425$

At node $S_2 = 0.40 \times 1.17 + 0.6 \times 1.12 = 1.1400$

At node $S_3 = 0.20 \times 1.30 + 0.25 \times 1.20 + 0.40 \times 1.10$

$+ 0.10 \times 1.0 + 0.05 \times 0.95 = 1.1475$

At node $S_4 = 1.0 \times 1.0850 = 1.0850$

Conclusion:

Hence from calculation it is clear that maximum EMV is at node S3. So, company should recommend their customers about investing in Real Estate.

Working Note:

For drawing decision tree

In this problem bank has four investment proposals for their clients; shares, bonds, real estate and saving certificates, so first we draw one decision node and four decision branches. Now we need to check possible outcomes of each decision so we draw four chance nodes S_1, S_2, S_3 and S_4. Now, from S_1 draw three chance branches, as three options are possible from this chance node. From S_2 two options are possible so we draw two chance nodes. Similarly, from S_3 the bank has five and from S_4 has only one possible outcome. In this way, we will draw the decision tree.

For calculating EMV:

To calculate EMV we will use roll back technique and start from the extreme end. First we calculate EMV for each chance branch by multiplying possible outcome by its respective probability. Now for each chance node we add all EMVs. To take the final decision we select the maximum EMV.

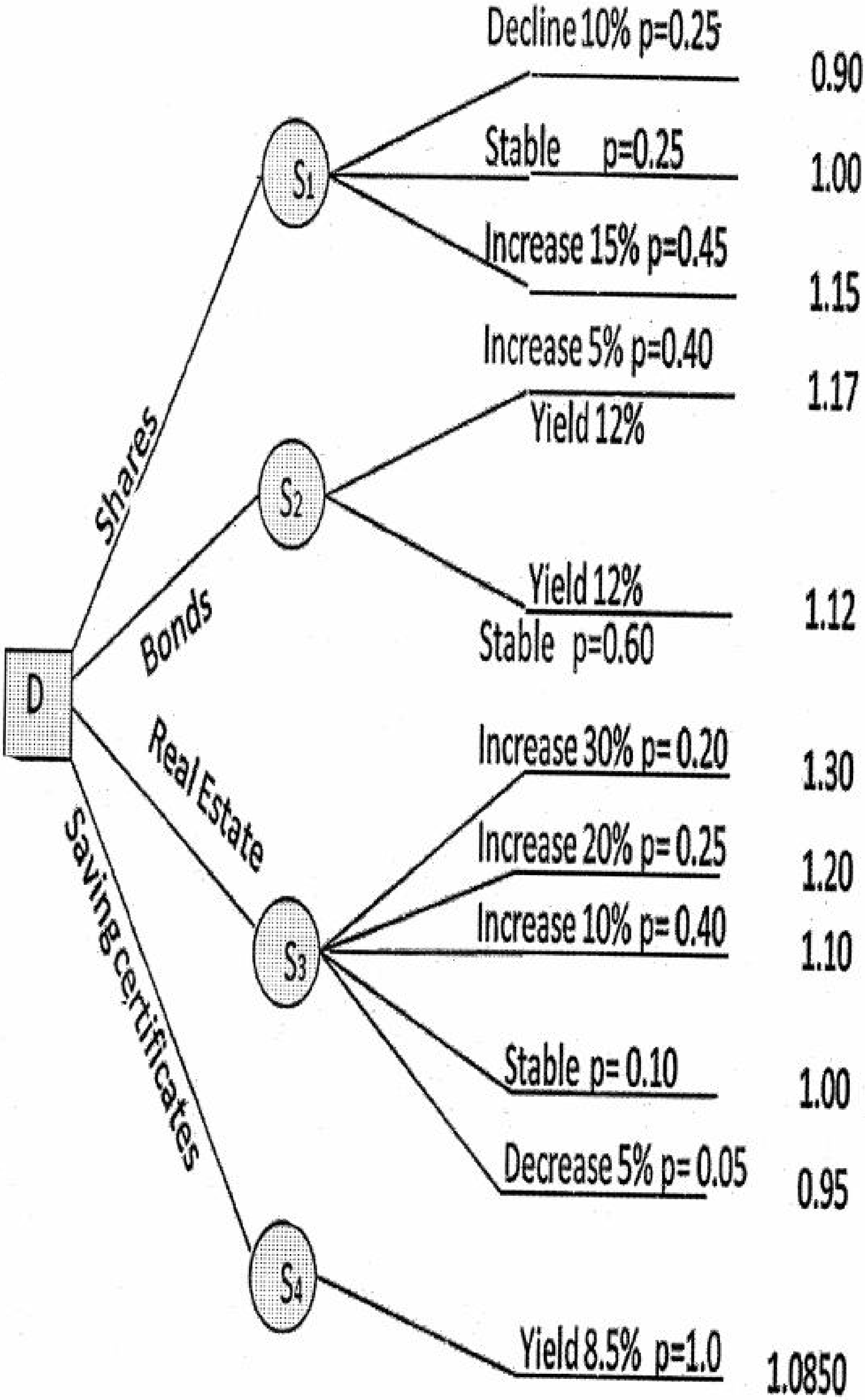


Fig: Decision tree for Investment Problem

Example 4:-

A finance manager is considering drilling a well. In the past only 70% of wells drilled were successful at 20 metres depth in that area. Morever, on finding no water at 20 metres, some persons in that area drilled further upto 25 metres but only 20% struck water at that level. The prevelling cost of drilling is Rs. 500 per metre. The finance manager estimated that in case he does not get water in his own well, he will have to pay Rs 15,000 to buy water from outside for the same period of getting water from the well.

The following decisions are considered:

i) Do not drill any well

ii) Drill upto 20 metres

iii) If no water is found at 20 metres, drill further upto 25 metres.

Draw an appropriate decision tree and determine the finance manager's optimum strategy.

Solution:-

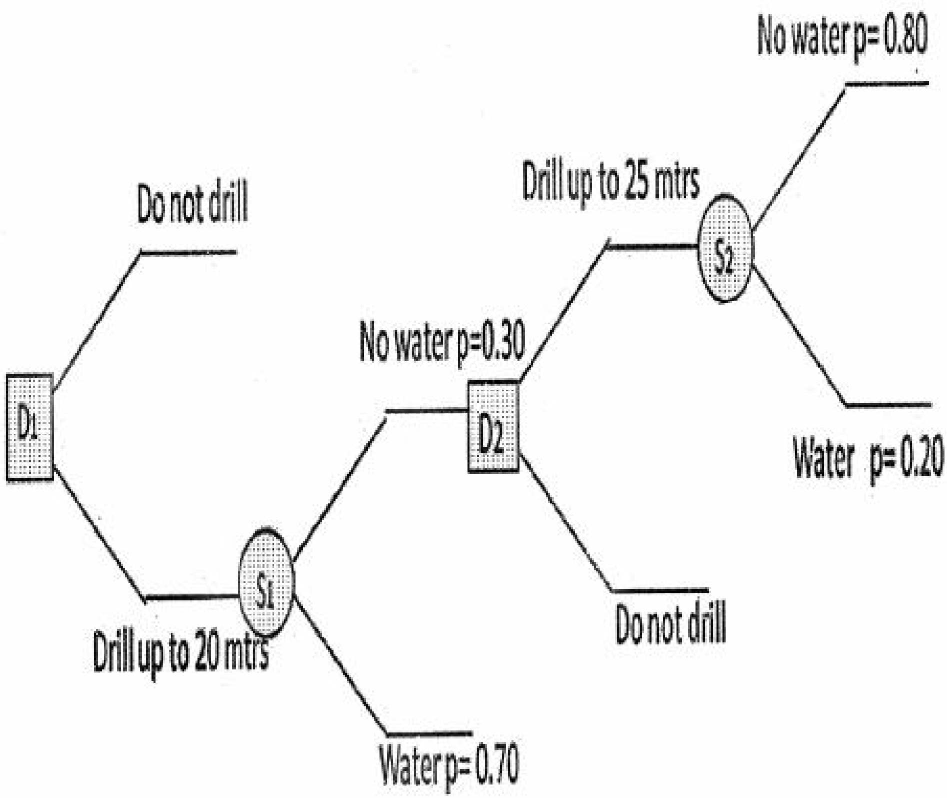


Fig: Decision tree for Drilling Problem

Calculation for EMV:

First, we calculate decision node D_2

At chance node S_2 = 0.20 x (500 x 25) + 0.80 x{15,000 + (500 x 25)}

= 2500 + 0.80x 27,500

= **Rs. 24,500**

If we do not drill or do not get water we have to purchase it at Rs. 15,000

Now, for decision node D_1

At chance node S_1 = 0.70 x (20 x 500) + 0.30 x 24,500

= **Rs. 14,350**

This can be summarized in a table as,

Node	Possibility	Calculation	Total cost (Rs.)	Decision suggested
D_2	Stop	15,000+20x500	25,000	Drill up-to 25mtrs
	Drill up-to 25 mtrs	0.8x(15000+25x500)=22,000 0.2x (500x25)=2,500	24,500	
D_1	Do not drill	15,000	15,000	Drill up-to 20 mtrs
	Drill up-to 20 mtrs	0.30x24,500=7,350 0.7x10,000=7000	14,350	

Conclusion:

From above calculation it is clear that, at decision node D1, if we drill up-to 20 mtrs it will incur less cost than do not drill. So at this stage we would like to drill till 20 mtrs. At decision node D2 if we further drill up-to 25mtrs it will incur less cost than if we do not drill. Thus, first we should drill till 20 mtrs and if no water is found we should further drill up-to 25 mtrs.

Example 5:-

A company has developed a new product in its RandD laboratory. The company has the options of setting up production facilities to market this product straight away. If the product is successful, then over the three years expected product life, the return will be Rs. 120 lakhs with a probability of 0.70. if the market does not respond favorable, then the return will be only Rs. 15 lakhs with probability of 0.30.

The company is considering whether it should test market this product building a small pilot plan. The chance that the test market will yield favorable response is 0.80. if the test market gives favorable response, then the chance of successful total market improves to 0.85.

If the test market gives poor response then the chance of success in the total market is only 0.30.

As before, the return from a successful market will be Rs.120 lakhs and from an unsuccessful market only Rs.15 lakhs. The installation cost to produce for the total market is Rs. 40 lakhs and the cost of test marketing pilot plan is Rs. 5lakhs. Using decision tree analysis, draw a decision tree diagram, carry out necessary analysis to determine the optimal decision.

Solution-

Calculation of maximum expected EMV

Decision node	Alternatives	Calculation	EMV (Rs. Lakh)	decision suggested
3	Set up facility Do not set up	0.85x120+(0.15x15)-40 0	64.25 0	Set up facility
2	Set up facility Do not set up	0.30x120+(0.70x15)-40 0	6.50 0	Set up facility
1	Set up facility Test market	0.70x120+(0.30x15)-40 0.80x64.25+(0.2x6.50)-5	48.5 47.7	Set up facility

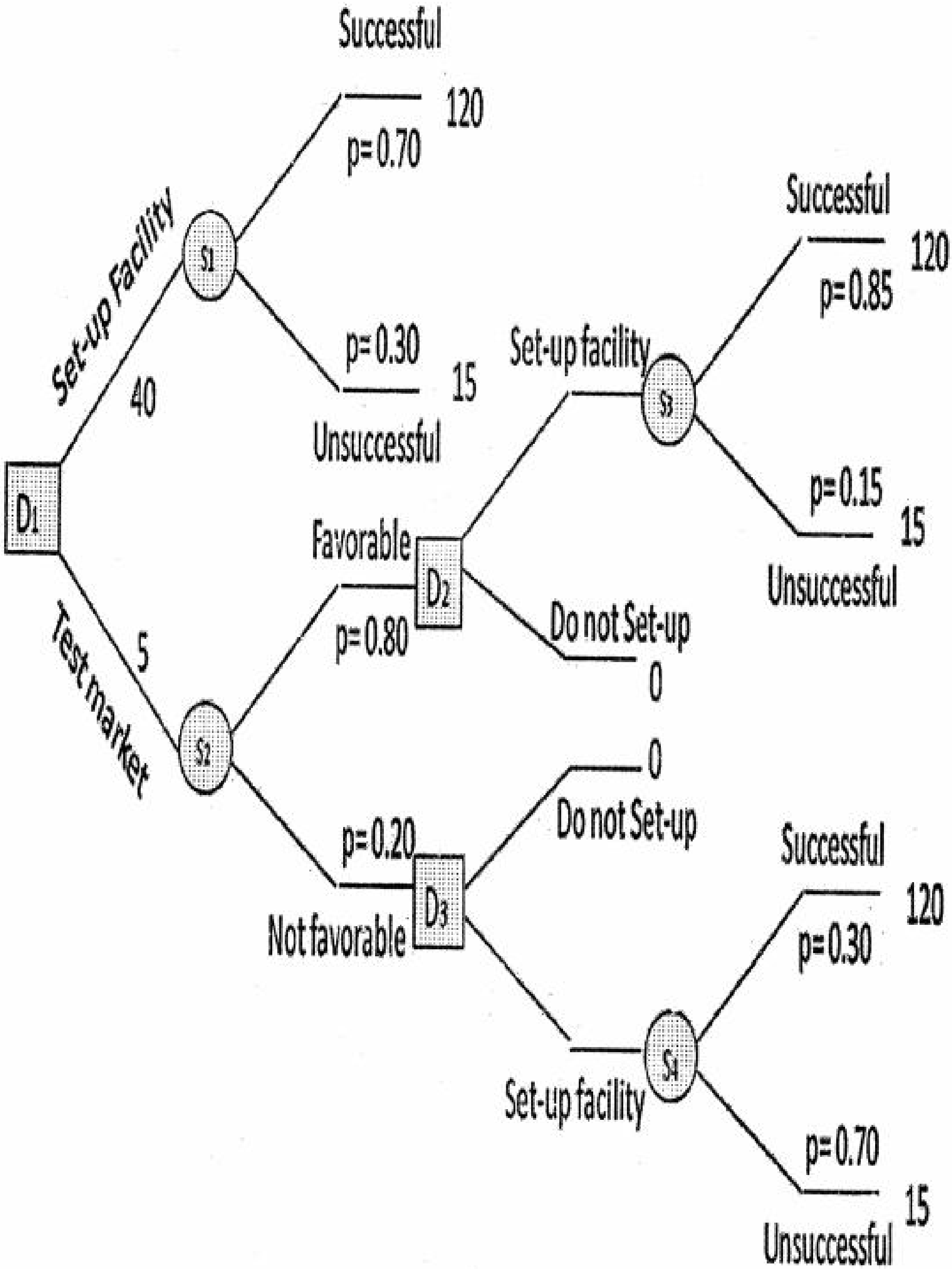


Fig: Decision Tree for Setting up Facility

Conclusion:

Maximum expected EMV at decision node 3 is Rs. 64.25Lakh, so we select Set up facility. In decision node 2 maximum expected EMV is Rs. 6.50Lakh, so again we select Set up facility. Finally at decision node 1 maximum expected EMV is Rs. 48.5Lakh so finally we select Set up facility.

12.7 ADVANTAGES OF DECISION TREE

- The decision tree approach structures the decision process and thus helps one in making a decision in a systematic manner.
- It is a very useful technique of showing and solving inter-related, sequential and multi-dimensional aspects of any major decision problem.
- It is a very useful technique in case where an initial decision and its results affect the sub-sequential decisions.
- It is very helpful in communicating the decision process to others.

12.8 DISADVANTAGES

- Sometime it becomes very complex problem in drawing and solving when three or more branches are associated with any decision or chance node.
- It provides the average valued solution.
- There can be inconsistency in assigning probability for different events.
- Non quantitative factors cannot be considered through decision tree approach, though those are equally important while taking decision.

12.9 KEY TERMS

Branch: Strategy or possibility available with decision maker.

Decision tree: A graphic representation of the sequences of action- event combination available to the decision maker

Multi-stage decision: A sequential decision making process where one decision is always based upon the outcome of previous decision.

Node: A point on decision tree on which decision or chance takes place.

Payoff: Quantitative value of all alternative courses of action.

State of nature: An event that is likely to occur in future over which decision maker do not have any control.

12.10 Self Assessment Questions

1. Discuss the difference between decision making under certainty, uncertainty and risk.

2. What do you mean by decision tree? How is it useful in decision making?

3. A company has a new product which they expect has great potential. Company has two options; either test the market or drop the product.

   If they test it, it will cost Rs. 50,000 and the response could be positive or negative with probability of 0.7 and 0.3 respectively. If it is positive they could market it with full scale or drop the product. If they market it with full scale, then the result might be low, medium or high demand and the respective net pay-off would be –Rs.1,00,000, Rs. 1,00,000 or Rs. 5,00,000. These outcomes have probabilities of 0.25, 0.55 and 0.20 respectively. If the result of the test market is negative, they have decided to drop the product. if at any point they drop the product there is a net gain of Rs. 25,000 from the scale of scrap. All financial values nave been discounted to the present.

4. A manufacturing company has just developed new product. On the basis of past experience a product such as this will either be successful, with expected gross return of Rs. 1,00,000 or unsuccessful, with expected gross return of Rs. 20,000. Similar product manufactured by company has a record of being successful about 50% of the times. The production and marketing cost of the new product are expected to be Rs. 50,000.

   The company is considering whether to market this new product or to drop it. Before making its decision, however, a test marketing effort can be conducted at a cost of Rs. 10,000. Based on the past experience test marketing results have been favorable about 70% of times. Furthermore, products tested favorable have been successful 80% of the time. However, when the test marketing result has been unfavorable, the product has only been successful 30% of the time. What course of action should the company pursue?

5. The Hi-bounce company manufactures guaranteed tennis balls. At the present time, approximately 10% of the tennis balls are defective. A defective ball leaving the factory cost the company Rs. 0.50 to honour its guarantee. Assume that all defective balls are returned at a cost of Rs. 0.10 per ball. The company can conduct a test, which always identifies both good and bad tennis balls. Draw a decision a tree and determine the optimal course of action and its expected cost.

6. An oil drilling company is considering the purchase of minerals rights on a property of Rs. 100lakhs. The price included test to indicate whether the property has type 'A' geological formation or type 'B', the company will be unable to tell the type of geological formation until the purchase is made. However, it is known that 40% of the land in this area has type 'A' formation and 60% has type 'B' formation. If the company decides to drill on the land it will cost Rs. 200 lakhs. If the company does drill it may hit an oil well or a dry hole. Drilling experience indicates that the probability of striking an oil well is 0.4 on type 'A' and 0.1 on type 'B'. Probability of hitting gas is 0.2 on type 'A' and 0.3 on type 'B', the estimated discounted cash value from an oil well is Rs. 1000lakhs and from a gas well is Rs.500lakhs, this includes everything except cost of mineral right and cost of drilling. Use the decision tree approach and recommend whether the company should purchase the mineral right?

12.11 Further Readings

1. N D Vohra : Quantitative Techniques in Management(Tata McGraw Hill)

2. V K Kapoor : Operations Research

3. Levine, Berenson, Krehbiel, Render, Stair, & Hanna : Quantitative Techniques for Management (Pearson)

4. Sharma , J.K : Operations Research : Theory and Applications (Macmillan India Ltd)

U.P.Rajarshi Tandon Open
University, Allahabad

MBA-1.4
Quantitative Techniques for Business Decisions

Block 4

विशेषज्ञ -समिति

1. Dr. Omji Gupta, Director SoMS UPRTOU, Allahabad
2. Prof. Arvind Kumar, Prof., Deptt. of Commerce, Lucknow University, Lucknow
3. Prof. Geetika, HOD, SoMS, MNNIT, Allahabad
4. Prof. H.K. Singh, Prof., Deptt. of Commerce, BHU, Varanasi

लेखक

Dr. Sanjay Mishra, Asso. Prof. MPJ Rohilkhand University, Bareilly.

सम्पादक

Prof. S.A.Ansari, Ex-Dean, Director and Head MONIRBA, University of Allahabad

परिमापक

अनुवाद की स्थिति में

मूल लेखक — अनुवाद

मूल सम्पादक — भाषा सम्पादक

मूल परिमापक — परिमापक

सहयोगी टीम

संयोजक Dr. Gaurav Sankalp, SoMS, UPRTOU, Allahabad.

प्रकाशन --उत्तर प्रदेश राजर्षि टण्डन मुक्त विश्वविद्यालय, इलाहाबाद

प्रकाशन- उत्तर प्रदेश राजर्षि टण्डन मुक्त विश्वविद्यालय, प्रयागराज की ओर से डॉ. अरूण कुमार गुप्ता, कुलसचिव द्वारा पुनः मुद्रित एवं प्रकाशित वर्ष-2020।

मुद्रक- चन्द्रकला यूनिवर्सल प्राइवेट लिमिटेड 42/7 जवाहर लाल नेहरू रोड, प्रयागराज-211002

Block 4 : Quantitative Techniques for Business Decisions

Block Introduction

Block four comprises of four units. Unit thirteen deals with programming techniques, Unit fourteen deals with duality in linear programming Unit fifteen explores transportation prooblem while unit sixteen highlights optimal solution of transportation problem and game theory.

UNIT 13 Programming Techniques

Structure

13.0 Objectives

After studying this unit, you would be able to understand the:

- linear programming problems;
- type of linear programming problems that occur in business;
- formulation of linear programming problems;
- solution of linear programming problems; and
- assumptions underlying linear programming.

13.1 Introduction

Everyday managers come across situations in which decisions are to be made regarding allocation of resources like men, machine, material, money, etc to achieve certain production objectives; or decision is to be made regarding allocation of advertising budget in different media so to achieve maximum exposure to potential buyers; or decision is to be made regarding selection of an investment portfolio from a variety of stock and bond investment alternatives so as to maximize the return on investment; or decision is to be made regarding most efficient assignment of employees in different working shifts so as to minimize the total number of employees and at the same time satisfying the staffing requirements; or decision is be made regarding the transportation of goods from different warehouses located in different places to different markets situated in different places so as to minimize the total transportation cost at the same time satisfying the demand of different markets; etc.

All these decision making situations and many more are basically problems related with either maximization or minimization of some objective subject to certain constraints or restrictions. These constraints or restrictions normally refer to the availability of limited resources under which an optimum decision is to be made.

Linear Programming is a technique related with the allocation of limited resources among competing alternatives so as to achieve the objective of maximization of profit, sales, etc. or minimization of cost, risk, etc.

13.2 Linear Programming

According to David Smith, "Linear Programming is an optimization technique for finding an optimal (maximum or minimum) value of a function, called objective function, of several independent variables, the variables being subject to various restrictions (or constraints) expressed as equations or inequalities. The term 'linear' indicates that the function to be maximized or minimized is linear in nature and that the corresponding constraints be represented by a system of linear inequalities or linear equations involving the variables."

" A method of planning and operation involved in the construction of a model of a real situation containing the following elements: (a) variables representing the available choices, and (b) mathematical expressions (i) relating the variables to the controlling conditions , and (ii) reflecting the criteria to be used in measuring the benefits derivable from each of the several possible plans, and (iii) establishing the objective. The method may be so devised as to ensure the selection of the best from a large number of alternatives." - Kohlar

Thus, in simple terms, Linear Programming could be defined as the technique of developing a mathematical model of problem for finding out the best alternative in order to achieve a particular objective, which is either maximization of profit, sales, etc. or minimization of cost, risk, etc., subject to certain constraints or limitations. The term linear implies that the relationships representing the objective function and constraints are linear in nature.

13.3 Formulation of Linear Programming Problems

In order to formulate a problem as Linear Programming Problem (LPP) one has to clearly understand and define the problem in language form. The problem is to be defined in very clear and unambiguous terms so as to have true picture of the problem. The language should be clear enough to bring out the overall objective which needs to be achieved in terms of maximization or minimization. The constraints which appear to restrict the attainment of overall objective should also be spelled out.

Once the problem has been stated in language form then it needs to be converted into mathematical form so that it could be solved using graphical method or simplex method. Conversion into mathematical form or developing a mathematical model of the problem requires the following:

- defining the decision variables

- identifying the contribution coefficients (c_js) related with decision variables
- identifying the physical rate of substitution coefficients (a_{ij}s)
- finding out the resource availability or requirements i.e. the right hand side of constraints (b_js)
- stating the objective of the problem in mathematical terms (objective function)
- stating the constraints in mathematical terms in the form of equations or inequalities
- stating the non-negative restrictions related with decision variables

13.3.1 Example of LP model formulation : The Maximization Case

13.3.1.1 Example: A furniture manufacturing firm is involved in the manufacturing of wooden tables and wooden chairs. Manufacturing of both tables and chairs require certain number of labour hours (carpenters hours) and certain amount of wood. Each table requires 4 labour hours and 10 units of wood whereas each chair requires 6 labour hours and 8 units of wood. In a week 150 labour hours and 250 units of wood are available. Profit per table sold is Rs 70 and per chair sold is Rs 50.

Formulate this problem as linear programming problem to determine the number of tables and chairs to be produced in a week so as to maximize the profit of the firm. Assume that there is no marketing constraint so that all that is produced could be sold.

Solution

Objective Function : The first step in LPP formulation is to identify the goal and state it in terms of the objective function. In this case the goal is profit maximization, which would be achieved by producing and selling optimum number of tables and chairs. Let x_1 be the number of tables produced and sold in the market per week and x_2 is the number of chairs produced and sold in the market per week. Then the total profit, Z per week will be $70x_1 + 50x_2$ as unit contribution per table and per chair i

Rs 70 and Rs 50, respectively. Since we want to maximize our total profit then this could be written as Maximize $Z = 70x_1 + 50x_2$.

Here Maximize $Z = 70x_1 + 50x_2$ is known as objective function and x_1 and x_2 are known as decision variables.

Constraints : The second step in LPP formulation is to identify the constraints i.e. the limited availability of resources and express them mathematically in the form of equations or inequalities. In this case each unit of table requires 4 labour hours and each unit of chair requires 6 labour hours. So total number of labour hours used if x_1 number of tables and x_2 number of chairs are produced and sold in a week will be $4x_1 + 6x_2$. Since the total number of labour hours used in a week cannot be more than 150 (labour hours available per week) so we can express this constraint as $4x_1 + 6x_2$ d" 150.

Similarly, as each unit of table requires 10 units of wood and each unit of chair requires 8 units of wood. So the total number of units of wood used if x_1 number of tables and x_2 number of chairs are produced and sold in a week will be $10x_1 + 8x_2$. Since the total number of units of wood used in a week cannot be more than 250 (units of wood available per week) so we can express this constraint as $10x_1 + 8x_2$ d" 250.

Non-Negative Restrictions : As x_1 and x_2, the number of tables and chairs produced and sold in the market per week, cannot have negative values so this can be expressed by stating that x_1 and x_2 only assume values that are greater than or equal to zero. This is expressed symbolically as x_1 e" 0 and x_2 e" 0.

Now the appropriate complete formulation of the given problem in LP format is as follows:

Maximize $Z = 70x_1 + 50x_2$	Profit
Subject to	
$4x_1 + 6x_2$ d" 150	Labour hours constraint
$10x_1 + 8x_2$ d" 250	Wood constraint
x_1, x_2 e" 0	Non-negative restrictions

13.3.2 Example of LP model formulation : The Minimization Case

13.3.2.1Example: A patient visits a doctor for consultation regarding some weakness related problem. The doctor tells the patient that there is deficiency of Vitamins V_1 and V_2 in his body. A daily minimum intake of 40 units of V_1 and 50 units of V_2 is must for a healthy body and any amount in excess of these is not harmful for the body. On inquiry the patient comes to know that vitamins V_1 and V_2 are found in two different foods F_1 and F_2. One unit of F_1 contains 2 units of V_1 and 5 units of V_2 whereas one unit of F_2 contains 4 units of V_1 and 2 units of V_2. One unit of F_1 and F_2 costs Rs 30 and Rs 35 respectively.

Formulate this problem as linear programming problem to determine the optimal number of F_1 and F_2 to be purchased and consumed, which minimizes the total cost but at the same time meets the daily minimum requirements of vitamins V_1 and V_2.

Solution

Objective Function : The first step in LPP formulation is to identify the goal and state it in terms of the objective function. In this case the goal is cost minimization. Let x_1 and x_2 be the number of units of F_1 and F_2 to be purchased and consumed daily. Then the total cost, Z, will be $30x_1 + 35x_2$ as unit cost per F_1 and F_2 is Rs 30 and Rs 35, respectively. Since we want to minimize our total cost then this could be written as Minimize $Z = 30x_1 + 35x_2$.

Here Minimize $Z = 30x_1 + 35x_2$ is known as objective function and x_1 and x_2 are known as decision variables.

Constraints : The second step in LPP formulation is to identify the constraints and express them mathematically in the form of equations or inequalities. In this case each unit of F_1 contains 2 units of V_1 and each unit of F_2 contains 4 units of V_1. So total number of units of V_1 consumed if x_1 and x_2 number of units of F_1 and F_2 are purchased and consumed daily will be $2x_1 + 4x_2$. Since the total number of units of V_1 consumed daily cannot be less than 40 so we can express this constraint as $2x_1 + 4x_2$ e" 40.

Similarly, as each unit of F_1 contains 5 units of V_2 and each unit of F_2 contains 2 units of V_2. So total number of units of V_2 consumed if x_1 and x_2 number of units of F_1 and F_2 are purchased and consumed daily will be $5x_1 + 2x_2$. Since the total number of units of V_2 consumed daily cannot be less than 50 so we can express this constraint as $5x_1 + 2x_2$ e″ 50.

Non-Negative Restrictions : As x_1 and x_2, the number of units of F_1 and F_2 to be purchased and consumed daily, cannot have negative values so this can be expressed by stating that x_1 and x_2 only assume values that are greater than or equal to zero. This is expressed symbolically as x_1 e″ 0 and x_2 e″ 0.

Now the appropriate complete formulation of the given problem in LP format is as follows:

Minimize $Z = 30x_1 + 35x_2$	Cost
Subject to	
$2x_1 + 4x_2$ e″ 40	Vitamin V_1 constraint
$5x_1 + 2x_2$ e″ 50	Vitamin V_2 constraint
x_1 , x_2 e″ 0	Non-negative restrictions

13.4 General Model of Linear Programming Problems

In general terms, a linear programming problem can be written as Optimize (Maximize or Minimize) $Z = c_1x_1 + c_2x_2 + \ldots\ldots\ldots + c_nx_n$ (Objective function)

Subject to the constraints

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n$ d″ = e″ b_1

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n$ d″ = e″ b_2

....

....

....

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n$ d″ = e″ b_m

$x_1, x_2, \ldots\ldots\ldots, x_n$ e" 0 Non-negative restrictions

where c_j, a_{ij}, b_j (i = 1, 2, ...,m; j = 1, 2, ..., n) are constants and x_js are decision variables.

13.5 Basic Assumptions of Linear Programming

A LP model is based on certain assumptions. These are as follows:

- **Certainty** - We assume that conditions of certainty exist, i.e. the numerical values of the parameters involved are known with certainty and do not change during the period being studied. It is assumed that the coefficients of the objective function, the coefficients of the inequality/equality constraints and resources availability are known with certainty.

- **Continuity** – It is assumed that the decision variables are continuous. It means that solutions need not be in whole numbers. Instead, they are divisible and may take any fractional value. In production problems, we often define variables as the number of units produced per week or per month, and fractional values (for e.g. 5.6 tables and 3.5 chairs) would simply mean that there is work in progress. Something that was started in one week can be finished in the next.

- **Proportionality** – One of the basic assumptions in linear programming is that proportionality exists in the objective function and the inequality/equality constraints. It means that if one unit of a product contributes Rs 25 in profit then 10 units will give a profit of Rs 250. Similarly if production of one unit of a product takes five labour hours then the production of 10 units will take 50 labour hours.

- **Additivity** – This assumption implies that the total of all activities equals the sum of the individual activities. If a production of one table generated a profit of Rs 350 and the production of one chair Rs 250, then the total profit would be the sum of these two, i.e. Rs 600. Similarly if one table uses 10 units of wood and one chair 6 units then total units of wood used will be 16 if one table and one chair are manufactured.

- **Nonnegative variables** – A linear programming model assumes that the decision variables do not assume negative values. The assumption is realistic as negative values of physical quantities are impossible; you simply cannot produce a negative number of chairs, tables, shirts, lamps, or computers.

13.6 Simplex Method Procedure for Solving LP Type Maximization Type Problems

1. Formulate the LP problem's objective function and constraints.
2. Add slack variables to each less-than-or-equal-to constraint and to the problem's objective function.
3. The coefficient of these slack variables in objective function is zero.
4. Find an initial basic feasible solution by putting all the decision variables equal to zero in all the constraints.
5. Develop an initial simplex tableau with slack variables in the basis as shown below:

		C_j	c_1	c_2	---	c_n	0	0	--- 0	minimum ratio
C_B	Basic variables B	Solution Values x_B	x_1	x_2	---	x_n	s_1	s_2	--- s_m	
0	s_1	b_1	a_{11}	a_{12}	---	a_{1n}	1	0	— 0	
0	s_2	b_2	a_{21}	a_{22}	---	a_{2n}	0	1	— 0	
-	-	-	-	—	-	-	-	—	-- -	
-	-	-	—	-	-	-	—	-	-- -	

0	s_m	b_m	a_{m1}	a_{m2}	---	a_{mn}	-	-	-	1	
	Z_j	0	0		0	0	0		0		
$C_j - Z_j$	c_1	c_2	---	c_n	0	0		0			

6. Choose the variable with the greatest positive $C_j - Z_j$ to enter the solution. This is the pivot column.

7. Determine the solution mix variable to be replaced and the pivot row by selecting the row with the smallest (non-negative) ratio of the quantity-to-pivot column substitution rate. This row is the pivot row.

8. The element that lies at the intersection of pivot row and pivot column is known as pivot element or pivot number.

9. Calculate the new values for the pivot row by simply dividing all the elements of pivot row by pivot number.

10. Calculate the new values for the other row(s). All remaining row(s) are calculated as follows :

    (New row numbers) = (Numbers in old row) – [(Number above or below pivot number) **x** (Corresponding number in the new row, that is, the row replaced in step 3)] .

11. Calculate the Z_j and $C_j - Z_j$ values for this tableau. If all numbers in the $C_j - Z_j$ row are 0 or negative, an optimal solution has been reached. If this is not the case, return to Step 6.

Example 13.6.1 : Maximize $Z = x_1 + 4x_2 + 5x_3$

Subject to constraints

$3x_1 + 3x_3$ d″ 22

$x_1 + 2x_2 + 3x_3$ d″ 14

$3x_1 + 2x_2$ d″ 14 ;

Non-negative restrictions x_1, x_2, x_3 e″0

Solution

Introduce slack variables s_1, s_2, s_3 in the constraints and write the L.P. problem in the standard form as follows:

Maximize $Z= x_1 + 4x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to constraints

$3x_1 + 3x_3 + s_1 = 22$

$x_1 + 2x_2 + 3x_3 + s_2 = 14$

$3x_1 + 2x_2 + s_3 = 14$

and $x_1, x_2, x_3, s_1, s_2, s_3$ e″ 0.

An initial basic feasible solution is :

$x_1 = x_2 = x_3 = 0, s_1 = 22, s_2 = 14, s_3 = 14$

The initial Simplex Table is as follows:

		C_j	1	4	5	0	0	0	Minimum ratio x_B/x_3
C_B	Basic Variables B	Solution Values x_B	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	22	3	0	3	1	0	0	22/3
0	s_2	14	1	2	3*	0	1	0	14/3
0	s_3	14	3	2	0	0	0	1	
		Z_j	0	0	0	0	0	0	
		$C_j - Z_j$	1	4	5	0	0	0	

Since all $C_j - Z_j$ are not 0 or negative therefore an optimal solution has not been reached. In this case as 5 is the most positive value hence x_3 becomes the entering variable and as minimum ratio is associated with second row hence s_2 becomes the leaving variable.

Now we carry out the necessary calculations and construct the next simplex table.

		C_j	1	4	5	0	0	0	Minimum ratio x_B/x_2

C_B	Basic Variables B	Solution Values x_B	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	8	2	-2	0	1	-1	0	
5	x_3	14/3	1/3	2/3	1	0	1/3	0	(14/3)/(2/3)
0	s_3	14	3	2*	0	0	0	1	14/2
		Z_j	5/3	10/3	5	0	5/3	0	
		$C_j - Z_j$	-2/3	2/3	0	0	-5/3	0	

Since again all $C_j - Z_j$ are not 0 or negative therefore an optimal solution has not been reached. In this case as 2/3 is the most positive value hence x_2 becomes the entering variable and as minimum ratio is associated with third row hence s_3 becomes the leaving variable.

Now we carry out the necessary calculations and construct the next simplex table.

		C_j	1	4	5	0	0	0
C_B	Basic Variables B	Solution Values x_B	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	22	5	0	0	1	-1	1
5	x_3	0	-2/3	0	1	0	1/3	-1/3
4	x_2	7	3/2	1	0	0	0	½
		Z_j	8/3	4	5	0	5/3	1/3
		$C_j - Z_j$	-5/3	0	0	0	-5/3	-1/3

Since all entries in the $C_j - Z_j$ row of third simplex table are either zero or negative, optimal solution has been obtained, and the maximum value of Z occurs when $x_2 = 7$ and $x_1 = x_3 = 0$. And the value of Z is 28.

13.7 Simplex Method Procedure for Solving LP Minimization Type Problems (Big – M Method)

Minimization problems are quite similar to maximization problems the only difference being that in case of minimization problems the cost is to be minimized instead of profit being maximized. This introduces change in the way $C_j - Z_j$ values are interpreted. A negative $C_j - Z_j$ value indicates that the total cost will decrease if that variable is selected to enter the basis. Thus the new variable to enter the basis column will be the one having the most negative $C_j - Z_j$. In minimization problems an optimal solution is reached when all the numbers in $C_j - Z_j$ row are positive or zero.

The exact procedure for solving minimization problems is as follows:

1. Formulate the LP problem's objective function and constraints.
2. Introduce slack variables in each less-than-or-equal-to constraint; artificial variable in equal-to constraints; and surplus and artificial variables both in greater-than-or-equal-to constraints and introduce these variables in objective function.
3. The coefficient of slack and surplus variables in objective function is zero. The coefficient of artificial variables in objective function is M. M is a very large number, so large that M minus anything or M plus anything is M only. That's why this method is also known as Big – M method.
4. Find an initial basic feasible solution by putting all the decision variables and surplus variables equal to zero in all the constraints.
5. Develop an initial simplex tableau with slack and artificial variables in the basis.
6. Choose the variable with the greatest negative $C_j - Z_j$ to enter the solution. This is the pivot column.
7. Determine the solution mix variable to be replaced and the pivot row by selecting the row with the smallest (non-negative) ratio of

the quantity-to-pivot column substitution rate. This row is the pivot row.

8. The element that lies at the intersection of pivot row and pivot column is known as pivot element or pivot number.

9. Calculate the new values for the pivot row by simply dividing all the elements of pivot row by pivot number.

10. Calculate the new values for the other row(s). All remaining row(s) are calculated as follows :

    (New row numbers) = (Numbers in old row) – [(Number above or below pivot number) x (Corresponding number in the new row, that is, the row replaced in step 3)] .

11. Calculate the Z_j and $C_j - Z_j$ values for this tableau. If all numbers in the $C_j - Z_j$ row are 0 or positive, an optimal solution has been reached. If this is not the case, return to Step 6.

Note : - In case of maximization type of LPP if artificial variable is to be introduced then coefficient of artificial variable in objective function is – M.

Example 13.7.1 : Minimize $Z = 40x_1 + 24x_2$

Subject to constraints

$20x_1 + 50x_2$ e″ 4800

$80x_1 + 50x_2$ e″ 7200

Non-negative restrictions x_1, x_2 e″ 0

Solution

Subtracting surplus variables from the two constraints and adding artificial variables in the same, then we can write problem as

Minimize $Z = 40x_1 + 24x_2 + 0s_1 + 0s_2 + M A_1 + M A_2$

Subject to constraints

$$20x_1 + 50x_2 - s_1 + A_1 = 4800$$

$$80x_1 + 50x_2 - s_2 + A_2 = 7200$$

Non-negative restrictions x_1, x_2 e″ 0

An initial basic feasible solution is :

$x_1 = x_2 = 0,\ s_1 = s_2 = 0,\ A_1 = 4800,\ A_2 = 7200$

The initial Simplex Table is as follows:

		C_j	40	24	0	0	M	M	Minimum ratio x_B/x_2
C_B	Basic Variables B	Solution Values x_B	x_1	x_2	s_1	s_2	A_1	A_2	
M	A_1	4800	20	50*	-1	0	1	0	96
M	A_2	7200	80	50	0	-1	0	1	144
		Z_j	100M	100M	-M	-M	M	M	
		$C_j - Z_j$	40 – 100M	24 – 100M	M	M	0	0	

Since all $C_j - Z_j$ are not 0 or positive therefore an optimal solution has not been reached. In this case as 24 – 100M is the most negative value hence x_2 becomes the entering variable and as minimum ratio is associated with first row hence A_1 becomes the leaving variable.

Now we carry out the necessary calculations and construct the next simplex table.

		C_j	40	24	0	0	M	M	Minimum ratio x_B/x_1
C_B	Basic VariablesB	Solution Values x_B	x_1	x_2	s_1	s_2	A_1	A_2	
24	x_2	96	2/5	1	-1/50	0	1/50	0	240
M	A_2	2400	60*	0	1	-1	-1	1	40
		$C_j - Z_j$	152/5 – 60M	0	12/25 -M	M	2M – 12/25	0	

Since again all $C_j - Z_j$ are not 0 or positive therefore an optimal solution has not been reached. In this case as 152/5 – 60M is the most negative value hence x_1 becomes the entering variable and as minimum ratio is associated with second row hence A_2 becomes the leaving variable.

Now we carry out the necessary calculations and construct the next simplex table.

C_B	Basic Varia blesB	C_j Solution Values x_B	40 x_1	24 x_2	0 s_1	0 s_2	M A_1	M A_2	Minimum ratio x_B/s_1
24	x_2	80	0	1	-2/75	1/50	2/75	-1/50	-3000
40	x_1	40	1	0	1/60*	-1/60	-1/60	1/60	2400
		$C_j - Z_j$	0	0	-2/75	38/75	M + 2/75	M + 38/75	

Again all $C_j - Z_j$ are not 0 or positive therefore an optimal solution has not been reached. In this case as -2/75 is the most negative value hence s_1 becomes the entering variable and as minimum ratio is associated with second row hence x_1 becomes the leaving variable.

Now we carry out the necessary calculations and construct the next simplex table.

		C_j	40	24	0	0	M	M
C_B	Basic Varia blesB	Solution Values x_B	x_1	x_2	s_1	s_2	A_1	A_2
24	x_2	144	8/5	1	0	-1/50	0	1/50
0	s_1	2400	60	0	1	-1	-1	1
		$C_j - Z_j$	8/5	0	0	12/25	M	M – 12/25

Since all entries in the $C_j - Z_j$ row of simplex table are either zero or positive, optimal solution has been obtained, and the minimum value of Z^* occurs when $x_2 = 144$ and $x_1 = 0$. And the minimum value of Z is 3456.

Example 13.7.2 : Solve the following linear programming problem using simplex method.

$$\text{Maximize } Z = 2x_1 + 4x_2$$

Subject to constraints

$2x_1 + x_2$ d" 18

$3x_1 + 2x_2$ e" 30

$x_1 + 2x_2 = 26$

Non-negative restrictions x_1, x_2 e" 0

Solution

The inequalities can be rewritten introducing slack surplus, and artificial variables as follows:

Maximize $Z = 2x_1 + 4x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

Subject to the constraints

$2x_1 + x_2 + s_1 = 18$

$3x_1 + 2x_2 - s_2 + A_1 = 30$

$x_1 + 2x_2 + A_2 = 26$

Non-negative restrictions x_1, x_2 e" 0

The initial Simplex Table is as follows:

		C_j	2	4	0	0	-M	-M	Minimum ratio x_B/x_2
C_B	Basic Variables B	Solution Values x_B	x_1	x_2	s_1	s_2	A_1	A_2	
0	s_1	18	2	1	1	0	0	0	18
-M	A_1	30	3	2	0	-1	1	0	15
-M	A_2	26	1	2*	0	0	0	1	13
		Z_j	-4M	-4M	0	M	-M	-M	
		$C_j - Z_j$	2 + 4M	4 + 4M	0	-M	0	0	

Since the column x_2 has the maximum positive $C_j - Z_j$ value, x_2 enters the basis and A_2 leaves it. Table below shows the second simplex table.

		C_j	2	4	0	0	-M	-M	Minimum ratio x_B/x_1
C_B	Basic Varia blesB	Solution Values x_B	x_1	x_2	s_1	s_2	A_1	A_2	
0	s_1	5	1.5	0	1	0	0	-0.5	10/3
-M	A_1	4	2*	0	0	-1	1	-1	2
4	x_2	13	0.5	1	0	0	0	0.5	26
		Z_j	-2M	4	0	M	-M	M+2	
		$C_j - Z_j$	2M	0	0	-M	0	-2M - 2	

Since the $C_j - Z_j$ value of x_1 is positive, x_1 enters the basis and A_1 leaves the basis. The third simplex tableau is shown below.

		C_j	2	4	0	0	-M	-M
C_B	Basic Varia blesB	Solution Values x_B	x_1	x_2	s_1	s_2	A_1	A_2
0	s_1	2	0	0	1	0.75	-0.75	0.25
2	x_1	2	1	0	0	-0.5	0.5	-0.5
4	x_2	12	0	1	0	0.25	-0.25	0.75
		Z_j	0	4	0	0	0	2
		$C_j - Z_j$	0	0	0	0	-M	-M – 2

Since all entries in the $C_j - Z_j$ row of simplex table are either zero or negative, optimal solution has been obtained, and the maximum value of Z occurs when $x_1 = 2$ and $x_2 = 12$. And the maximum value of Z is 52.

13.8 Key Words

1. Artificial Variable- A variable that has no meaning in a physical sense, but acts as a tool to help generate an initial L.P. solution.

2. Basic Variables- the set of variables that are in the solution (i.e. have positive, non-zero values) and are listed in the product mix column.

3. Basis- The set of basic variables comprises a basic solution to an L.P.

4. Basic Coefficients: the coefficients in the objective function which correspond to a set of basic variables.

5. Cj – Zj- The row containing the net profit or loss that will result by bringing one unit of a variable indicated in that column into the solution of a L.P. problem.

6. Degeneracy- A condition that arises when there is a tie in the values used to determine which variable will enter the solution next. It can lead to cycling back and forth between two non-optimal solutions.

7. Degenerate Basic Feasible Solution- A basic feasible solution with fewer than m positive variables (hence more than n-m zero variables).

8. Iteration- In the simplex method, the move from one corner to an adjacent corner.

9. Iterative Procedure- A process (algorithm) that repeats the same steps over and over again.

10. Non-basic Variables- Variables not in the solution mix or basis. Non-basic variables are equal to zero.

11. Optimal Solution- A solution that is best for the given problem.

12. Pivot Column- The column with the largest positive number in the Cj – Zj row of a maximization problem, or the largest negative Cj – Zj value in a minimization problem. It indicates which variable will enter the solution next.

13. Pivot Row- The row corresponding to the variable that will leave the basis in order to make room for the variable entering (as indicated by the new pivot column). This is the smallest positive ratio found by dividing the quantity column values by the pivot column values for each row.

14. Pivot Number- The element at the inter-section of the pivot row and the pivot column.

15. Simplex Method- An algorithm for solving L.P. problems investigates feasible corner points only, always maintaining or

improving the objective function, until an optimal solution is obtained.

16. Simplex Table- A table for keeping track of calculations at each iteration of the simplex method.

17. Slack Variable- A variable added to less than or equal to constraints in order to create an equality for a simplex method. It represents a quantity of unused resources.

18. Surplus Variable- A variable inserted in a greater than or equal to constraint to create an equality. It represents the amount of resource usage above the minimum required usage.

19. Unboundedness- A condition describing L.P. maximization problems having solutions that can become infinitely large without violating any stated constraints.

20. Zj Row- The row containing the opportunity cost of bringing one unit of a variable into the solution of linear programming problem.

13.9 Self Assessment Questions

13.9.1 A firm machines and drills two castings X and Y. The time required to machine and drill one casting including machine set up time is as follows:

Casting	*Machine hours*	*Drilling hours*
X	04	02
Y	02	05

There are two lathe and three drilling machines. The working week is of 40 hours; there is no lost time or over time. Variables costs for both castings are Rs 120 per unit while total fixed costs amount to Rs 1000 per week. The selling price of casting X is Rs 300 per unit and that of Y is Rs 360 per unit. There are no limitations on the number of X and Y casting that can be sold. The company wishes to maximize its profit. Formulate a linear programming model for the problem.

13.9.2 A manufacturing company produces two types of products : the super and the regular.Resource requirements for production are given below in the table. There are 1600 hours of assembly worker hours available per week. 700 hours of paint time and 300 hours of inspection time. Regular customers will demand at least 1500 units of the regular type and 90 of the super type.

Product type	Profit contribution (Rs.)	Assembly Time (hr.)	Paint Time (hr.)	Inspection Time (hr.)
Regular	5.0	1.2	0.8	0.2
Super	75	1.6	0.9	0.2

Formulate and solve the given linear programming problem to determine the product mix on a weekly basis.

13.9.3 Two products are manufactured by passing sequentially through three machines. Time per machine allocated to the two products is limited to 10 hours per day. The production time and profit per unit of each product are:

Product	Machine 1	Machine 2	Machine 3	Profit (Rs.)
1	12	10	5	20
2	4	8	10	30

(i) Find the optimal mix of the two products using simplex method.

(ii) Identify the machine(s) with the abundant capacity at the optimal solution.

(iii) For each machine with full utilisation, determine worth per unit increase in the capacity.

(iv) Which of the three machines should be given highest priority for capacity increase?

13.9.4 An advertising agency wishes to reach two type of audiences :

Customers with annual income greater than Rs. 15,000 (target audience A) and customers with annual income less than 15,000 (target audience B). The total advertising budget is Rs. 2,00,000. One programme of T.V. advertising costs Rs. 50,000: one programme on radio advertising costs Rs. 20,000. For contact reasons, at least three programmes ought to be on T.V, and the number of radio programmes must be limited to five. Surveys indicate that a single T.V programmes reaches 4,50,000 customers in target audience A and 50,000 in target audience B. One radio programme reaches 20,000 in target audience A and 80,000 in target audience B. Determine the media mix to maximize the total reach.

13.9.5 Solve the following using Simplex Method:

Maximize $Z = 15x_1 + 25x_2$

Subject to constraints:

$7x_1 + 6x_2$ e″ 20

$8x_1 + 5x_2$ d″ 30

$3x_1 - 2x_2 = 18$

x_1, x_2 e″ 0

13.9.6 Solve the following using Big M Method

Minimize $Z = 2x_1 + x_2$

Subject to constraints:

$3x_1 + x_2 = 3$

$4x_1 + 3x_2$ e″ 6

$x_1 + 2x_2$ d″ 3

x_1, x_2 e″ 0

13.9.7 Solve the following using Big M Method

Minimize $Z = 4x_1 + 3x_2$

Subject to constraints:

$200x_1 + 100x_2$ e″ 4000

$x_1 + 2x_2$ e″ 50

$40x_1 + 40x_2$ e″ 1400

x_1, x_2 e″ 0

13.10 Further Readings

1. N D Vohra : Quantitative Techniques in Management(Tata McGraw Hill)
2. V K Kapoor : Operations Research
3. Taha : Operations Research (Pearson)
4. Sharma, J.K : Operations Research : Theory and Applications (Macmillan India Ltd).

UNIT 14 DUALITY IN LINEAR PROGRAMMING

Structure

14.0 Objectives

After studying this unit, you would be able to understand the:

- duality;
- formulations of dual problems;
- importance of dual problems;
- economic interpretation of dual; and
- relationship between Optimal Solutions of Primal and Dual.

14.1 Introduction

With every Linear Programming problem (LPP) is associated another Linear programming problem known as Dual, which is the mirror image of the original linear Programming problem. The original Linear Programming problem is called the primal problem.

In other words it means that we can look at the same data from two different angles and can state two intimately related related LPPs. The LPP which is stated first is called the primal problem and the second LPP is called the dual problem. In fact either of the problems could be taken as primal as both of them originate from the same data; and consequently the other becomes its dual.

The dual contains economic information which can prove to be highly useful to management, and it may also be easier to solve, in terms of less computation, than the primal problem. Generally, if the LPP primal involves maximizing the profit function subject to less-than-or-equal-to resource constraints, the dual will involve minimizing total opportunity costs subject to greater-than-equal-to product profit constraints. Formulating the dual problem from primal is not complex, and once it is formulated, the solution procedure is exactly the same as for any LPP.

14.2 Dual Formulation

In order to formulate the dual from primal it is necessary to state the LPP in standard form. By standard form is meant that all the variables in the problem should be non-negative and all the constraints should be of 'less than or equal to type' if the problem is of maximization type, and 'greater than or equal to type' if the problem is of minimization type.

If the primal LPP is not in standard form and there is some variable which is unrestricted in sign i.e. it could be positive, negative or zero, then it should be replaced by the difference of two non-negative variables.

If some of the constraints are not in right direction i.e. in case of maximization type of LPP we have a constraint in the 'greater than or equal to type form' or in case of minimization type of LPP we have a constraint in the 'less than or equal to type form', then in order to express the LPP in standard form we multiply the corresponding constraint with -1 so that the direction of inequality is reversed.

For example, a maximization type primal LPP in standard form is written as follows:

Maximize $Z = c_1x_1 + c_2x_2 + \dots\dots + c_nx_n$ (Objective function)

Subject to the constraints

$$a_{11}x_{1} + a_{12}x_{2} + \ldots\ldots\ldots\ldots\ldots + a_{1n}x_{n} \; d'' \; b_{1}$$

$$a_{21}x_{1} + a_{22}x_{2} + \ldots\ldots\ldots\ldots\ldots + a_{2n}x_{n} \; d'' \; b_{2}$$

$$\ldots \qquad \ldots \qquad \qquad \ldots \quad \ldots$$

$$\ldots \qquad \ldots \qquad \qquad \ldots \quad \ldots$$

$$a_{m1}x_{1} + a_{m2}x_{2} + \ldots\ldots\ldots\ldots\ldots + a_{mn}x_{n} \; d'' \; b_{m}$$

and $x_1, x_2, \ldots\ldots\ldots, x_n \; e'' \; 0$ (Non-negative restrictions)

The dual LPP associated with this primal LPP could be expressed as follows:

Minimize $Z^* = b_1y_1 + b_2y_2 + \ldots\ldots\ldots\ldots\ldots + b_my_m$ (Objective function)

Subject to the constraints

$$a_{11}y_{1} + a_{21}y_{2} + \ldots\ldots\ldots\ldots\ldots + a_{m1}y_{m} \; e'' \; c_{1}$$

$$a_{12}y_{1} + a_{22}y_{2} + \ldots\ldots\ldots\ldots\ldots + a_{m2}y_{m} \; e'' \; c_{2}$$

$$\ldots \qquad \ldots \qquad \qquad \ldots \quad \ldots$$

$$\ldots \qquad \ldots \qquad \qquad \ldots \quad \ldots$$

$$a_{1n}y_{1} + a_{2n}y_{2} + \ldots\ldots\ldots\ldots\ldots + a_{mn}y_{m} \; e'' \; c_{n}$$

and $y_1, y_2, \ldots\ldots\ldots, y_m \; e'' \; 0$ (Non-negative restrictions)

Rules for Constructing Dual from Primal

1. If the primal is a maximization type problem then the dual is minimization type problem and vice versa.
2. The number of variables in dual is equal to the number of constraints in primal and vice versa.
3. The right hand side values of the primal constraints become the dual's objective function coefficients.
4. The primal objective function coefficients become the right hand side values of dual constraints.

5. The transpose of the primal constraints coefficients become the dual constraint coefficients.

6. Constraint inequality signs are reversed.

Example 14.2.1 – Write the dual of the LPP given below.

Maximize $Z = 20x_1 + 50x_2$

Subject to the constraints

$5x_1 + 4x_2$ d″ 20

$3x_1 + 5x_2$ d″ 15

x_1, x_2 e″ 0

Solution

In accordance with the above rules since the primal is of maximization type therefore the objective function of dual shall be of minimization type. The number of variables in dual shall be equal to the number of constraints in primal i.e. in this case it will be 2. The RHS values 20 and 15 of primal constraints shall become the coefficients of dual variables in objective function. Therefore, we can write the objective function of dual as

Minimise $Z^* = 20y_1 + 15y_2$

The number of variables in primal shall be equal to the number of constraints in dual. So number of constraints in dual will be 2. Additionally the coefficients of primal variables in objective function shall become the RHS values of dual constraints i.e. in this case it will be 20 and 50. The transpose of the primal constraints coefficients shall become the dual constraint coefficients and all the constraints shall be of greater than or equal to type since objective function is of minimization type. Therefore, we can write the constraints of dual as

$5y_1 + 3y_2$ e″ 20

$4y_1 + 5y_2$ e″ 50

And finally non-negative restrictions

y_1, y_2 e″ 0.

Example 14.2.2 – Write the dual of the LPP given below.

Minimize $Z = 10x_1 + 20x_2 + 15x_3$

Subject to the constraints

$25x_1 + 40x_2 + 5x_3$ e″ 120

$30x_1 + 15x_2 + 10x_3$ e″ 85

x_1, x_2, x_3 e″ 0

Solution

In accordance with the above rules since the primal is of minimization type therefore the objective function of dual shall be of maximization type. The number of variables in dual shali be equal to the number of constraints in primal i.e. in this case it will be 2. The RHS values 120 and 85 of primal constraints shall become the coefficients of dual variables in objective function. Therefore, we can write the objective function of dual as

Maximise $Z^* = 120y_1 + 85y_2$

The number of variables in primal shall be equal to the number of constraints in dual. So number of constraints in dual will be 3. Additionally, the coefficients of primal variables in objective function shall become the RHS values of dual constraints i.e. in this case it will be 10, 20 and 15. The transpose of the primal constraints coefficients shall become the dual constraint coefficients and all the constraints shall be of less than or equal to type since objective function is of maximization type. Therefore we can write the constraints of dual as

$25y_1 + 30y_2$ d″ 10

$40y_1 + 15y_2$ d″ 20

$5y_1 + 10y_2$ d″ 15

And finally non-negative restrictions

y_1, y_2 e″ 0.

Example 14.2.3 -- Write the dual of the LPP given below.

Maximize $Z = 7x_1 + 10x_2 + 4x_3$

Subject to the constraints

$5x_1 + 10x_2 + 4x_3$ d″ 60

$25x_1 + 20x_2 + 15x_3$ e″ 120

$10x_1 + 5x_2 + 8x_3 = 85$

x_1, x_2, x_3 e″ 0

Solution

In order to write dual, first of all we state primal in standard form. The objective is of maximization type, therefore, all the constraints should be in the form of less than or equal to type inequality. Here, we find that the second constraint is of greater than or equal to type and the third constraint is in the form of equation. Therefore we multiply second constraint $25x_1 + 20x_2 + 15x_3$ e″ 120 with -1 and replace third constraint $10x_1 + 5x_2 + 8x_3 = 85$ with two weak inequalities $10x_1 + 5x_2 + 8x_3$ d″ 85 and $10x_1 + 5x_2 + 8x_3$ e″ 85; and then multiply $10x_1 + 5x_2 + 8x_3$ e″ 85 with -1. Now after carrying out these changes we restate the primal in standard form as given below:

Maximize $Z = 7x_1 + 10x_2 + 4x_3$

Subject to the constraints

$5x_1 + 10x_2 + 4x_3$ d″ 60

$- 25x_1 - 20x_2 - 15x_3$ d″ - 120

$10x_1 + 5x_2 + 8x_3$ d″ 85

$- 10x_1 - 5x_2 - 8x_3$ d″ - 85

x_1, x_2, x_3 e″ 0

In accordance with the above rules, since the primal is of maximization type therefore the objective function of dual shall be of minimization type. The number of variables in dual shall be equal to the number of constraints in primal i.e. in this case it will be 4. The RHS values 60, -120, 85 and -85 of primal constraints shall become the coefficients of dual variables in objective function. Therefore we can write the objective function of dual as

$$\text{Minimise } Z^* = 60y_1 - 120y_2 + 85y_3 - 85y_4$$

The number of variables in primal shall be equal to the number of constraints in dual. So number of constraints in dual will be 3. Additionally, the coefficients of primal variables in objective function shall become the RHS values of dual constraints i.e. in this case it will be 7, 10 and 4. The transpose of the primal constraints coefficients shall become the dual constraint coefficients and all the constraints shall be of greater than or equal to type since objective function is of minimization type. Therefore, we can write the constraints of dual as

$$5y_1 - 25y_2 + 10y_3 - 10y_4 \geq 7$$

$$10y_1 - 20y_2 + 5y_3 - 5y_4 \geq 10$$

$$4y_1 - 15y_2 + 8y_3 - 8y_4 \geq 4$$

And finally non-negative restrictions

$$y_1, y_2, y_3, y_4 \geq 0.$$

If we put, $y = y_3 - y_4$ in the above dual then we can rewrite dual as

$$\text{Minimise } Z^* = 60y_1 - 120y_2 + 85\,y$$

Subject to constraints

$$5y_1 - 25y_2 + 10\,y \geq 7$$

$$10y_1 - 20y_2 + 5\,y \geq 10$$

$$4y_1 - 15y_2 + 8\,y \geq 4$$

Non-negative restrictions $y_1, y_2 \geq 0$

and y is unrestricted in sign.

Relationship between Primal and Dual Problems

PRIMAL	DUAL
Maximisation	Minimisation
No. of variables	No. of constraints
No. of constraints	No. of variables
Less than equal to type constraint	Non-negative variable

Equal to type constraint	Unrestricted variable
Unrestricted variable	Equal to type constraint
Objective function coeff. for jth variable	RHS constant for jth constraint
RHS constant for ith constraint	Objective function coeff. for ith variable

14.3 Economic Interpretation of Dual

A furniture manufacturing firm is involved in the manufacturing of wooden tables and wooden chairs. Manufacturing of both tables and chairs require certain number of labour hours (carpenters hours) and certain amount of wood. Each table requires 4 labour hours and 10 units of wood whereas each chair requires 6 labour hours and 8 units of wood. In a week 150 labour hours and 250 units of wood are available. Profit per table sold is Rs 70 and per chair sold is Rs 50. Determine the number of tables and chairs to be produced in a week so as to maximize the profit of the firm. Assume that there is no marketing constraint so that all that is produced could be sold.

Let x_1 be the number of tables produced and sold in the market per week and x_2 is the number of chairs produced and sold in the market per week. Then the complete formulation of the given problem in LP format is as follows:

Maximize $Z = 70x_1 + 50x_2$ Profit

Subject to the constraints

$4x_1 + 6x_2$ d″ 150 Labour hours constraint

$10x_1 + 8x_2$ d″ 250 Wood constraint

x_1, x_2 e″ 0 Non-negative restrictions

The objective of the firm is to maximize profit subject to the labour hours and wood constraints.

Suppose an individual approaches the firm and wants on rental basis one week labour hours and wood. Let y_1 and y_2 be the per unit rent of labour hours and wood which can be charged by the firm. Then the total rent received by the firm will be $150\,y_1 + 250\,y_2$. The firm wants to have an idea about the minimum acceptable rent on which it can lend its facilities for one week. So the objective is to find the minimum value of $150\,y_1 + 250\,y_2$.

Here the constraints of the firm are that the money which it receives on rental basis should be more than or equal to the money which it can earn by employing the same resources on production of tables and chairs.

The manufacturing of one unit of table requires 4 labour hours and 10 units of wood. Thus the total rental from these amounts of resources should be greater than or equal to the profit obtained from one unit of table i.e. $4\,y_1 + 10\,y_2$ e″ 70.

Similarly, one unit of chair requires 6 labour hours and 8 units of wood. Thus the total rental from these amounts of resources should be greater than or equal to the profit obtained from one unit of chair i.e. $6\,y_1 + 8\,y_2$ e″ 50.

The money received on rental basis cannot be negative. Therefore we can restate the problem of the firm as follows:

Minimise $Z^* = 150\,y_1 + 250\,y_2$

Subject to the constraints

$4\,y_1 + 10\,y_2$ e″ 70

$6\,y_1 + 8\,y_2$ e″ 50

y_1, y_2 e″ 0

This problem is the dual of the given problem.

14.4 Relationship between Optimal Solutions of Primal and Dual

Once the dual problem has been solved we can read the optimal solution of the primal problem from the optimal solution table of the dual and vice versa. The steps to be followed for reading the optimal solution of primal problem are as follows;

- The slack and surplus variables in the dual problem correspond with the basic primal variables in the optimal solution.
- The net evaluations corresponding to the columns of slack/surplus variables with changed sign/directly gives the optimal values of basic primal variables.
- The value of the objective functions of both dual and primal is same for optimal solutions.

Example 14.4.1 : A manufacturing company makes three products, each of which requires three operations as part of the manufacturing process. The company can sell all the products it can manufacture but its production is limited by the capacity of its operations centers. Additional data concerning the company is given in the following table.

Manufacturing Requirements (Hours/Unit)

Product Selling	Center1	Center2	Center3	Cost Price
A	1	3	2	11 15
B	3	4	1	12 20
C	2	2	2	10 16
Hrs Available	160	120	80	

What should be the product mix? Write the dual of the given problem and read its solution from the optimal solution table of primal.

Solution

Let X_1, X_2 and X_3 be the number of units of the products A, B and C to be produced. The objective function is

Maximize $Z = 4X_1 + 8X_2 + 6X_3$

Subject to constraints

$X_1 + 3X_2 + 2X_3$ d″ 160

$$3X_1 + 4X_2 + 2X_3 \text{ d'' } 120$$

$$2X_1 + X_2 + 2X_3 \text{ d'' } 80$$

$$X_1, X_2, X_3 \text{ e'' } 0$$

We add the slack variables to the constraints

$$X_1 + 3X_2 + 2X_3 + S_1 = 160$$

$$3X_1 + 4X_2 + 2X_3 + S_2 = 120$$

$$2X_1 + X_2 + 2X_3 + S_3 = 80$$

Now we solve it using simplex method. The calculations are shown in the simplex tables ahead.

First simplex table

		C_j	4	8	6	0	0	0	Minimum ratio x_B/x_2
C_B	Basic Varia blesB	Solution Values x_B	X_1	X_2	X_3	S_1	S_2	S_3	
0	S_1	160	1	3	2	1	0	0	160/3
0	S_2	120	3	4*	2	0	1	0	30
0	S_3	80	2	1	2	0	0	1	80
		$C_j - Z_j$	4	8	6	0	0	0	

Second simplex table

		C_j	4	8	6	0	0	0	Minimum ratio x_B/x_2
C_B	Basic Varia blesB	Solution Values x_B	X_1	X_2	X_3	S_1	S_2	S_3	
0	S_1	70	-1.25	0	0.5	1	-0.75	0	140
8	X_2	30	0.75	1	0.5	0	0.25	0	60
0	S_3	50	1.25	0	1.5*	0	-0.25	1	33.33
		$C_j - Z_j$	-2	0	2	0	-2	0	

Third simplex table

		C_j	4	8	6	0	0	0
C_B	Basic Variables B	Solution Values x_B	X_1	X_2	X_3	S_1	S_2	S_3
0	S_1	53.33	-1.67	0.00	0.00	1.00	-0.67	-0.33
8	X_2	13.33	0.33	1.00	0.00	0.00	0.33	-0.33
6	X_3	33.33	0.83	0.00	1.00	0.00	-0.17	0.67
		$C_j - Z_j$	-3.67	0.00	0.00	0.00	-1.67	-1.33

This is the optimum table. Since all the Cj – Zj are negative or zero. The optimum values are $X_1 = 0$, $X_2 = 13.33$, and $X_3 = 33.33$.

The dual of this problem has the objective of minimizing the opportunity cost of not using the resources in an optimal manner. Let's call the dual variables that it will attempt to solve for Y_1, Y_2 and Y_3. Thus, each constraint in the primal problem will have a corresponding variable in the dual program. Also, each decision variable in the primal problem will have a corresponding constraint in the dual problem.

The primal problem is

Maximize $Z = 4X_1 + 8X_2 + 6X_3$

Subject to constraints

$X_1 + 3X_2 + 2X_3$ d" 160

$3X_1 + 4X_2 + 2X_3$ d" 120

$2X_1 + X_2 + 2X_3$ d" 80

X_1, X_2, X_3 e" 0

Therefore, the dual is

Minimize $Z^* = 160\,Y_1 + 120\,Y_2 + 80\,Y_3$

Subjects to constraints

$Y_1 + 3Y_2 + 2Y_3$ e″ 4

$3Y_1 + 4Y_2 + Y_3$ e″ 8

$2Y_1 + 2Y_2 + 2Y_3$ e″ 6

Y_1, Y_2, Y_3 e″ 0.

For finding the values of dual variables Y_1, Y_2 and Y_3 we look at final simplex table of primal problem i.e. Third Simplex table given above.

In this table net evaluations corresponding to the columns of slack variables with changed sign directly gives the optimal values of dual variables. Hence the values of dual variables are Y_1=0, Y_2 = 1.67 and Y_3 = 1.33.

Example 14.4.2 : A firm wishes to get atleast 160 million 'audience exposures,' the number of times one of the ads is seen or heard by a person. Additionally, the firm wants at least 60 million of these exposures to involve persons with family income of over Rs 10,000 a month and atleast 80 million of these exposures to involve persons between 18 and 40 years of age. The relevant information pertaining to magazine and television is given below:

	Magazine	Television
Cost per ad. (Rs. Thousand)	40	200
Audience per ad. (million)	4	40
Audience per ad. with monthly income over Rs 10000/- (million)	3	10
Audience per ad. in the age group 18 – 40 years (million)	8	10

The firm wishes to know the number of ads to be released in each media so as to keep ad expenditure minimum. Formulate LPP and solve it. Also write the dual of the given problem and read its solution from the optimal solution table of primal.

Solution

Let X_1 and X_2 be the number of ads to be released in magazine and TV. Then the primal problem could be written as follows:

Minimize $Z = 40X_1 + 200X_2$

Subject to constraints

$4X_1 + 40X_2$ e" 160

$3X_1 + 10X_2$ e" 60

$8X_1 + 10X_2$ e" 80

X_1, X_2 e" 0

With appropriate surplus and artificial variables the problem can be restated as follows:

Minimize $Z = 40X_1 + 200X_2$

Subject to constraints

$4X_1 + 40X_2 - S_1 + A_1 = 160$

$3X_1 + 10X_2 - S_2 + A_2 = 60$

$8X_1 + 10X_2 - S_3 + A_3 = 80$

$X_1, X_2, X_3, S_1, S_2, S_3, A_1, A_2, A_3$ e" 0

The calculations are shown in the simplex tables ahead.

First simplex table

	C_j		40	200	0	0	0	M	M	M	Minimum
C_B	Basic Variables B	Solution Values x_B	X_1	X_2	S_1	S_2	S_3	A_1	A_2	A_3	ratio x_B/x_2
M	A_1	160	4	40*	-1	0	0	1	0	0	4
M	A_2	60	3	10	0	-1	0	0	1	0	6
M	A_3	80	8	10	0	0	-1	0	0	1	8
	$C_j - Z_j$		40 −15M	200 −60M	M	M	M	0	0	0	

Second simplex table

		C_j	40	200	0	0	0	M	M	Minimum ratio x_B/x_1
C_B	Basic Variables B	Solution Values x_B	X_1	X_2	S_1	S_2	S_3	A_2	A_3	
200	X_2	4	1/10	1	-1/40	0	0	0	0	40
M	A_2	20	2	0	1/4	-1	0	1	0	10
M	A_3	40	7*	0	1/4	0	-1	-1/4	1	40/7
		$C_j - Z_j$	20 –9M	0	5-M/2	M	M	5M/4	0	

Third simplex table

		C_j	40	200	0	0	0	M	Minimum ratio x_B/s_3
C_B	Basic Variables B	Solution Values x_B	X_1	X_2	S_1	S_2	S_3	A_2	
200	X_2	24/7	0	1	-1/35	0	1/70	1/280	240
M	A_2	60/7	0	0	5/28	-1	2/7*	1/14	30
40	X_1	40/7	1	0	1/28	0	-1/7	-1/28	-
		$C_j - Z_j$	0	0	30/7-5M/28	M	20/7 –2M/7	–5/7 –13M/14	

Fourth simplex table

		C_j	40	200	0	0	0
C_B	Basic Variables B	Solution Values x_B	X_1	X_2	S_1	S_2	S_3
200	X_2	3	0	1	-21/560	1/20	0

0	S_3	30	0	0	5/8	-7/2	1
40	X_1	10	1	0	1/8	-1/2	0
		$C_j - Z_j$	0	0	5/2	10	0

This is the optimum table. The optimum values are $X_1 = 10$ and $X_2 = 3$ i.e. 10 ads should be released in magazine and 3 ads in TV. The total expenditure will be 40x10 + 200x3 = 1000 (Rs thousand)

The primal problem is

Minimize $Z = 40X_1 + 200X_2$

Subject to constraints

$4X_1 + 40X_2$ e″ 160

$3X_1 + 10X_2$ e″ 60

$8X_1 + 10X_2$ e″ 80

X_1, X_2 e″ 0

Therefore, the dual is

Maximize $Z^* = 160\,Y_1 + 60\,Y_2 + 80\,Y_3$

Subjects to constraints

$4Y_1 + 3Y_2 + 8Y_3$ d″ 40

$40Y_1 + 10Y_2 + 10Y_3$ d″ 200

Y_1, Y_2, Y_3 e″ 0.

For finding the values of dual variables Y_1, Y_2 and Y_3 we look at final simplex table of primal problem i.e. Fourth Simplex table given above.

In this table net evaluations corresponding to the columns of surplus variables directly give the optimal values of dual variables. Hence the values of dual variables are $Y_1 = 5/2$, $Y_2 = 10$ and $Y_3 = 0$.

14.5 Characteristics of the Duai Problem

- Dual of the dual is primal.
- If any of the problems has an infeasible solution then the value of the objective function of the other is unbounded.

- If either the primal or dual has a finite optimal solution then the other one also possesses the same.
- The optimal values of the objective functions of primal and dual are same.

14.6 Key Words

1. **Primal Problem** – The original linear programming problem stated from the data is called primal problem.
2. **Dual Problem** - Corresponding to every primal problem there is another linear programming problem formulated from the parameters of the original problem known as the dual problem.
3. **Dual Variables** - The decision variables of the dual linear programming problem are known as dual variables.
4. **Shadow Price** - Shadow Price of a resource is the change in the optimal value of the objective function per unit increase of the resource. The coefficients of slack variables in the $C_j - Z_j$ row.
5. **Unrestricted Variable** – A variable which can have positive, negative or zero value is known as unrestricted variable.

14.7 Self Assessment Questions

14.7.1 How can dual problem be used in management decision making?

14.7.2 Discuss the characteristics and advantages of duality.

14.7.3 Discuss the relationship between primal and dual solution.

14.7.4 Obtain the dual of the following LPP

Maximize $Z = 70x_1 + 30x_2 + 44x_3$

Subject to the constraints

$15x_1 + 30x_2 + 14x_3$ d″ 160

$25x_1 + 10x_2 + 12x_3$ d″ 140

x_1, x_2, x_3 e″ 0

14.7.5 Obtain the dual of the following LPP

Minimize $Z = 10x_1 + 20x_2$

Subject to the constraints

$3x_1 + 2x_2$ e″ 18

$10x_1 + 5x_2$ e″ 45

$7x_1 + 6x_2$ e″ 30

x_1, x_2 e″ 0

14.7.6 Obtain the dual of the following LPP

Maximize $Z = 8x_1 + 10x_2 + 5x_3$

Subject to the constraints

$x_1 \quad - x_3$ d″ 4

$2x_1 + 4x_2$ d″ 12

$x_1 + x_2 + x_3$ e″ 2

$3x_1 + 2x_2 - x_3 = 8$

x_1, x_2, x_3 e″ 0

14.7.7 A farmer has 1000 acres of land on which he can grow corn, wheat or soya beans. Each acre of corn requires Rs 100 for preparation, 7 man-days of work, and yields a profit of Rs 300. An acre of wheat requires Rs 120 for preparation, 8 man-days of work, and yields a profit of Rs 400. An acre of soya beans costs Rs 70 to prepare, requires 8 man-days of work, and yields a profit of Rs 200. If the farmer has Rs 100,000 for preparation and can count on 8,000 man-days of work, how many acres should be allotted to each crop to maximize profit? Also write the dual of the problem and read its solution from the solution of primal.

14.7.8 A firm can produce three different plastic valves. Their production process involves work in three different chambers. Chamber 1 can process valves for 1200 minutes each week, Chamber 2 can process valves for 900 minutes each week and Chamber 3 tests the valves and can work for 1300 minutes each week.

The three valve types, profit per valve and the time in minutes required in each chamber are:

Valve Type	*Time required in minutes*			Profit per valve (Rs)
	Chamber 1	Chamber 2	Chamber 3	
A	5	7	4	40
B	3	2	10	55
C	2	4	5	45

Find how many valves of each type should be produced by the firm in order to maximize its profit? Also write the dual of the problem and read its solution from the solution of primal.

14.8 Further Readings

1. N D Vohra : Quantitative Techniques in Management(Tata McGraw Hill)
2. V K Kapoor : Operations Research
3. Taha : Operations Research (Pearson)
4. Sharma , J.K : Operations Research : Theory and Applications (Macmillan India Ltd)

UNIT 15 TRANSPORTATION PROBLEM

Structure

15.0 Objectives

After studying this unit, you would be able to understand the:

- transportation problem;
- Methods of solving transportation problems;
- use of North West Corner Rule for finding initial basic feasible solution;
- use of Lowest Cost Method for finding initial basic feasible solution; and
- use of Vogel's Approximation Method for finding initial basic feasible solution.

15 .1 Introduction

The transportation problem is concerned with finding an optimal distribution plan for a single commodity from different sources (origins or points of supply) to different destinations (or points of demand). In a typical transportation problem commodity is available at a number of sources from where it is to be transported to a number of destinations and the unit transportation cost between different origins and destinations is known. The objective is to find the distribution plan for transporting the commodity from sources to destinations that satisfies the supply and demand conditions and at the same time minimizes the total transportation cost. A typical transportation problem is depicted in Figure 3.1 given below:

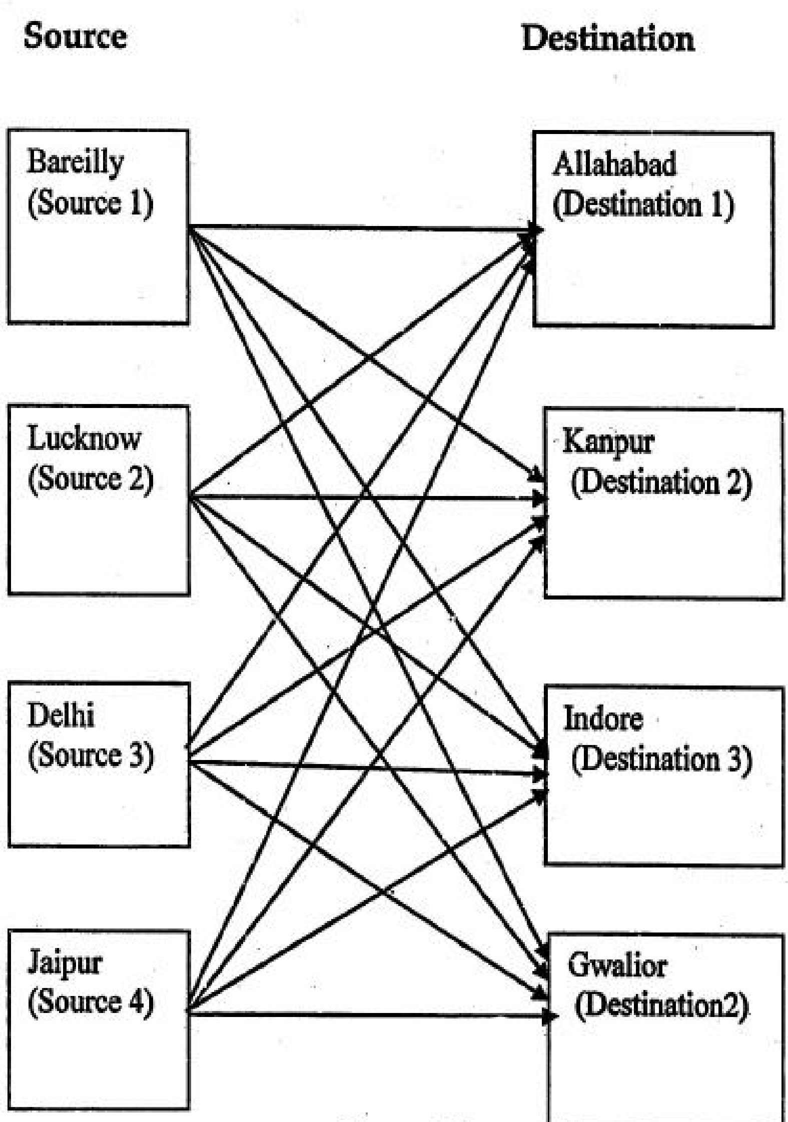


Figure 3.1

Here source indicates the place from where transportation will begin and destination indicates the place where the commodity has to reach.

15.2 Mathematical Formulation

Let us consider the 4 sources (origins) as O_1, O_2, O_3 & O_4 and the 4 destinations as D_1, D_2, D_3 & D_4 respectively. Let $a_i {}^{3} 0$, i= 1,2, ...4, be the amount available at the i^{th} origin O_i and let the amount required at the j^{th} destination D_j be $b_j {}^{3} 0$, j= 1,2,....4.

Let the cost of transporting one unit from i^{th} origin to j^{th} destination be c_{ij}, i= 1,2, ...4, j=1,2,....4. If $x_{ij} {}^{3} 0$ be the amount to be transported from i^{th} origin to jth destination, then the problem is to determine x_{ij} so as to

Minimize $Z = c_{11}x_{11} + c_{12}x_{12} + \ldots\ldots\ldots + c_{44}x_{44}$

Subject to the constraints

$$x_{11} + x_{12} + \ldots\ldots\ldots + x_{14} = a_1$$

$$x_{21} + x_{22} + \ldots\ldots\ldots + x_{24} = a_2$$

$$\ldots \quad \ldots \quad \ldots$$

$$x_{14} + x_{24} + \ldots\ldots\ldots + x_{44} = b_4$$

$$x_{11}, \ldots\ldots\ldots, x_{44} e''0$$

In general terms, let $O_1, O_2, \ldots\ldots\ldots$ Om be origins and D_1, D_2, ...D_n be the destinations respectively. Let a_i 0, i= 1,2, ...m, be the amount available at the i^{th} origin O_i and let the amount required at the j^{th} destination D_j be $b_j {}^{3} 0$, j= 1,2,....n.

Let the cost of transporting one unit from i^{th} origin to j^{th} destination be c_{ij}, i= 1,2, ...m, j=1,2,....n. If $x_{ij} {}^{3} 0$ be the amount to be transported from i^{th} origin to jth destination, then the problem is to determine x so as to

Minimise $Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$

Subject to the constraint

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1,2,\ldots m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1,2, \ldots n$$

and $x_{ij} \geq 0$, for all i and j.

This linear programming problem (LPP) is called a Transportation Problem.

The transportation model can also be portrayed in a tabular form by means of a transportation table as shown below:

To / From	D_1	D_2	....	......	D_n	Supply
O_1	$C_{11}(X_{11})$	$C_{12}(X_{12})$	....	....	$C_{1n}(X_{1n})$	a_1
O_2	$C_{21}(X_{21})$	$C_{22}(X_{22})$	....	....	$C_{2n}(X_{2n})$	a_2
....	....	....				
....	....	....	....	....	....	....
O_m	$C_{m1}(X_{m1})$	$C_{m2}(X_{m2})$	....	....	$C_{mn}(X_{mn})$	a_m
Demand	b_1	b_2	....	....	b_n	

Table No. 15.1

15.3 North West Corner Rule

The North West Corner Rule is a method for finding a basic feasible solution of a transportation problem where the basic variables are selected from the North – West corner of the table.

- The various steps involved in North West Corner Rule are:
- Select the north west (upper left-hand) corner cell of the transportation table and make maximum possible allocation in this cell i.e. allocate as many units as possible in this cell which is equal to the minimum between available supply and demand requirements $\{x_{11} = \min(a_1, b_1)\}$.
- Adjust the supply and demand numbers in the respective rows and columns.

- If the supply of the first row is exhausted then move down to the first cell in the second row and make maximum possible allocation in this cell i.e. If $b_1 > a_1$, the supply of origin O_1 is exhausted but the demand at D_1 is not satisfied. So move down to the second row, and make the second allocation:

$$x_{21} = \min(a_2, b_1 - x_{11}) \text{ in the cell } (2,1).$$

- If the demand for the first cell is satisfied then move horizontally to the next cell in the second column and make maximum possible allocation in this cell i.e. If $a_1 > b_1$, then the demand at D_1 is satisfied but the supply of origin O_1 is not exhausted. So move to second column in the first row and make the second allocation:

$$x_{12} = \min(a_1 - x_{11}, b_2) \text{ in the cell } (1,2).$$

- If for any cell supply equals demand then the next allocation is made moving diagonally to the cell in the next row and next column i.e. If $a_1 = b_1$, then the demand at D_1 is satisfied and at the same time the supply of origin O_1 is also exhausted. So move to the cell (2,2) in the second row and second column and make the allocation:

$x_{22} = \min(a_2, b_2)$ in the cell (2,2).

- Continue this procedure in the zigzag manner until the total available supplies are fully exhausted and total demand requirements are also satisfied, so that the south-east corner of the table is reached.

15.4 Example of North West Corner Rule

A manufacturer has three plants P_1, P_2 & P_3 situated in three different cities – Allhabad, Lucknow and Bareilly, manufacturing the same product. These plants supply to three warehouses W_1, W_2 & W_3 located in three different cities – Delhi, Agra and Gorakhpur. The cost of transporting one unit from different plants to different warehouses in rupees along with available supply and demand is given in the table below:

		Warehouse			
		W_1	W_2	W_3	Supply
Plant	P_1	6	8	10	150
	P_2	7	11	11	175
	P_3	4	5	12	275
	Demand	200	100	300	600

Table No. 3.2

Determine how many units of the product are transported from each plant to each warehouse using North West Corner Rule.

Solution

The first allocation is made to the cell in the upper left-hand corner of the table i.e. cell P_1 W_1 (i.e., the "northwest corner"). The amount allocated is the most possible, subject to the supply and demand constraints for that cell. This amount is 150 units, since that is the maximum that can be supplied by P_1 at Allhabad city, even though 200 units are demanded by W_1 at Delhi.

Thus the supply of P_1 gets exhausted but still 50 units are required at W_1.

We next allocate at cell P_2W_1. The amount allocated at P_2W_1 is minimum of 175 units, the supply available at P_2, and 50 units, the amount now demanded at W_1. Because 50 units is the minimum of the two therefore the amount allocated at cell P_2W_1 is 50 units.

The third allocation is made in the same way as the second allocation at cell P_2W_2. The most that can be allocated is either 100 units (the amount demanded at W_2) or 125 units (175 units minus the 50 units allocated at cell P_2W_1). The smaller amount, 100 units, is allocated at cell P_2W_2.

The fourth allocation is made on similar lines at cell P_2W_3. The amount allocated at cell P_2W_3 is 25 units. Similarly the fifth allocation of 275 units is made at cell P_3W_3.

To / From	W_1	W_2	W_3	Supply
P_1	6 150	8	10	150
P_2	7 50	11 100	11 25	175
P_3	4	5	12 275	275
Demand	200 100		300	600

Table No. 15.3

These allocations, which exhaust all the available supplies and satisfy the total demand, provide the solution of the problem.

The total transportation cost for this initial solution in rupees is

6*150+7*50+11*100+11*25+12*275 = Rs.5,925/-

15.5 Lowest Cost Method

In lowest cost method or least cost method or matrix minima method we make allocation at cell having the least cost in the transportation table. It means that the first allocation is made in the cell for which cost is minimum in the transportation table. Then allocation is given in next minimum cost and so on. It means lower cost cells are given priority over higher cost cells.

In the North West Corner Rule, there is no priority to lower cost cells. We make allocations at north west corner cell of the table without considering the cost involved. Hence, generally, the initial solution obtained by least cost method is better than the solution obtained by North West Corner Rule.

The various steps involved in least cost method are:

- Find the cell having the least cost in the transportation table.
- If the least cost is unique, then make maximum possible allocation in this cell. If the least cost is not unique i.e. there is a tie in the minimum cost then conceptually either of the cells can be selected. However a better initial solution is obtained if we select the cell where greater allocation can be made.
- If the supply is exhausted delete the row and adjust the demand and

  if the demand is satisfied then delete the column and adjust the supply.
- Repeat the above steps for the reduced transportation table until all the supplies are exhausted and all the demands are met.

15.6 Example of Lowest Cost Method

If we try to solve the transportation problem stated in example 3.4 using lowest cost method then the first allocation is made to the cell in the table having the lowest cost i.e. cell P_3W_1 where unit transportation cost is Rs 4. The amount allocated is the most possible, subject to the supply and demand constraints for that cell. This amount is 200 units, since that is the maximum that is demanded by W_1 at Delhi even though 275 units can be supplied by P_3 at Bareilly city. Thus the demand of W_1 is satisfied but still 75 units are available at P_3. This allocation is shown below in Table No. 3.4

To/From	W_1	W_2	W_3	Supply
P_1	6	8	10	150
P_2	7	11	11	175
P_3	4 200	5	12	275
Demand	200	100	300	600

Table No. 15.4

Now we delete all the remaining cells in column W_1. The next allocation is made in the reduced table to the cell that has the minimum cost. This cell is P_3W_2 where unit transportation cost is Rs 5. The maximum possible allocation in this cell is 75 units (minimum of 75 units, now the supply available at P_3, and 100 units, the amount demanded at W_2). This allocation is shown below in Table No. 3.5

ToFrom	W_1	W_2	W_3	Supply
P_1	6	8	10	150
P_2	7	11	11	175
P_3	4 200	5 75	12	275
Demand	200	100	300	600

Table No. 3.5

Now we delete all the remaining cells in row P_3. The third allocation is made to cell P_1W_2, which has the minimum cost of Rs.8 in the reduced transportation table. The maximum possible allocation in this cell is 25 units (minimum of 150 units, the supply available at P_1, and 25 units, the amount nowdemanded at W_2).

In a similar manner the fourth allocation of 125 units is made to cell P_1W_3, and the last allocation of 175 units is made to cell P_2W_3. These allocations, which exhaust all the available supplies and satisfy the total demand, provide the solution of the problem as shown below in Table No. 3.6

To/From	W_1	W_2	W_3	Supply
P_1	6	8 25	10 125	150
P_2	7	11	11 175	175
P_3	4 200	5 75	12	275
Demand	200	100	300	600

Table No. 15.6

The total transportation cost for this initial solution in rupees is

4*200+8*25+5*75+10*125+11*175 = Rs.4,550/-

This total cost of Rs.4,550/- is lower than the cost of Rs.5,925 for the North West Corner Rule solution. It is on account of the fact that the North West Corner Rule does not consider cost at all in making allocations— the lowest cost method does. Thus, the initial solution achieved by using the lowest cost method is usually better than the solution obtained by North West Corner Rule.

15.7 Vogel's Approximation Method (VAM)

Vogel's Approximation Method (VAM) is not quite as simple as the North West Corner Rule but it helps in finding an initial solution which is nearer to the optimal solution. As a matter of fact, in some cases the initial solution happens to be the optimal solution. In this method allocations are made on the basis of the opportunity or extra cost (penalty) that would be incurred if allocations are not made in the cells having minimum unit transportation costs in different rows and different columns.

The various steps involved in Vogel's Approximation Method are:

Calculate the penalties for each row and each column by taking the difference between the smallest and the next smallest costs minimum extra amount which is to be paid for not making the allocation at the cell having the minimum unit transportation cost in corresponding row or corresponding column. This penalty also represents the difference between the distribution cost on the best route in the row or column and the second best route in the row or column.

- Select the row or column with the largest penalty and allocate as much as possible in the cell having the least transportation cost in the selected row or column. If a there is a tie between penalties then it can be broken by selecting the cell where maximum allocation can be made.
- Adjust the supply and demand; and delete the row if supply is exhausted or delete the column if demand is satisfied.
- Recompute the penalties for the reduced transportation table and repeat the aforesaid mentioned steps until all the rim requirements have been met (i.e. the entire available supplies at various origins have been exhausted and total demand at various destinations have been satisfied)

15.8 Example of Vogel's Approximation Method (VAM)

Again if we try to solve the transportation problem stated in example 3.4 using VAM then the first step is to find out the penalties for each row and each column by taking the difference between the smallest and the next smallest costs associated with each row and each column respectively. The penalty costs for our example are shown at the right and at the bottom of Table No. 3.7.

To/From	W_1	W_2	W_3	Supply	Penalty
P_1	6	8	10	150	(2)
P_2	7	11	11	175	(4)
P_3	4	5	12	275	(1)
Demand	200	100	300	600	
Penalty	(2)	(3)	(1)		

Table No. 15.7

Now we select the row or column where the penalty is highest. In Table No. 3.7 highest penalty cost of Rs.4 is associated with row 2. We select row 2 and make maximum possible allocation at cell where the cost is lowest in this row. This cell is P_2W_1 where the cost is Rs.7. The maximum possible allocation in this cell is 175 units (minimum of 175 units, the supply available at P_2, and 200 units, the amount demanded at W_1). Now we delete the remaining cells of row 2 and will not consider them for future allocations as the supply of P_2 has exhausted after this allocation. After this all the penalty costs are recomputed for the reduced table. The allocation and penalty costs are shown in Table No. 3.8

ToFrom	W_1	W_2	W_3	Supply	Penalty
P_1	6	8	10	150	(2)
P_2	7 175	11	11	175	--
P_3	4	5	12	275	(1)
Demand	200	100	300	600	
Penalty	(2)	(3)	(2)		

Table No. 15.8

Next, we repeat the previous step and select the row or column with the highest penalty cost, which is now column 2 with a penalty cost of Rs. 3 (see Table no....). The cell in column 2 with the lowest cost is P_3W_2. The maximum possible allocation in this cell is 100 units (minimum of 275 units, the supply available at P_3, and 100 units, the amount demanded at W_2). Now we delete the remaining cells of column 2 and will not consider them for future allocations as the demand of W_2 has been satisfied after this allocation. After this all the penalty costs are recomputed for the reduced table. The allocation and penalty costs are shown in Table No. 3.9

ToFrom	W_1	W_2	W_3	Supply	Penalty
P_1	6	8	10	150	(4)
P_2	7 175	11	11	175	--
P_3	4	5 100	12	275	(8)
Demand	200	100	300	600	
Penalty	(2)	---	(2)		

Table No. 15.10

Next the highest penalty cost is Rs.8 associated with row 3. The cell P_3W_1 has the minimum cost of Rs.4 in row 3. The maximum possible allocation in this cell is 25 units (minimum of 175 units, the supply now available at P_3, and 25 units, the amount now being demanded at W_2). Now we delete the remaining cells of column 1 and will not consider them for future allocations as the demand of W_1 has been satisfied after this allocation. After this all the penalty costs are recomputed for the reduced table. The allocation and penalty costs are shown in Table No. 3.10

ToFrom	W$_1$	W$_2$	W$_3$	Supply	Penalty
P$_1$	[6]	[8]	[10]	150	---
P$_2$	[7] 175	[11]	[11]	175	--
P$_3$	[4] 25	[5] 100	[12]	275	--
Demand	200	100	300	600	
Penalty	--	---	(2)		

Table No. 3.11

Notice in table no. 3.10 above that only column 3 has a penalty cost. Rows 1 and 3 have only one feasible cell, so a penalty does not exist for these rows. Thus, the last two allocations are made in column 3. First, 150 units are allocated to cell P_1W_3 because it has the lowest cell cost and then150 units are allocated to cell P_3W_3 These allocations, which exhaust all the available supplies and satisfy the total demand, provide the solution of the problem as shown below in Table No. 3.11

ToFrom	W$_1$	W$_2$	W$_3$		Supply
P$_1$	[6]	[8]	[10] 150		150
P$_2$	[7] 175	[11]	[11]		175
P$_3$	[4] 25	[5] 100	[12] 150	275	
Demand	200	100	300	600	

Table No. 15.11

The total transportation cost for this initial solution in rupees is

7*175+4*25+5*100+10*150+12*150 = Rs.5,125/-

Like the lowest cost method, VAM typically results in a lower cost for the initial solution than does the northwest corner rule.

15.9 Optimality Test

Once an initial solution has been obtained, the next step is to check whether it's an optimal solution or not. An optimal solution is one for which total transportation cost is minimum i.e. the total transportation cost for no other solution can be less than the transportation cost obtained through optimal solution. In order to check whether the current solution is optimal or not two methods are available – one is known as Modified Distribution Method (MODI) and the other is known as Stepping Stone Method. In both the methods we try to evaluate each unoccupied cell (represents unused route) in the transportation table in terms of an opportunity of reducing total transportation cost. The basic solution principle in a transportation problem is to determine whether a transportation route not at present being used (i.e., an empty cell) would result in a lower total cost if it were used.

The two methods - modified distribution (MODI) method and stepping stone method have been discussed in detail in the next unit of study material.

15.10 Key Words

1. Transportation problem – The problem which is concerned with finding an optimal distribution plan for a single commodity from different sources (origins or points of supply) to different destinations (or points of demand).
2. Origin – It is the location from which shipments are dispatched.
3. Destination – It is the place where the shipments are to reach.
4. Unit transportation cost – The cost of transporting one unit from an origin to a destination.
5. Feasible Solution – A solution that satisfies the rim requirements

of the transportation table (i.e. supply and demand requirements) and also satisfies the non-negative restrictions.

6. Basic Feasible Solution – A feasible solution of(m x n) transportation problem where the total number of allocations is equal to (m x n – 1)is known as basic feasible solution.

7. North West Corner Rule – The North West corner rule is a method for finding a basic feasible solution of a transportation problem where the basic variables are selected from the North – West corner of the table.

8. Lowest cost method –Lowest cost method is a method for finding a basic feasible solution of a transportation problem where we make allocation at cell having the least cost in the transportation table.

9. Vogel's Approximation Method (VAM) – It is a method for finding a basic feasible solution of a transportation problem in which allocations are made on the basis of the opportunity or extra cost (penalty) that would incur if allocations are not made in the cells having minimum unit transportation costs in different rows and different columns.

10. Penalty – It the difference between the smallest and the next smallest costs associated with each row and each column.

5.11 Self Assessment Questions

15.11.01 Determine the initial feasible solution using north west corner rule to the following transportation problem where Oi and Dj represent i^{th} origin and j^{th} destination, respectively.

		Destination				
		D_1	D_2	D_3	D_4	Supply
Origin	O_1	6	4	1	5	140
	O_2	8	9	2	7	160
	O_3	4	3	6	2	50
	Demand	60	100	150	40	350

15.11.02 A firm owns facilities at six places. It has manufacturing plants at places A, B and C with daily production of 50, 40 and 60 units respectively. At point D, E and F it has three warehouses with daily demands of 20, 95 and 35 units respectively. Per unit shipping costs are given in the following table. The firm wants to minimize its total transportation cost, find the initial feasible solution using North West Corner Rule?

		Warehouse		
		D	E	F
Plant	A	6	4	1
	B	3	8	7
	C	4	4	2

15.11.03 A company has factories at F_1, F_2 and F_3 which supply to warehouses at W_1, W_2 and W_3. Weekly factory capacities are 200, 160 and 90 units, respectively. Weekly warehouse requirement are 180, 120 and 150 units, respectively. Unit shipping costs (in rupees) are as follows:

	W_1	W_2	W_3	Supply
F_1	16	20	12	200
F_2	14	8	18	160
F_3	26	24	16	90
Demand	180	120	150	450

Determine the initial feasible solution using North West Corner Rule for this company.

15.11.04 Luminous lamps have three factories - F_1, F_2, and F_3 with production capacity 30, 50, and 20 units per week respectively. These units are to be shipped to four warehouses W_1, W_2, W_3 and W_4 with requirement of 20, 40, 30 and 10 units per week respectively. The transportation costs per unit between factories and warehouses are given below:

	W_1	W_2	W_3	W_4	
F_1	1	2	1	4	30
F_2	3	3	2	1	50
F_3	4	2	5	9	20
	20	40	30	10	100

Find the initial feasible solution using North West Corner Rule for this company.

15.11.05 A company wants to supply materials from three plants to three new projects. Project I requires 50 truck loads, project II requires 40 truck loads and project III requires 60 truck loads. Supply capacities for the plants P_1, P_2 and P_3 are 50, 55 and 45 truck loads. The table of transportation costs are given below :

	I	II	III
P_1	7	10	12
P_2	8	12	7
P_3	4	9	10

Determine the initial feasible solution using lowest cost method for this company.

15.11.06 There are three sources which store a given product. The sources supply these products to four dealers. The capacities of the sources and the demands of the dealers are given. Capacities $S_1 = 150$, $S_2 = 40$, $S_3 = 80$, Demands $D_1 = 90$, $D_2 = 70$, $D_3 = 50$, $D_4 = 60$. The unit transportation cost matrix is given as follows:

	D_1	D_2	D_3	D_4	Capacity
S_1	27	23	31	69	150
S_2	10	45	40	32	40
S_3	30	54	35	57	80
Demand	90	70	50	60	

Find how many units are transported from which source to which dealer using lowest cost method.

15.11.07 There are three factories F_1, F_2, F_3 situated in different areas with supply capacities as 200, 400 and 350 units respectively. The items are shipped to five markets M_1, M_2, M_3, M_4 and M_5 with demands as 150, 120, 230, 200, 250 units respectively. The unit transportation cost matrix is given as follows :

	M_1	M_2	M_3	M_4	M_5
F_1	2	5	6	4	7
F_2	4	3	5	8	8
F_3	4	6	2	1	5

Determine the initial basic feasible solution using Vogel's Approximation Method.

15.11.08 Identical products are produced in four factories and sent to four warehouses for delivery to the customers. The costs of transportation, capacities and demands are given as below:

	W_1	W_2	W_3	W_4	Capacity
F_1	9	6	11	5	200
F_2	4	5	8	5	150
F_3	7	8	4	6	350
F_4	3	3	10	10	250
Demand	260	150	340	200	

Determine the initial basic feasible solution using Vogel's Approximation Method.

15.11.09 Find the initial basic feasible solution to the following transportation problem for which the unit transportation costs are given below using (a) North West Corner Rule (b) Lowest Cost Method, and (c) Vogel's Approximation Method. Also compare the solutions.

	A	B	C	D	E	Supply
I	9	9	7	6	9	90
II	11	6	6	8	3	20
III	4	7	6	12	5	50
IV	8	5	3	4	6	50
Demand	80	60	20	40	10	

15.12 Further Readings

1. N D Vohra : Quantitative Techniques in Management(Tata McGraw Hill)
2. V K Kapoor : Operations Research
3. Taha : Operations Research (Pearson)
4. Sharma , J.K : Operations Research : Theory and Applications (Macmillan India Ltd)

UNIT 16 OPTIMAL SOLUTION OF TRANSPORTATION PROBLEM & GAME THEORY

Structure

16.0 Objectives

After studying this unit, you would be able to understand the:

- MODI method and its use in finding optimal solution;
- Stepping Stone method and its use in finding optimal solution;
- special cases that occur in transportation problems;

- game theory;
- two person zero sum game; and
- method to solve two person zero sum games.

16.1 Introduction

The basic objective behind mathematical formulation of any business decision making problem is to solve it and find the best possible solution. In transportation problems also after presenting them through a mathematical model we try to find an initial solution using North West Corner Rule or Lowest Cost Method or Vogel's Approximation Method (VAM). Once we have the initial solution we try to test whether it is the best solution i.e. optimal solution or not. If the given solution is not optimal then we keep on improving it until we reach on the optimal solution. For testing optimality and finding optimal solution we can use any of the two methods, namely, Modified Distribution (MODI) method or Stepping Stone method. Both the methods use the concept of opportunity cost associated with empty cells i.e. unoccupied cells in transportation table, to find optimality of the given solution.

16.2 Modified Distribution (MODI) Method

The Modified Distribution (MODI) Method is easier and more efficient than the stepping stone method. It makes use of the concept of dual variables for finding the opportunity cost associated with empty cells and thereby evaluating the empty cells.

The various steps involved in Modified Distribution method are:

- Determine an initial basic feasible solution containing m+n-1 occupied cells using any one of the three methods, namely, North West Corner Rule or Lowest Cost Method or Vogel's Approximation Method.
- Determine the values of dual variables, u_i (i=1,.......,m) and v_j (j=1.........,n) associated with each row and each column of transportation table by using the relation $u_i + v_j = c_{ij}$,which holds

good for occupied cells. Since there are m+n-1 occupied cells therefore there are m+n-1 equations involving u_i, v_j & c_{ij}. Hence in order to find the values of all u_i & v_j we start with arbitrarily assigning any value to one of the dual variables.

Compute the opportunity cost associated with each unoccupied cell using the relationship $\ddot{A}ij = c_{ij} - (u_i + v_j)$.

Check the sign of each opportunity cost. If the opportunity costs of all the unoccupied cells are positive then this signifies that the current solution is optimum solution and the problem possesses unique optimum solution. If the opportunity costs of all the unoccupied cells are either positive or zero then this signifies that the current solution is optimum solution and the problem possesses multiple optimal solutions. On the other hand, if one or more unoccupied cells have negative opportunity cost, then this signifies that the current solution is not an optimum solution.

Select the unoccupied cell with the largest negative opportunity cost. The variable corresponding to this cell is to be included in the next solution i.e. in the next transportation table this cell will become occupied.

Draw a closed path or loop starting from the unoccupied cell selected in the previous step. In order to make the loop we start with the unoccupied cell by drawing an arrow from this cell to an occupied cell in the same row or column. Then we move vertically or horizontally to another occupied cell by drawing another arrow. We repeat this procedure of moving vertically or horizontally to occupied cells drawing arrows until we reach the unoccupied cell from which we have started. In this process of moving from one occupied cell to another we always move vertically or horizontally but never diagonally; and step over empty and, if required, over occupied cells without changing them. It is to be noted that the right angle turn in this path is permitted only at occupied cells and at the original unoccupied cell.

- Assign a plus (+) sign at the unoccupied cell, place alternate minus (-) signs and plus (+) signs on each corner cell of the loop.
- Select the smallest quantity found in occupied cells containing minus signs. Add this quantity to all cells on the loop with plus signs; subtract the quantity from all cells with minus signs. In this way an unoccupied cell becomes an occupied cell and we get a new solution.
- Again we repeat the whole procedure of testing solution and improving the existing solution until an optimum solution is obtained.

16.3 Example of Modified Distribution (MODI) Method

Suppose there is a company having three factories manufacturing the same commodity which are required to be transported to three stockists. The availability and demand at factories and stockists along with unit transportation costs (in rupees) involved are given below in the table. Find the optimum transportation schedule from factories to stockists and also calculate the total transportation cost.

Factories	Stockists			Monthly Capacity
	X	Y	Z	
A	4	8	8	56
B	16	24	16	82
C	8	16	24	77
Monthly Demand	72	102	41	

Applying North West Corner Rule we find an initial solution which is given below in table.

Factories	Stockists			Monthly Capacity
	X	Y	Z	
A	4 (56)	8	8	56
B	16 (16)	24 (66)	16	82
C	8	16 (36)	24 (41)	77
Monthly Demand	72	102	41	

In order to test whether this solution is optimal or not we calculate the values of dual variables u_i and v_j associated with each row and column; and thereafter we find the opportunity costs associated with each unoccupied cell. The values of u_i and v_j and opportunity costs (shown in parenthesis at upper right corner of the cell) are given below in table.

Factories	Stockists			Monthly Capacity	u^1
	X	Y	Z		
		(-4)	(-12)		
A	4 (56)	8	8 (-16)	56	0
B	16 (16) (0)	24 (66)	16	82	12
C	8	16 (36)	24 (41)	77	4
Monthly Demand	72	102	41		
vj	4	12	20		

A look at opportunity costs reveals that the current solution is not optimal and the variable corresponding to cell (2,3) becomes the entering variable. We draw a closed loop starting from this cell as shown in above table. Then we carry out the necessary calculations and reach on the revised solution which is given below in table along with the values of u_i and v_j and opportunity costs.

Factories	Stockists			Monthly Capacity	u_i
	X	Y	Z		
		(-4)	(4)		
A	4 (56)	8	8	56	0
B	16 (16)	24 (25)	16 (41)	82	12
C	(0) 8	16 (77)	(16) 24	77	4
Monthly Demand	72	102		41	
vj	4	12	4		

A look at opportunity costs reveals that the current solution is not optimal and the variable corresponding to cell (2,3) becomes the entering variable. We draw a closed loop starting from this cell as shown in above table. Then we carry out the necessary calculations and reach on the revised solution which is given below in table along with the values of u_i and v_j and opportunity costs.

Factories	Stockists			Monthly Capacity	u_i
	X	Y	Z (4)		
A	4 (31)	8 (25) (4)	8	56	0
B	16 (41) (-4)	24	16 (41) (12)	82	12
C	8	16 (77)	24	77	8
Monthly Demand	72	102	41		
vj	4	8	4		

Again the current solution is not optimal and the variable corresponding to cell (3,1) becomes the entering variable. We draw a closed loop starting from this cell as shown in above table. Then we carry out the necessary calculations and reach on the revised solution which is given below in table along with the values of u_i and v_j and opportunity costs.

Factories	Stockists			Monthly Capacity	u_i
	X	Y	Z		
A	(4) 4	8 (56)	(8) 8	56	0
B	16 (41)	(0) 24	16 (41)	82	16
C	8 (31)	16 (46)	(16) 24	77	8
Monthly Demand	72	102	41		
vj	0	8	0		

As the opportunity costs of all the unoccupied cells are either positive or zero then this signifies that the current solution is optimum solution and the problem possesses multiple optimal solutions.

The transportation cost for this optimal solution is 8x56 + 16x41 + 16x41 + 8x31 + 16x46 = Rs. 2744/-

16.4 Stepping Stone Method

The stepping stone method is an iterative procedure for moving from an initial solution to optimum solution. In this method we try to evaluate the cost effectiveness of shipping goods via transportation routes not currently being used. In this method we try to find out, "What would happen to total shipping costs if one unit is transported through a route not currently being used?" If the total cost decreases then this route should be included in the solution i.e. some units must be transported using this route. If the total cost increases then this route should not be included in the solution i.e. no unit must be transported using this route. We repeat this procedure for all the routes not currently being used. If for all the routes not currently being used the total cost increases then this signifies that the current solution is optimum solution otherwise not. If the current solution is not optimum then we try to improve it by making use of loops until we reach on optimum solution. The exact steps to be followed in stepping stone method are:

- Find an initial basic feasible solution using any of the three methods.
- Select an unoccupied cell (route not currently being used).
- Draw a closed path or loop starting from the unoccupied cell selected in the previous step. In order to make the loop we start with the unoccupied cell by drawing an arrow from this cell to an occupied cell in the same row or column. Then we move vertically or horizontally to another occupied cell by drawing another arrow. We repeat this procedure of moving vertically or horizontally to occupied cells drawing arrows until we reach the unoccupied cell from which we have started. The cells at the turning points are called 'Stepping Stones' on the path.
- Assign a plus (+) sign at the unoccupied cell, place alternate minus (-) signs and plus (+) signs on each corner cell of the loop.
- Find out the 'net change in the cost' along the loop or closed path by adding together the unit costs occurring at cells with a plus sign and then subtracting the unit costs occurring at cells with a minus sign.
- Repeat this procedure of finding the 'net change in the cost' along the loop or closed path for all the unoccupied cells (i.e. unused routes).
- If the 'net change in the cost' for all the unoccupied cells (i.e. unused routes) is positive or zero than this means that the current solution

is optimum solution. If not, then it means that the current solution is not optimum and it could be improved.

- Select the cell having the highest negative 'net change in the cost.' This cell becomes the entering cell i.e. this route should be included in the solution. Draw a closed path or loop starting from this unoccupied cell. Assign a plus (+) sign at the unoccupied cell, place alternate minus (-) signs and plus (+) signs on each corner cell of the loop.
- Select the smallest quantity found in occupied cells containing minus signs. Add this quantity to all cells on the loop with plus signs; subtract the quantity from all cells with minus signs. In this way an unoccupied cell becomes an occupied cell and we get a new solution.
- Again we repeat the whole procedure of testing solution and improving the existing solution until an optimum solution is obtained.

16.5 Example of Stepping Stone Method

Again we look at the initial solution of the previous example which is given below in table.

Factories	Stockists			Monthly Capacity
	X	Y	Z	
A	4 (56)	8	8	56
B	16 (16)	24 (66)	16	82
C	8	16 (36)	24 (41)	77
Monthly Demand	72	102	41	

In order to test whether this solution is optimal or not we find net change in the cost for all the unoccupied cells. The net change in the cost for unoccupied cells (1,2), (1,3), (2,3), and (3,1) comes out to be -4 (8-24+16-4), -12 (8-24+16-24+16-4), -16 (16-24+16-24) and 0 (8-16+24-16), respectively. This means that the current solution is not optimal and the variable corresponding to cell (2,3) becomes the entering variable.

Factories	Stockists			Monthly Capacity
	X	Y	Z	
A	4 (56)	(-4) 8	(-4) 8	56
B	16 (16) (0)	24 (66)	16(41) (16)	82
C	8	16 (77)	24	
Monthly Demand	72	102	41	
vj	4	12	20	

We draw a closed loop starting from this cell as shown in above table. Then we carry out the necessary calculations and reach on the revised solution which is given below in table along with the net change in the cost for unoccupied cells.

Factories	Stockists			Monthly Capacity
	X	Y	Z	
A	4 (56)	(-4) 8	(4) 8	56
B	16 (16) (0)	24 (25)	16 (41) (16)	82
C	8	16 (77)	24	77
Monthly Demand	72	102	41	
vj	4	12	4	

A look at the net change in the cost for unoccupied cells reveals that the current solution is not optimal and the variable corresponding to cell (1,2) becomes the entering variable. We draw a closed loop starting from this cell as shown in above table. Then we carry out the necessary calculations and reach on the revised solution which is given below in table along with the net change in the cost for unoccupied cells.

Factories	Stockists			Monthly Capacity
	X	Y	Z	
A	4 (31)	8 (25)	8 (4)	56
B	16 (41)	24 (4)	16 (41)	82
C	8 (-4)	16 (77)	24 (12)	77
Monthly Demand	72	102	41	
vj	4	8	4	

Again the current solution is not optimal and the variable corresponding to cell (3,1) becomes the entering variable. We draw a closed loop starting from this cell as shown in above table. Then we carry out the necessary calculations and reach on the revised solution which is given below in table along with the net change in the cost for unoccupied cells.

Factories	Stockists			Monthly Capacity
	X	Y	Z	
A	4 (4)	8 (56)	8 (8)	56
B	16 (41)	24 (0)	16 (41)	82
C	8 (31)	16 (46)	24 (16)	77
Monthly Demand	72	102	41	
vj	0	8	0	

As the net change in the cost for all the unoccupied cells are either positive or zero then this signifies that the current solution is optimum solution and the problem possesses multiple optimal solutions.

The transportation cost for this optimal solution is 8x56 + 16x41 + 16x41 + 8x31 + 16x46 = Rs. 2744/-

16.6 Special Cases in Transportation Problems

While solving transportation problems certain special type of transportation problems could arise which require slight modification in the approach for solving them. The different special cases and the approach for solving them are discussed as follows.

16.6.1 Unbalanced Transportation Problems

So far the methods for determining an initial solution and an optimal solution have been demonstrated within the context of a balanced transportation problem i.e. transportation problem in which total supply is equal to the total demand. In real life, however, more often than not we come across situations in which total supply available, at all the sources, is not equal to the total demand of all the destinations i.e. an unbalanced transportation problem. This unbalanced transportation problem could be solved easily by using the methods discussed earlier after converting it into a balanced transportation problem.

Supply more than Demand - If the total supply is more than the total demand then we introduce a dummy destination with demand exactly equal to the surplus capacity or surplus availability or surplus supply. The transportation cost to dummy destination from different sources is zero as actually no shipments will be made from different sources to dummy destination in reality.

For example, let there are three factories with capacities 250, 150 & 100 units respectively and three warehouses with demands equal to 200, 100 & 150 units respectively. Here since the total supply/capacity is 500 units, more than the total demand, which is 450 units, by 50 units so we introduce a dummy destination (warehouse) with demand equal to

50 units and unit transportation cost from different sources to this destination as zero. After introducing this dummy destination and solving the transportation problem using North West Corner Rule we arrive on the initial basic feasible solution which shown below in the table.

ToFrom	Ware house-I	Ware house -II	Ware house -III	Dummy Ware house	Total Capacity of Factories
Factory - 1	12 (200)	15 (50)	10	0	250
Factory – 2	14	17 (50)	8 (100)	0	150
Factory - 3	11	14	12 (50)	0 (50)	100
Total Demand of Warehouses	200	100	150	50	500

Supply less than Demand - If the total supply is less than the total demand then we introduce a dummy source with supply exactly equal to the excess demand. The transportation cost from dummy source to different destinations is zero as actually no shipments will be made from dummy source to different destinations in reality.

For example, let there are three factories with capacities 200, 100 & 150 units respectively and three warehouses with demands equal to 250, 150 & 100 units respectively. Here since the total demand is 500 units, more than the total supply, which is 450 units, by 50 units so we introduce a dummy source (factory) with supply/capacity equal to 50 units and unit transportation cost from this source to different destinations as zero. After introducing this dummy source and solving the transportation problem using North West Corner Rule we arrive on the initial basic feasible solution which shown below in the table.

ToFrom	Ware house -I	Ware house -II	Ware house -III	Dummy Ware-house	Total Capacity of Factories
Factory - 1	12 (200)	15 (50)	10	0	250
Factory – 2	14	17 (50)	8 (100)	0	150
Factory - 3	11	14	12 (50)	0(50)	100
Total Demandof Warehouses	200	100	150	50	500

16.6.2 Multiple Optimal Solutions

In transportation problem the optimal solution obtained may or may not be unique. If the opportunity costs (in case of MODI method) of all the unoccupied cells are positive then this signifies that the current solution is optimum solution and the problem possesses unique optimum solution. If the opportunity costs of all the unoccupied cells are either positive or zero then this signifies that the current solution is optimum solution but the problem possesses multiple optimal solutions. The alternate optimal solution can be obtained by shipping goods through any of the unoccupied cell having zero opportunity cost. To get the alternate optimal solution we follow the same procedure which is followed for getting improved solution .i.e. we draw a loop starting from the unoccupied cell having zero opportunity cost and try to allocate maximum possible amount in this unoccupied cell. The new solution which we get in this manner is alternate optimal solution.

For example, Let there be four factories with capacities 200, 400, 300 & 200 units respectively and four warehouses with demands equal to 350, 450, 200 & 100 units respectively. The unit transportation costs in rupees along with availability and demand is shown below in table.

ToFrom	Ware house -I	Ware house -II	Ware house -III	Ware house -IV	Total Capacity of Factories
Factory - 1	3	3	3	6	200
Factory – 2	4	5	3	5	400
Factory - 3	4	4	4	6	300
Factory - 4	5	2	3	7	200
Total Demand of ware-house	350	450	200	100	500

After solving the above transportation problem we reach on the optimal solution which is shown below in table along with the opportunity costs associated with each unoccupied cell. The total transportation cost for this solution is Rs. 3700/-.

ToFrom	Ware house -I	Ware house -II	Ware house -III	Ware house -IV	Total Capacity of Factories
Factory - 1	(0) 3(200)	3	(1) 3	(2) 6	200
Factory – 2	4(100)	(1) 5	3 (200)	3 (100)	400
Factory - 3	4(50)	4(350)	(1) 4	(1) 6	300
Factory - 4	(3) 5	2(200)	(2) 3	(4) 7	100
Total Demand of ware-house	350	450	200	100	500

16.6.3 Maximization Type Transportation Problems

Normally a typical transportation problem is of minimization type i.e. a transportation problem in which unit transportation costs are given and transportation schedule from different origins to different destinations is to be determined which minimizes the total transportation cost. But sometimes we also come across transportation problems which are of maximization type i.e. transportation problems in which unit profit between different origins and different destinations is given, instead of unit transportation cost, and we have to find the transportation schedule that maximizes the total profit.

The procedure followed for solving maximization type of transportation problem is same as the one which is followed for solving minimization type of transportation problem but the method is not directly applied on the maximization type of transportation problem. Firstly the maximization type of transportation problem is converted into an equivalent minimization type of transportation problem, and then the transportation method is applied. For converting the problem from maximization type to minimization type we subtract all the profit values from the highest profit and thus reach on the opportunity loss matrix. Now the usual transportation method is applied on this opportunity loss table. Once we reach on the optimal solution of the minimization type of problem we interpret the solution in terms of the original profit table and calculate the maximum profit.

For example: Let there be three factories with capacities 11, 13, & 19 units respectively and four sales depots with requirements equal to 6, 10, 12 & 15 units respectively. The unit profit in rupees for different factory – sales depot combination is shown below in table. Find the number of units to be transported from different factories to different sales depots which maximizes the total profit.

To/From	Sales Depot-I	Sales Depot-II	Sales Depot-III	Sales Depot-IV	Capacity
Factory - 1	22	27	18	30	11
Factory – 2	26	25	29	20	13
Factory - 3	11	16	25	2	19
Requirements	6	10	12	15	

As this is a maximization type of transportation problem so we first convert it into an equivalent minimization type of transportation problem by subtracting all the profit values from the highest profit 30 and thus reach on the opportunity loss table as shown below

To/From	Sales Depot-I	Sales Depot-II	Sales Depot-III	Sales Depot-IV	Capacity
Factory - 1	8	3	12	0	11
Factory – 2	4	5	1	10	13
Factory - 3	19	14	5	28	19
Requirements	6	10	12	15	

Now we apply the usual transportation method on this table and reach on the optimal solution shown below in table.

To/From	Sales Depot-I	Sales Depot-II	Sales Depot-III	Sales Depot-IV	Capacity
Factory - 1	8	3	12	0(11)	11
Factory – 2	4(6)	5(3)	1	10(4)	13
Factory - 3	19	14(7)	5(12)	28	19
Requirements	6	10	12	15	

In order to calculate the maximum profit the original profit table is concerned. The maximum profit for the optimal solution is: 30 x 11 +26 x 6 +25 x 3 +20 x 4 +16 x 7 +25 x 12 = Rs 1053/-

16.6.4 Degeneracy in Transportation Problem

When the number of occupied cells in the solution of transportation problem is less than (m + n – 1), where 'm' is the number of rows and 'n' is the number of columns in the transportation table, then the solution is called a degenerate solution. If the number of occupied cells in the solution is equal to (m + n – 1) then the solution is called a non - degenerate solution.

The degenerate solution or degeneracy could occur either at the initial solution or at any subsequent solution. The problem with degenerate solution is that it cannot be tested for optimality. If we use stepping stone method for checking optimality then this method fails as there are not enough occupied cells for tracing closed loops for all the unoccupied cells (empty cells). On the other hand we cannot use MODI method for checking optimality as we cannot determine all the u_i and v_j values.

In order to solve degenerate problem, first of all degeneracy is removed by introducing an infinitesimally small amount to one or more empty cell (as the case be) and treating the cell as occupied cell. This infinitesimally small amount normally represented by Ý (epsilon) is small enough so as not to violate the supply or demand constraints but significant enough to convert the unoccupied cell into occupied cell. The only thing to be kept in mind while introducing Ý is that it should be introduced in an independent unoccupied cell, originating from which a closed loop cannot be traced. This epsilon is kept in the solution until degeneracy is removed or final solution is reached, whichever occurs first.

Degeneracy in an initial solution - For example, let there are three warehouses with capacities 250, 150 & 100 units respectively and three markets with demands equal to 200, 200 & 100 units respectively. The unit transportation cost along with supply and demand is shown in table below:

To/From	Market - I	Market - II	Market - III	Capacity of Warehouses
Warehouse - 1	12	15	10	250
Warehouse – 2	14	17	8	150
Warehouse - 3	11	14	12	100
Demand of Markets	200	200	100	500

After solving the transportation problem using North West Corner Rule we arrive on the initial basic feasible solution which is shown below in the table.

To/From	Market - I	Market - II	Market - III	Capacity of Warehouses
Warehouse - 1	12 (200)	15 (50)	10	250
Warehouse – 2	14	17 (150)	8	150
Warehouse - 3	11	14	12 (100)	100
Demand of Markets	200	200	100	500

This solution is a degenerate solution as the number of occupied cells is 4, which is less than 5 (3+3-1). So we introduce epsilon in cell (2,3) and treat this cell as occupied cell for subsequent step of the solution procedure. The initial non degenerate solution after introduction of epsilon is shown below in table.

To/From	Market - I	Market - II	Market - III	Capacity of Warehouses
Warehouse - 1	12 (200)	15 (50)	10	250
Warehouse – 2	14	17 (150)	8 (Ý)	150
Warehouse - 3	11	14	12 (100)	100
Demand of Markets	200	200	100	500

Degeneracy in later solution stages - For example, let there are four warehouses with capacities 50, 100, 70 & 30 units respectively and three markets with demands equal to 50, 80 & 100 units respectively. The unit transportation cost along with supply and demand is shown in table below:

To/From	Market - I	Market - II	Market - III	Capacity of Warehouses
Warehouse - 1	7	3	6	50
Warehouse – 2	4	6	8	100
Warehouse - 3	5	8	4	70
Warehouse - 4	8	4	3	30
Demand of Markets	50	80	100	

After introducing dummy market with demand equal to 20 units and solving the transportation problem using VAM we arrive on non - degenerate initial basic feasible solution which is shown below in the table.

To/ From	Market - I	Market - II	Market - III	Dummy Market	Capacity of Ware houses
Warehouse - 1	7	3 (50)	6	0	50
Warehouse – 2	4 (50)	6 (30)	8 (20)	0	100
Warehouse - 3	5	8	4 (50)	0 (20)	70
Warehouse - 4	8	4	3 (30)	0	30
Demand of Markets	50	80	100	20	

We use MODI method and calculate the opportunity costs associated with each unoccupied cell. The initial solution along with opportunity costs is shown below in table.

ToFrom	Market - I	Market - II	Market - III	Dummy Market	Capacity of Ware houses
Warehouse - 1	[6] 7	3 (50)	[1] 6	[-1] 0	50
Warehouse – 2	4 (50)	6 (30)	8 (20)	[-4] 0	100
Warehouse - 3	[5] 5	[6] 8	4 (50)	0 (20)	70
Warehouse - 4	[9] 8	[3] 4	3 (30)	[1] 0	30
Demand of Markets	50	80	100	20	

Since there are some negative opportunity costs associated with unoccupied cells therefore the current solution is not optimum. We pick cell (2,4) as it contains the largest negative opportunity cost. The variable corresponding to this cell is to be included in the next solution i.e. in the next transportation table this cell will become occupied. We draw a closed path or loop starting from this unoccupied cell shown below in table.

To/ From	Market -I	Market -II	Market -III	Dummy Market	Capacity of Ware houses
Warehouse - 1	[6] 7	3 (50)	[1] 6	[-1] 0	50
Warehouse – 2	4 (50)	6 (30)	8 {-} (20)	[-4] 0 (+)	100
Warehouse - 3	[5] 5	[6] 8	4 {+} (50)	0 {-} (20)	70
Warehouse - 4	[9] 8	[3] 4	3 (30)	[1] 0	30
Demand of Markets	50	80	100	20	

The maximum amount which can be transported through cell (2,4) is 20 units and the next solution is shown below in table. This new solution is degenerate as the number of occupied cells is 6 which is less than 7 (m+n-1). This happened because one cell (2,4) became occupied but two cells (2,3) and (3,4) became unoccupied, simultaneously.

To/From	Market - I	Market - II	Market - III	Dummy Market	Capacity of Warehouses
Warehouse - 1	7	3 (50)	6	0	50
Warehouse – 2	4 (50)	6 (30)	8	0 (20)	100
Warehouse - 3	5	8	4 (70)	0	70
Warehouse - 4	8	4	3 (30)	0	30
Demand of Markets	50	80	100	20	

In order to solve this we introduce epsilon in cell (3,4) and treat this cell as occupied cell for subsequent step of the solution procedure.

After carrying out the necessary steps of optimality testing, we finally reach on optimal solution shown below in table.

ToFrom	Market -I	Market -II	Market -III	Dummy Market	Capacity of Ware houses
Warehouse - 1	7	3 (50)	6	0	50
Warehouse – 2	4 (50)	6 (30)	8	0 (20)	100
Warehouse - 3	5	8	4 (70)	0	70
Warehouse - 4	8	4 (Ý)	3 (30)	0	30
Demand of Markets	50	80	100	20	

The total transportation cost for this solution is Rs 900/-

16.6.5 Prohibited Routes

Some times in transportation problems we find that certain routes could not be used due to unfavourable conditions like floods, strike, collapsed bridge, etc. When this happens the problem is said to have prohibited routes. To handle such problem we assign a very high cost (say M) to the prohibited route and solve the transportation problem using the methods discussed earlier in usual manner. The assignment of very high cost to the prohibited route ensures that this route does not become a part of our optimal solution.

16.7 Game Theory

In real life it is a common observation that decisions are not made in isolation whether we talk of individuals, business organizations, teams, armies, nations, etc. The decisions taken and their outcome depend upon the decisions taken by others i.e. to say that we operate in an environment of interdependence where what decision/action/strategy we adopt depends upon the decision/action/strategy adopted by others. Examples are chess game played between two players, business organizations competing in the market, diplomats negotiating a treaty, war between nations, management – workers negotiations, etc.

Game theory, which dates back to 1944 when John Von Neumann and Oskar Morgenstern published their book "Theory of Games and Economic Behaviour," is the branch of decision theory that attempts to study decision making situations where two or more rational opponents are involved under conditions of conflict and competition. In this theory one's alternative action is determined after taking into consideration all possible alternatives available to the opponent. This theory tells the processes and principles to be followed by which a particular action should be chosen while dealing with rational opponents.

A game refers to a situation in which two or more competitors (opponents/players/decision makers) are competing to achieve their respective goals or objectives and their fates are intertwined i.e. the outcome of their decision/move/strategy/action depends upon the decision/move/strategy/action adopted by the others.

Consider the famous example of a game known as the Prisoner's Dilemma. Suppose the police has arrested two people whom they know have committed an armed robbery together. The two people are interrogated by police in separate rooms and are each told the following:

- If you both confess, you will each go to jail for 10 years.
- If only one of you confesses, he gets only 1 year and the other gets 25 years.
- If neither of you confesses, you each get 3 years in jail.

In this case the dilemma of the prisoner is whether to confess or not? As the outcome of his action depends upon the action taken by the other and his problem is that he does not know what action the other has taken or will take.

16.8 Basic Concepts

The game model which is used for solving competing and conflicting situations makes use of the following concepts:

Number of players: When the number of players (persons/competitors) involved in a game is two then it is called two person game while when the number of players (persons/competitors) playing is 'n' then the game is known as 'n' person game.

Payoff: Payoff refers to the outcome of the game when the players involved adopt a particular strategy and counter strategy.

Payoff matrix: The payoffs in terms of gains/losses associated with different combinations of strategies and counter strategies adopted by different players can be represented in a matrix form, known as payoff matrix, in which the row designations are the strategies/actions available to player A and column designations are the strategies/actions available to player B.

Player B

$$
\text{Player A} \quad
\begin{array}{c}
\\
A_1 \\
A_2 \\
- \\
A_i \\
- \\
A_m
\end{array}
\begin{pmatrix}
B_1 & B_2 & - & B_j & - & - & B_n \\
a_{11} & a_{12} & . & a_{1j} & . & . & a_{1n} \\
a_{21} & a_{22} & . & a_{2j} & . & . & a_{2n} \\
- & - & - & - & - & - & - \\
a_{i1} & a_{i2} & . & a_{ij} & . & . & a_{in} \\
- & - & - & - & - & - & - \\
a_{m1} & a_{m2} & . & a_{mj} & . & . & a_{mn}
\end{pmatrix}
$$

In the payoff matrix, shown above, A_1, A_2 ..... A_m are the strategies/ actions which player A can adopt and B_1, B_2 ..... B_n are the strategies/ actions which player B can adopt. a_{ij} is the payoff/outcome which results for player A when A adopts strategy A_i and B adopts counter strategy B_j.

Strategy: A strategy is the course of action which a player can choose/adopt during the game. The two types of strategies are (i) Pure Strategy and (ii) Mixed Strategy

Pure Strategy - If a player adopts the same strategy each time then it is known as pure strategy. In case of pure strategy both the players know precisely what strategy or counter strategy the other player is going to adopt i.e. a deterministic condition exists in which the objective is to maximize the gains or minimize the losses.

Mixed Strategy – In case of mixed strategy a player adopts a combination of strategies. In this the opponent is always guessing as to which action/strategy will be adopted by the other in a particular situation i.e. a probabilistic condition exists in which the objective is to maximize the gains or minimize the losses.

Principle of Dominance – In a game sometimes on comparison of strategies available to players we find that some of their strategies dominate certain other strategies in all the situations i.e. in all situations the outcomes that result from adopting a particular strategy are better or same as the outcomes associated with the adoption of some other strategy.

For example, in case of player A, for the payoff matrix given below, we find that in all circumstances the outcomes associated with adoption of strategy A_1 are more than or equal to the outcomes associated with adoption of strategy A_2. So in this case A will never adopt A_2 and we say that strategy A_1 dominates strategy A_2 or strategy A_2 is dominated by strategy A_1.

		Player B		
		B_1	B_2	B_3
Player A	A_1	7	6	8
	A_2	4	3	8
	A_3	10	5	-2

In case of player B, for the payoff matrix given above, we find that in all circumstances the outcomes associated with adoption of strategy B_2 are less than the outcomes associated with adoption of strategy B_1 i.e. the losses of player B are less when it adopts strategy B_2 in place of strategy B_1. So in this case B will never adopt B_1 and we say that strategy B_2 dominates strategy B_1 or strategy B_1 is dominated by strategy B_2.

This principle of dominance helps us in reducing the size of the matrix by eliminating strategies that are dominated by other strategies and thus helps us in solving the game.

Two-Person, Zero-Sum Game - A game in which just two players (player A and player B) are involved is known as 'two-person, zero-sum game', if the gains of one player are equal to the losses of the other i.e. in a 'two-person, zero-sum game' the sum of the gains of two players is always equal to zero. In this type of game the gains of one player are exactly equal to the losses of other player. Two-person, zero-sum games are also known as rectangular games as these are generally presented through a payoff matrix in a rectangular form.

For example, there are two firms A and B competing with each other to gain market share. Both these firms have three strategies available to them, namely, A_1 (low advertising), A_2 (medium advertising), A_3 (heavy advertising) and B_1 (low advertising), B_2 (medium advertising), B_3 (heavy advertising), respectively. The payoffs corresponding to different combinations of strategy and counter strategy adopted by the competitors is given below in table no. For example, medium advertising by A i.e. A adopts strategy A_2 and low advertising by B i.e. B adopts strategy B_1 will result in 4 percent increase in market share for A or 4 percent decrease in market share for B. Similarly, heavy advertising by A i.e. A adopts strategy A_3 and heavy advertising by B i.e. B adopts strategy B_3 will result in 2 percent decrease in market share for A or 2 percent increase in market share for B

		Player B		
		B_1	B_2	B_3
Player A	A_1	7	6	8
	A_2	4	3	8
	A_3	10	5	-2

The above payoff matrix representing the payoffs for a two person zero sum game has been developed from A's point of view – a positive payoff indicates that firm A gains market share at the expense of firm B and a negative payoff indicates that firm B gains market share at the expense of A.

16.9 Solution of Games with Saddle Point

In order to solve the problem represented by table no. we try to find out which strategy/strategies will be adopted by the two players/firms so as to maximize its gains (firm A) or minimize its losses (firm B) being fully aware about the strategies which are available to itself and the competitor; and also the resulting payoffs.

One approach followed by competitors, to solve the problem, is to adopt the best strategy assuming that the worst will happen. In this case A thinks that if it adopts strategy A_1 then B will adopt the best counter strategy B_2 and A's payoff will be 6 (percent increase in market share); if it adopts strategy A_2 then B will again adopt B_2 as it is the best counter strategy and A's payoff will be 3; and if it adopts strategy A_3 then B will adopt the best counter strategy B_3 and A's payoff will be -2 i.e. 2 percent decrease in market share. Keeping these payoffs in mind A will choose the strategy which will result in maximum payoff out of these minimum payoffs i.e. A will choose strategy A_1. This approach of selecting strategy in which we pick the maximum payoff out of the minimum payoffs is known as **maximin strategy**.

Similarly, B will be cautious in its approach and thinks that if it adopts strategy B_1 then A will adopt the best counter strategy A_3 and A's payoff will be 10 (percent increase in market share); if it adopts strategy B_2 then A will adopt A_1 as it is the best counter strategy and A's payoff will be 6; and if it adopts strategy B_3 then A will adopt either A_1 or A_2 and A's payoff in either case will be 8. Keeping these payoffs in mind B will choose the strategy which will result in minimum payoff out of these maximum payoffs to A i.e. B will choose strategy B_2. This approach of selecting strategy in which we pick the minimum payoff out of the maximum payoffs is known as **minimax strategy**.

In this case the payoff for A when it follows maximin strategy is 6, which is same as the payoff that accrues to A when B adopts best strategy using minimax criterion. Thus when A adopts pure strategy A_1 and B adopts pure strategy B_2 the payoff is 6, which is known as **the value of the game** and represents the final payoff to the winning player. This payoff or point where maximin value is equal to minimax value is known as the **saddle point** and the game is solved.

Operationally in order to determine the saddle point we adopt the following process:

- Choose the minimum element of every row of the payoff matrix and mark them through circles.
- Choose the maximum element of every column of the payoff matrix and mark them by squares.

- If there seems there an element in the payoff matrix with a circle and a square jointly then that position is known as saddle point and the element is the value of the game.

16.10 Solution of Games without Saddle Point

The problems in which maximum of row minima is not equal to minimum of column maxima are mixed strategy problems. In these problems saddle point does not exist and the players use a combination of strategies instead of pure strategy. The players employ two or more strategies with some definite probabilities associated with each of them and selecting one of them at a time.

A mixed strategy problem could be solved using algebraic method, graphical method or linear programming method.

16.10.1 Algebraic Method – Let there be a two person zero sum game with the following payoff matrix.

		Player B		
	Strategy	B_1	B_2	Probability
Player A	A_1	a_{11}	a_{12}	p
	A_2	a_{21}	a_{22}	1 - p
	Probability	q	1 - q	

It is not possible to further reduce this matrix applying principle of dominance. In this game player A adopts strategies A_1 and A_2 with probabilities p and 1 – p; and player B adopts strategies B_1 and B_2 with probabilities q and 1 – q.

Expected payoff of player A when B adopts strategy B_1 is equal to $a_{11}\,p + a_{21}\,(1-p)$ and expected payoff of player A when B adopts strategy B_2 is equal to $a_{12}\,p + a_{22}\,(1-p)$.

Now A shall assign that value to p which will make its expected payoff equally good whether B adopts strategy B_1 or B_2. This value of p could be found by equating the two expected payoffs i.e.

$$a_{11}\,p + a_{21}\,(1\text{-}p) = a_{12}\,p + a_{22}\,(1\text{-}p)$$

$$\Longrightarrow \quad p = (a_{22} - a_{21}) \;/\; \{(a_{11}+a_{22}) - (a_{12}+a_{21})\}$$

and $$1 - p = (a_{11} - a_{12}) \;/\; \{(a_{11}+a_{22}) - (a_{12}+a_{21})\}$$

Similarly by equating the expected payoffs of player B, we get

$$q = (a_{22} - a_{12}) \;/\; \{(a_{11}+a_{22}) - (a_{12}+a_{21})\}$$

and $$1 - q = (a_{11} - a_{21}) / \{(a_{11}+a_{22}) - (a_{12}+a_{21})\}$$

The value of the game is obtained by putting the value of p in one of the equations relating to expected gain of player A, which comes out to be

$$V = (a_{11}a_{22} - a_{21}a_{12}) / \{(a_{11}+a_{22}) - (a_{12}+a_{21})\}$$

For example, there are two firms A and B competing with each other to gain market share. Both these firms have three strategies available to them, namely, A_1 (low advertising), A_2 (medium advertising), A_3 (heavy advertising) and B_1 (low advertising), B_2 (medium advertising), B_3 (heavy advertising), respectively. The payoffs corresponding to different combinations of strategy and counter strategy adopted by the competitors is given below in table no.

		Player B		
	Strategy	B_1	B_2	B_3
Player A	A_1	7	6	8
	A_2	4	3	8
	A_3	10	9	-2

On applying principle of dominance we find that in case of player A, strategy A_1 dominates strategy A_2 so we delete second row corresponding to A_2. In case of player B we find that strategy B_2 dominates strategy B_1 so we delete first column corresponding to B_1. The new reduced payoff matrix which cannot be further reduced is given below:

		Player B	
	Strategy	B_2	B_3
Player A	A_1	6	8
	A_3	9	-2

Solving this payoff matrix applying algebraic method we find

$$p = (-2 - 9) \ / \ \{(6 - 2) - (8 + 9)\}$$

or $\quad p = -11 \ / - 13$

or $\quad p = 11/13 \quad \text{and} \quad 1 - p = 2/13$

$$q = (-2 - 8) \ / \ \{(6 - 2) - (8 + 9)\}$$

or $\quad q = -10 \ / - 13$

or $\quad q = 10/13 \quad \text{and} \quad 1 - q = 3/13$

The value of the game,

$$V = (-12 - 72) \ / \ \{(4) - (17)\}$$

or $\quad V = - 84 \ / - 13$

or $\quad V = 84/13$

16.10.2 Graphical Method – Now we shall talk about game theory problems in which the payoff matrix is either in the form of 2 x n or m x 2 i.e. one of the players has only two strategies to choose from and other player is having more than two strategies from which it can choose. It could be that the original problem was having payoff matrix in this form or it has been reduced to this form applying principle of dominance.

The graphical method is used to reduce this 2 x n or m x 2 payoff matrix to 2 x 2 payoff matrix by identifying and eliminating dominated strategies and then solving the problem using algebraic method. We shall illustrate the solution procedure of 2 x n or m x 2 games with the help of examples.

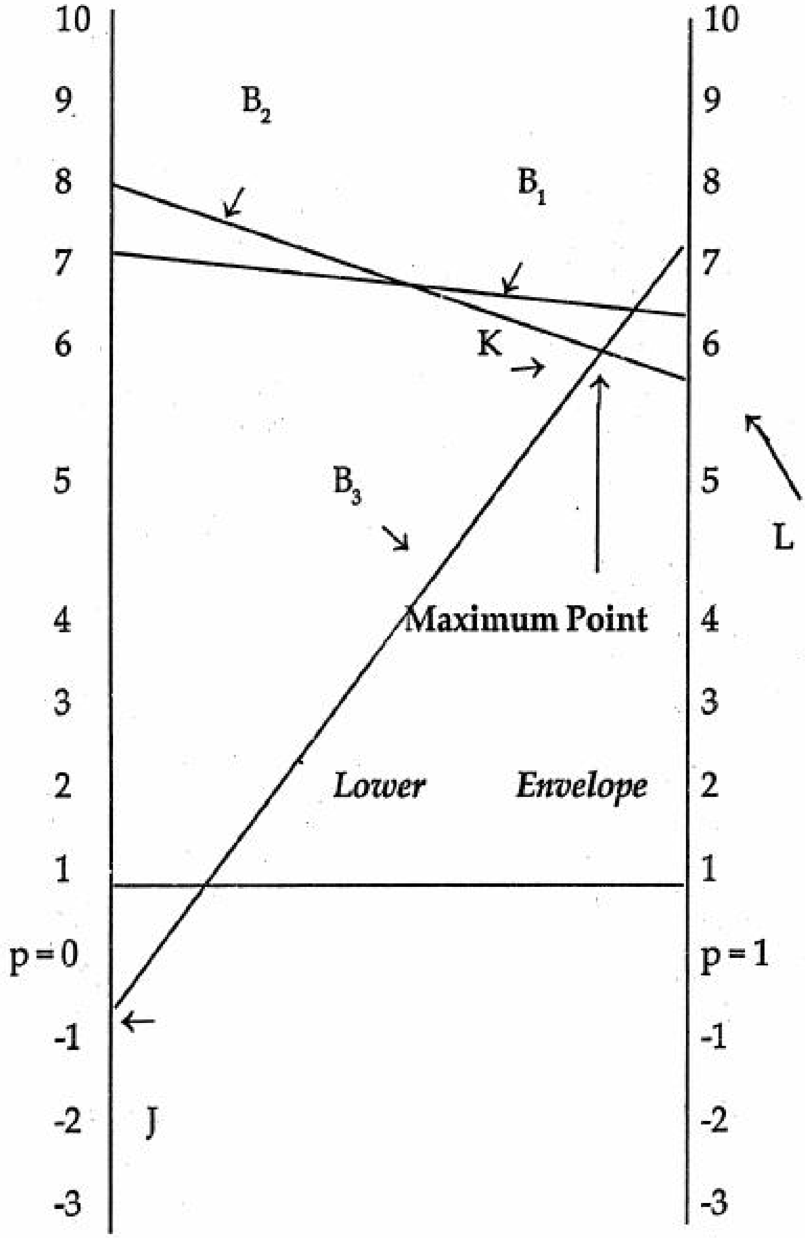
2 x n payoff matrix problem – Lets again take the example of two firms A and B competing with each other to gain market share with slightly different payoff matrix which is given below in table no.

		Player B		
	Strategy	B_1	B_2	B_3
Player A	A_1	7	6	8
	A_2	4	3	8
	A_3	8	9	-2

On applying principle of dominance we find that in case of player A, strategy A_1 dominates strategy A_2 so we delete second row corresponding to A_2. The new reduced payoff matrix which cannot be further reduced is given below:

		Player B		
	Strategy	B_1	B_2	B_3
Player A	A_1	7	6	8
	A_3	8	9	-2

The expected payoffs of player A when it uses A_1 & A_2 with probabilities p & 1 – p; and B adopts B_1, B_2 and B_3 are $7p + 8(1-p)$, $6p + 9(1-p)$ and $8p - 2(1-p)$, respectively. We can represent these by graphically plotting each payoff as function of p. For plotting we take p on x axis and payoffs on y axis and as shown below:



The lines marked B_1, B_2 and B_3 represent the expected payoffs of player A when B adopts strategies B_1, B_2 and B_3 for different values of p. The boundary of lower envelope marked by JKL represent the minimum payoffs which will accrue to A irrespective of the strategy adopted by B for different values of p. A will like to maximize its minimum payoff. In the figure this maximum payoff occurs at point K of lower envelope.

This point occurs at the intersection of lines related with strategies B_2 and B_3. Hence the two strategies that player B will be adopting are B_2 and B_3. Therefore now the 2 x 2 payoff matrix which needs to be solved is

		Player B	
	Strategy	B_2	B_3
Player A	A_1	6	8
	A_3	9	-2

The values of probabilities and the value of game after using algebraic method are:

p = 11/13 and 1 – p = 2/13; and the value of game V = 84/13

m x 2 payoff matrix problem – We can solve mx2 payoff matrix problem on similar lines as we have solved 2xn problem. The difference being that in case of mx2 problem we try to find the two strategies that player A will adopt from m strategies available to it. In order to find these strategies we represent the payoffs that accrue to player B when A adopts different strategies graphically by taking payoffs on y axis and q on x axis. In this case we are concerned with upper envelope as B tries to minimize its maximum losses. Then we try find the lowest point on this upper envelope. The two lines that intersect at this point represent the two strategies that will be adopted by player A. After this we can apply algebraic method for finding probabilities and value of the game.

16.10.3 Linear Programming Method -

The linear programming method is used in solving mixed strategies games of greater than (2 x 2) size. Consider an m x n payoff matrix in which player A has m strategies and player has n strategies. The elements of pay off matrix be $\{(a_{ij}) ; i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n\}$. Let p_i be the probability of selection of strategy i by player A and q_j be the probability of selection of strategy j by player B. This game could be defined as linear programming problem with the objective to maximize value of game V with following constraints:

$$a_{11}p_1 + a_{21}p_2 + \ldots\ldots\ldots + a_{m1}p_m \; e'' \; V$$
$$a_{12}p_1 + a_{22}p_2 + \ldots\ldots\ldots + a_{m2}p_m \; e'' \; V$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{1n}p_1 + a_{2n}p_2 + \ldots\ldots\ldots + a_{mn}p_m \; e'' \; V$$
$$p_1 + p_2 + \ldots\ldots + p_{m=1}$$
$$p_1, p_2, \ldots\ldots, p_m \; e'' \; 0$$

Assuming that V is positive (which would be if all a_{ij}'s are positive), we define a new variable $x_i = p_i/V$ and divide each constraint by V. Then we can restate linear programming problem as

Minimise $Z = 1/V$ or $x_1 + x_2 + \ldots\ldots\ldots + x_m$

Subject to the constraints

$$a_{11}x_1 + a_{21}x_2 + \ldots\ldots\ldots + a_{m1}x_m \; e'' \; 1$$

$a_{12}x_{1} + a_{22}x_{2} + \ldots\ldots\ldots\ldots\ldots + a_{m2}x_{m}$ e″ 1

....

$a_{1n}x_{1} + a_{2n}x_{2} + \ldots\ldots\ldots\ldots\ldots + a_{mn}x_{m}$ e″ 1

$x_1, x_2, \ldots\ldots\ldots, x_m$ e″ 0

Similarly, for player B the problem is to determine q_j so as to maximize (-V) or minimize V subject to

$a_{11}q_{1} + a_{12}q_{2} + \ldots\ldots\ldots\ldots\ldots + a_{1n}q_{n}$ d″ V

$a_{21}q_{1} + a_{22}q_{2} + \ldots\ldots\ldots\ldots\ldots + a_{2n}q_{n}$ d″ V

....

$a_{m1}q_{1} + a_{m2}q_{2} + \ldots\ldots\ldots\ldots\ldots + a_{mn}q_{n}$ d″ V

$q_1 + q_2 + \ldots\ldots\ldots + q_n = 1$

$q_1, q_2, \ldots\ldots\ldots, q_n$ e″ 0

Dividing each inequality and equality by V and putting $y_j = q_j/V$ we can restate the linear programming problem as

Maximise $Z = 1/V$ or $y_1 + y_2 + \quad + y_n$

Subject to the constraints

$a_{11}y_{1} + a_{12}y_{2} + \ldots\ldots\ldots\ldots\ldots + a_{1n}y_{n}$ d″ 1

$a_{21}y_{1} + a_{22}y_{2} + \ldots\ldots\ldots\ldots\ldots + a_{2n}y_{n}$ d″ 1

....

$a_{m1}y_{1} + a_{m2}y_{2} + \ldots\ldots\ldots\ldots\ldots + a_{mn}y_{n}$ d″ 1

$y_1, y_2, \ldots\ldots\ldots, y_n$ e″ 0

After this the problem could be solved in usual manner.

16.11 Key Words

1. Unbalanced transportation problem - A transportation problem in which total supply is not equal to the total demand is called an unbalanced transportation problem.
2. Degenerate Solution - When the number of occupied cells in the solution of transportation problem is less than $(m + n - 1)$, where 'm' is the number of rows and 'n' is the number of columns in the transportation table, then the solution is called a degenerate solution.
3. Game - A game refers to a situation in which two or more competitors (opponents/players/decision makers) are competing to achieve their respective goals or objectives and their fates are intertwined i.e. the outcome of their decision/move/strategy/action depends upon the decision/move/strategy/action adopted by the others.
4. Payoff - Payoff refers to the outcome of the game when the players involved adopt a particular strategy and counter strategy.
5. Payoff matrix - The payoffs in terms of gains/losses associated with different combinations of strategies and counter strategies adopted by different players can be represented in a matrix form known as payoff matrix.

6. **Strategy** - A strategy is a possible course of action open to the player.
7. Pure Strategy - If a player adopts the same strategy each time then it is known as pure strategy. In case of pure strategy both the players know precisely what strategy or counter strategy the other player is going to adopt.
8. Mixed Strategy – In case of mixed strategy a player adopts a combination of strategies. In this the opponent is always guessing as to which action/strategy will be adopted by the other in a particular situation.
9. Two-Person, Zero-Sum Game - A game in which just two players (player A and player B) are involved is known as 'two-person, zero-sum game', if the gains of one player are equal to the losses of the other i.e. in a 'two-person, zero-sum game' the sum of the gains of two players is always equal to zero.
10. Maximin Strategy - The approach of selecting strategy in which we pick the maximum payoff out of the minimum payoffs is known as maximin strategy.
11. Minimax Strategy - The approach of selecting strategy in which we pick the minimum payoff out of the maximum payoffs is known as minimax strategy.
12. **Saddle point** - If a payoff matrix has an entry that is simultaneously a maximum of row minima and a minimum of column maxima, then this entry is called the saddle point of the game.
13. **Value of the game** - The expected pay off when all the players of the game go after their optimal strategies is called as 'value of the game'. If this value is zero then the game is said to be fair.
14. n-person game - There are limited number of competitors such that n e" 2. In the case of n = 2, it is known as two-person game and in case of n > 2, it is known as n-person game.

16.12 Self Assessment Questions

16.12.1 Discuss how MODI method differ from Stepping Stone method.

16.12.2 A cement factory manager is considering the best way to transport cement from three manufacturing centres P, Q, R to depots A, B, C, D and E. The weekly production and demands alongwith transportation costs per tonne are given below:

		Depot					
		A	B	C	D	E	Supply
Manufacturing Centres	P	4	1	3	4	4	60
	Q	2	3	2	2	3	35
	R	3	5	2	4	4	40
	Demand	22	45	20	18	30	

Find the best distribution programme using MODI method.

16.12.3 A transportation problem involves the following costs, supply, and demand:

From	To				
	A	B	C	D	Supply
1	500	750	300	450	12
2	650	800	400	600	17
3	400	700	500	550	11
Demand	10	10	10	10	

Find the optimal solution using Stepping Stone method.

16.12.4 Steel mills in three cities produce the following amounts of steel:

Location	Weekly Production
A. Bareilly	150
B. Agra	210
C. Kanpur	320

These mills supply steel to four cities where manufacturing plants have the following demand:

Location	Weekly Demand
1. Delhi	130
2. Allahabad	70
3. Jaipur	180
4. Hissar	240

Shipping costs per ton of steel are as follows:

From	To			
	1	2	3	4
A	14	9	16	18
B	11	8	7	16
C	16	12	10	22

Because of a truckers' strike, shipments are at present prohibited from Bareilly to Jaipur.

a. Set up a transportation tableau for this problem and determine the initial solution. Identify the method used to find the initial solution.

b. Solve this problem using MODI.

c. Are there multiple optimal solutions? Explain. If so, identify them.

16.12.5 Discuss the importance of game theory to business decisions.

16.12.6 Discuss the maximin and minimax principles of game theory.

16.12.7 What is two person zero sum game? Can there be a non zero sum game also?

16.12.8 Find the saddle point for the following game:

	B_1	B_2	B_3	B_4
A_1	5	8	2	4
A_2	2	6	1	3

16.12.9 Solve the following game involving two firms A and B:

	B_1	B_2	B_3
A_1	10	5	-2
A_2	13	12	15

16.12.10 Solve the following game and find value of game:

	B_1	B_2	B_3
A_1	8	4	-2
A_2	-2	-1	3

16.12.11 Solve the following game and find value of game:

	B_1	B_2
A_1	-7	6
A_2	7	-4
A_3	-4	-2
A_4	8	-6

16.13 Further Readings

1. N D Vohra : Quantitative Techniques in Management(Tata McGraw Hill)
2. V K Kapoor : Operations Research
3. Levine, Berenson, Krehbiel, Render, Stair, & Hanna : Quantitative Techniques for Management (Pearson)
4. Sharma, J.K : Operations Research : Theory and Applications (Macmillan India Ltd)

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U.P.Rajarshi Tandon Open
University, Allahabad

MBA-1.4
Quantitative Techniques for Business Decisions

Block
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विशेषज्ञ -समिति

1. Dr. Omji Gupta, Director SoMS UPRTOU, Allahabad
2. Prof. Arvind Kumar, Prof., Deptt. of Commerce, Lucknow University, Lucknow
3. Prof. Geetika, HOD, SoMS, MNNIT, Allahabad
4. Prof. H.K. Singh, Prof., Deptt. of Commerce, BHU, Varanasi

लेखक

Dr. Gaurav Sankalp SOMS, UPRTOU, Allahabad

सम्पादक

Prof. S.A.Ansari, Ex-Dean, Director and Head MONIRBA, University of Allahabad

परिमापक

अनुवाद की स्थिति में

मूल लेखक	अनुवाद
मूल सम्पादक	भाषा सम्पादक
मूल परिमापक	परिमापक

सहयोगी टीम

संयोजक Dr. Gaurav Sankalp, SoMS, UPRTOU, Allahabad.

नोट : पाठ्य सामग्री में मुद्रित सामग्री के विचारों एवं आकड़ों आदि के प्रति विश्वविद्यालय उत्तरदायी नहीं है।

प्रकाशन --उत्तर प्रदेश राजर्षि टण्डन मुक्त विश्वविद्यालय, इलाहाबाद

प्रकाशन- उत्तर प्रदेश राजर्षि टण्डन मुक्त विश्वविद्यालय, प्रयागराज की ओर से डॉ. अरूण कुमार गुप्ता, कुलसचिव द्वारा पुनः मुद्रित एवं प्रकाशित वर्ष-2020।

मुद्रक- चन्द्रकला यूनिवर्सल प्राइवेट लिमिटेड 42/7 जवाहर लाल नेहरू रोड, प्रयागराज-211002

Block 5 : Quantitative Techniques for Business Decisions

Block Introduction

Block five comprises of four units. Unit seventeen explores the various areas of assignment problem and hungarian method. Unit Eighteen highlights Markov Chain and its application. Unit nineteen deals with point estimation, Internal estimation and time series while unit twenty deals with CPM, PERT and Queuing Theory

Unit 17- The Assignment Problem and the Hungarian Method

Unit Structure

17.0 Objectives

In this unit we will understand the assignment problems and Hungarian model, maxima and minima.

17.1 Introduction-

The Hungarian method is a combinatorial optimization algorithm that solve the assignment problem in polynomial time and which is anticipated afterward primal-dual methods. It was developed and published by Harold Kuhn in 1955, who gave the name "Hungarian method" because the algorithm was mainly based on the earlier works of two Hungarian mathematicians: DénesKnig and JenEgerváry.

James Munkres reviewed the algorithm in 1957 and observed that it is strongly polynomial. Since then the algorithm has been known also as Kuhn–Munkres algorithm or Munkres assignment algorithm. The time complexity of the original algorithm was $O(n^4)$, however Edmonds and Karp, and independently Tomizawa noticed that it can be modified to achieve an $O(n^3)$running time. Ford and Fulkerson extended the method to general transportation problems. In 2006, it was discovered that Carl Gustav Jacobi had solved the assignment problem in the 19th century, and the solution had been published posthumously in 1890 in Latin.

17.2 Theorem

If a number is added to or subtracted from all of the entries of any one row or column of a cost matrix, then on optimal assignment for the resulting cost matrix is also an optimal assignment for the original cost matrix.

17.3 The Hungarian Method

The following algorithm applies to the above theorem to a given n × n cost matrix to find an optimal assignment.

Step 1. Subtract the smallest entry in each row from all the entries of its row.

Step 2. Subtract the smallest entry in each column from all the entries of its column.

Step 3. Draw lines through appropriate rows and columns so that all the zero entries of the cost matrix are covered and the minimum number of such lines is used.

Step 4. Test for Optimality: (i) If the minimum number of covering lines is n, an optimal assignment of zeros is possible and we are finished. (ii) If the minimum number of covering lines is less than n, an optimal assignment of zeros is not yet possible. In that case, proceed to Step 5.

Step 5. Determine the smallest entry not covered by any line. Subtract this entry from each uncovered row, and then add it to each covered column. Return to Step 3.

17.3.1 Example

Mr. Sankalp works as a sales manager for a toy manufacturer, and he currently has three sales people on the road meeting buyers. His sales people are in Lucknow, Kanpur and Allahabad . He want them to travel to three other cities: Meerut, Agra and Aligarh The table below shows the cost of airplane tickets indollars between these cities.

From \ To	Meerut	Agra	Aligarh
Lucknow	250	400	350
Kanpur	400	600	350
Allahabad	200	400	250

Where should you send each of your sales people in order to minimize airfare?

Step 1. Subtract 250 from Row 1, 350 from Row 2, and200 from Row 3.

250	400	350		0	150	100
400	600	350		50	250	0
200	400	250		0	200	50

Step 2. Subtract 0 from Column 1, 150 from Column 2, and 0 from Column 3.

0	150	100		0	0	100
50	250	0		50	100	0
0	200	50		0	50	50

Step 3. Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines.

~~0 0 100~~

~~50 100 0~~

0 50 50

Step 4. Since the minimal number of lines is 3, an optimal assignment of zeros is possible and we are finished.

Since the total cost for this assignment is 0, it must be an optimal assignment.

0 0 100

50 100 0

0 50 50

Here is the same assignment made to the cost matrix.

250 400 350

400 600 350

200 400 250

Example 17.3.2: A construction company has four large bulldozers located at four different garages. The bull dozers are to be moved to four different construction sites. The distances in miles between the bulldozers and the construction sites are given below.

Bulldozer/Site	A	B	C	D
1	90	75	75	80
2	35	85	55	65
3	125	95	90	105
4	45	110	65	115

How should the bulldozers be moved to the construction sites in order to minimize the total distance travelled?

Step 1. Subtract 75 from Row 1, 35 from Row 2, 90 from Row 3, and 45 from Row 4.

90	75	75	80		15	0	0	5
35	85	55	65		0	50	20	25
125	95	90	105	H″	35	5	0	115
45	110	65	115		0	65	20	70

Step 3. Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines.

20 0 5 0

0 45 20 20

35 0 0 5

0 60 50 60

Step 4. Since the minimal number of lines is less than 4, we have to return to Step 5.

Step 5. Note that 20 is the smallest entry not covered by any line. Subtract 20 from each uncovered row.

20	0	5	0		20	0	5	0
0	45	20	20	H"	-20	25	0	0
35	0	0	5		35	0	0	5
0	60	50	60		20	40	30	40

Now Add 20 to covered column

20	0	5	0		40	0	5	0
"20	25	0	0		0	25	0	0
35	0	0	5	H"	55	0	0	5
-20	40	30	40		0	40	30	40

Now return to Step 3.

Step 3. Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines.

~~40 0 5 0~~

~~0 25 0 0~~

~~55 0 0 5~~

~~0 40 30 40~~

Step 4. Since the minimal number of lines is 4, an optimal assignment of zeros is possible and we are finished.

40 0 5 0

0 25 0 0

55 0 0 5

0 40 30 40

Since the total cost for this assignment is 0, it must be an optimal assignment.

Here is the same assignment made to the original cost matrix.

90	75	75	80
35	85	55	65
125	95	90	105
45	110	95	115

So we should send Bulldozer 1 to Site D, Bulldozer 2 to Site C, Bulldozer 3 to Site B, and Bulldozer 4 to Site A.

17.4 Maxima of a function

Let f(x) be a real function defined on an interval I, then, f(x) is said to have the maximum value in I, if there exists a point a in I such that

f(x) = f(a) for all x belongs to I.

17.5 Minima of a function

Let f(x) be a real function defined on an interval I, then, f(x) is said to have the maximum value in I, if there exists a point a in I such that

f(x) = f(a) for all x belongs to I.

Example 17.5.1: Find the maximum and minimum value of the

f(x)= sin3x + 4 for all x belongs to Real no.

Sol: We have,

f(x)= sin3x + 4 for all x belongs to R

Now -1d″sin3xd″1 for all x belongs to R

-1+4d″sin3x + 4 d″1+4 for all x belongs to R

3d″sin3x + 4 d″5 for all x belongs to R

3d″f(x)d″5 for all x belongs to R

Thus maximum value of f(x) is 5 and minimum value is 3.

17.6 Self Assessment Question

Q1. What do you understand by Hungarian Method? Describe in brief with suitable example.

Q2. Give important feature of Hungarian method.

Q3. Describe maxima and minima with the help of suitable graph.

Q4. Find the maximum and minimum value of following functions

a) f(x)=3x^2+ 6x+8 for all x belongs to Real no.

b) f(x)=-|x-1|+5 for all x belongs to Real no.

c) f(x)=x^3+1 for all x belongs to Real no.

d) f(x)=sin(sin x) for all x belongs to Real no.

17.7 Summary

The Assignment Problem: Suppose we have n resources to which we want to assign n tasks on a one-to-one basis. Suppose also that we know the cost of assigning a given resource to a given task. We wish to find an optimal assignment–one which minimizes total cost.

The Mathematical Model: Let ci, j be the cost of assigning the ith resource to the jth task. We define the cost matrix to be the n × n matrix

c1,1c1,2 · · · c1,n

C = c2,1c2,2 · · · c2,n

...

...

...

cn,1 cn,2 · · ·cn,n

An assignment is a set of n entry positions in the cost matrix, no two of which lie in the same row or column.The sum of the n entries of an assignment is its cost. An assignment with the smallest possible cost is called an optimal assignment.

The Hungarian Method: The Hungarian method is an algorithm which finds an optimal assignmentfor a given cost matrix.

17.8 Further Readings

1. N D Vohra : Quantitative Techniques in Management(Tata McGraw Hill)
2. V K Kapoor : Operations Research
3. Levine, Berenson, Krehbiel, Render, Stair, & Hanna : Quantitative Techniques for Management (Pearson)
4. Sharma , J.K : Operations Research : Theory and Applications (Macmillan India Ltd)

experiments. In principle, when we observe a sequence of chance experiments, all of the past outcomes could influence our predictions for the next experiment. For example, this should be the case in predicting a student's grades on a sequence of exams in a course. But to allow this much generality would make it very difficult to prove general results.

In 1907, A. A. Markov began the study of an important new type of chance process. In this process, the outcome of a given experiment can affect the outcome of the next experiment. This type of process is called a Markov chain.

18.2 Specifying a Markov Chain

We describe a Markov chain as follows: We have a set of *states*, S = *fs*1; *s*,,,,,,,,,,,,,,,; *srg*. The process starts in one of these states and moves successively from one state to another. Each move is called a *step*. If the chain is currently in state *si*, then it moves to state *sj* at the next step with a probability denoted by *pij*, and this probability does not depend upon which states the chain was in before the current state.

The probabilities *pij* are called *transition probabilities*. The process can remain in the state it is in, and this occurs with probabilit, *pii*. An initial probability distribution, defined on *S*, specifies the starting state. Usually this is done by specifying a particular state as the starting state.

R. A. Howard1 provides us with a picturesque description of a Markov chain as a frog jumping on a set of lily pads. The frog starts on one of the pads and then jumps from lily pad to lily pad with the appropriate transition probabilities.

18.2.1 Example According to Kamini, Shyam, and Tarun, the Land of Bundelkhand is blessed by many things, but not by good weather. They never have two nice days in a row. If they have a nice day, they are just as

Unit 18- Markov Chains

Unit Structure

18.0 Objective

In this unit we will learn the Markov's Model and Markov chain.

- Specifying a Markov Chain
- Transition Matrix
- Absorbing Markov Chains
- Canonical Form
- Ergodic Markov Chains

18.1 Introduction

Modern probability theory studies chance processes for which the knowledge of previous outcomes influences predictions for future experiments. In principle, when we observe a sequence of chance experiments, all of the past outcomes could influence our predictions for the next experiment. For example, this should be the case in predicting a student's grades on a sequence of exams in a course. But to allow this much generality would make it very difficult to prove general results.

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18.2.1 Example According to Kamini, Shyam, and Tarun, the Land of Bundelkhand is blessed by many things, but not by good weather. They never have two nice days in a row. If they have a nice day, they are just as likely to have sunny or raing the next day. If they have sunny or rain, they have an even chance of having the same the next day. If there is change from sunny or rain, only half of the time is this a change to a nice day. With this information we form a Markov chain as follows.

We take as states the kinds of weather R, N, and S. From the above information we determine the transition probabilities. These are most conveniently represented in a square array as

18.3 Transition Matrix

The entries in the first row of the matrix P in Example 18.3.1

represent the probabilities for the various kinds of weather following a rainy day. Similarly, the entries in the second and third rows represent the probabilities for the various kinds of weather following nice and sunny days, respectively. Such a square array is called the matrix of transition probabilities, or the transition matrix .

We consider the question of determining the probability that, given the chain is in state i today, it will be in state j two days from now. We denote this probability by $p(2)$ ij . In Example 18.3.1 , we see that if it is rainy today then the event that it is sunny two days from now is the disjoint union of the following three events: 1) it is rainy tomorrow and sunny two days from now, 2) it is nice tomorrow and sunny two days from now, and 3) it is sunny tomorrow and sunny two days from now. The probability of the first of these events is the product of the conditional probability that it is rainy tomorrow, given that it is rainy today, and the conditional probability that it is sunny two days from now, given that it is rainy tomorrow. Using the transition matrix **P**, we can write this product as $p11p13$. The other two events also have probabilities that can be written as products of entries of **P**. Thus, we have

$$p_{13} = p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33} :$$

This equation should remind the reader of a dot product of two vectors; we are dotting the first row of **P** with the third column of **P**. This is just what is done in obtaining the 1; 3-entry of the product of **P** with itself. In general, if a Markov chain has r states, then

$$p_{ij} = \sum_{k=1}^{r} p_{ik}\, p_{kj}$$

The following general theorem is easy to prove by using the above observation and induction.

18.3.1 Theorem Let **P** be the transition matrix of a Markov chain. The ijth entry $p(n)$ ij of the matrix **P**n gives the probability that the Markov chain, starting in state si, will be in state sj after n steps.

18.3.2 Example (Example 18.3.1 continued) Consider again the weather in the Land of Bundelkhand. We know that the powers of the transition matrix give us interesting information about the process as it evolves. We shall be particularly interested in the state of the chain after a large number of steps. The program Matrix Powers computes the powers of P.

We have run the program Matrix Powers for the Land of Bundelkhand example to compute the successive powers of P from 1 to 6. The results are shown in Table 18.3.1 . We note that after six days our weather predictions are, to three-decimal-place accuracy, independent of today's weather. The probabilities for the three types of weather, R, N, and S, are .4, .2, and .4 no matter where the chain started. This is an example of a type of Markov chain called a *regular* Markov chain. For this type of chain, it is true that long-range predictions are independent of the starting state.

$$p^1 = \begin{pmatrix} .500 & .250 & .250 \\ .500 & .000 & .500 \\ .250 & .250 & .500 \end{pmatrix}$$

$$p^2 = \begin{pmatrix} .438 & .188 & .375 \\ .375 & .250 & .375 \\ .375 & .188 & .438 \end{pmatrix}$$

$$p^3 = \begin{pmatrix} .406 & .203 & .391 \\ .406 & .188 & .406 \\ .391 & .203 & .406 \end{pmatrix}$$

$$p^4 = \begin{pmatrix} .402 & .199 & .398 \\ .398 & .203 & .398 \\ .398 & .199 & .402 \end{pmatrix}$$

$$p^5 = \begin{pmatrix} .400 & .200 & .399 \\ .400 & .199 & .400 \\ .399 & .200 & .400 \end{pmatrix}$$

$$p^6 = \begin{pmatrix} .400 & .200 & .400 \\ .400 & .200 & .400 \\ .400 & .200 & .400 \end{pmatrix}$$

First ,second, third row represents rain, nice ,and sunny days respectively while First ,second, third column represents rain, nice and sunny days respectively

Table 11.1: Powers of the Land of Bundelkhand transition matrix.

18.3.3 Theorem Let **P** be the transition matrix of a Markov chain, and let **u** be the probability vector which represents the starting distribution. Then the probability that the chain is in state *si* after *n* steps is the *i*th entry in the vector

$$\mathbf{u}^{(n)} = \mathbf{u}\mathbf{P}^n$$

We note that if we want to examine the behavior of the chain under the assumption that it starts in a certain state s_i, we simply choose **u** to be the probability vector with ith entry equal to 1 and all other entries equal to 0.

18.3.4 Example In the Land of Bundelkhand example (Example 11.1) let the initial probability vector **u** equal (1/3; 1/3; 1/3). Then we can calculate the distribution of the states after three days using Theorem 11.2 and our previous calculation of $\mathbf{P}^3$. We obtain

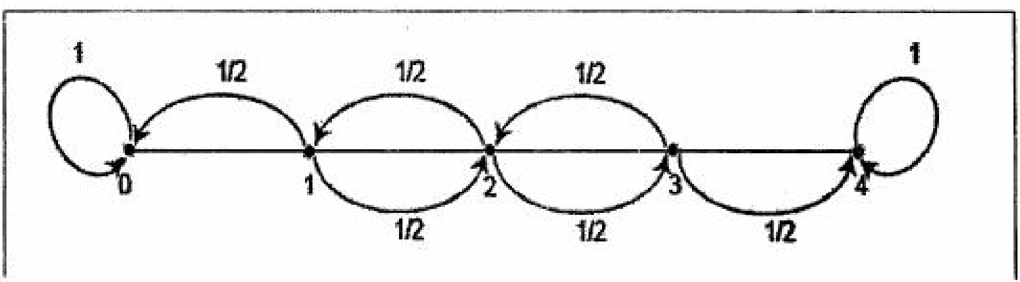
$$\mathbf{u}^{(3)} = \mathbf{u}\mathbf{P}^3 = (1/3; 1/3; 1/3) \begin{matrix} .406 & .203 & .391 \\ .406 & .188 & .406 \\ .391 & .203 & .406 \end{matrix}$$

$$= (.401, \ .188 \ \ .401)$$

18.4 Absorbing Markov Chains

The subject of Markov chains is best studied by considering special types of markov chains. The first type that we shall study is

called an *absorbing Markov chain.*



Drunkard's Walk

Definition A state s_i of a Markov chain is called *absorbing* if it is impossible to leave it (i.e., $p_{ii} = 1$). A Markov chain is *absorbing* if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state (not necessarily in one step).

Definition In an absorbing Markov chain, a state which is not absorbing is called *transient*.

18.5 Drunkard's Walk

18.5.1 Example A man walks along a four-block stretch of Park Avenue (see Figure 11.3). If he is at corner 1, 2, or 3, then he walks to the left or right with equal probability. He continues until he reaches corner 4, which is a bar, or corner 0, which is his home. If he reaches either home or the bar, he stays there.

We form a Markov chain with states 0, 1, 2, 3, and 4. States 0 and 4 are absorbing states. The transition matrix is then

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

The states 1, 2, and 3 are transient states, and from any of these it is possible to reach the absorbing states 0 and 4. Hence the chain is an absorbing chain. When a process reaches an absorbing state, we shall say that it is *absorbed*.

The most obvious question that can be asked about such a chain is: What is the probability that the process will eventually reach an absorbing state? Other interesting questions include: (a) What is the probability that the process will end up in a given absorbing state? (b) On the average, how long will it take for the process to be absorbed? (c) On the average, how many times will the process be in each transient state? The answers to all these questions depend, in general, on the state from which the process starts as well as the transition probabilities.

18.6 Canonical Form

Consider an arbitrary absorbing Markov chain. Renumber the states so that the transient states come first. If there are r absorbing states and t transient states, the transition matrix will have the following *canonical form*.

$$
\mathbf{P} = \begin{array}{c} \\ \text{TR.} \\ \text{ABS.} \end{array} \begin{array}{c|c} \text{TR.} & \text{ABS.} \\ \hline \mathbf{Q} & \mathbf{R} \\ \hline \mathbf{0} & \mathbf{I} \end{array}
$$

Here **I** is an r-by-r indentity matrix, **0** is an r-by-t zero matrix, **R** is a nonzero t-by-r matrix, and **Q** is an t-by-t matrix. The first t states are transient and the last r states are absorbing.

In Section 11.1, we saw that the entry $p(n)$ ij of the matrix **P**n is the probability of being in the state sj after n steps, when the chain is started in state si. A standard matrix algebra argument shows that **P**n is of the form

$$
\mathbf{P}^n = \begin{array}{c} \\ \text{TR.} \\ \text{ABS.} \end{array} \begin{array}{c|c} \text{TR.} & \text{ABS.} \\ \hline \mathbf{Q}^n & * \\ \hline \mathbf{0} & \mathbf{I} \end{array}
$$

where the asterisk . stands for the t-by-r matrix in the upper right-hand corner of **P**n: (This submatrix can be written in terms of **Q** and **R**, but the expression is complicated and is not needed at this time.) The form of $\mathbf{P}^n$ shows that the entries of $\mathbf{Q}^n$ give the probabilities for being in each of the transient states after n steps for each possible transient starting state. For our first theorem we prove that the probability of being in the transient states after n steps approaches zero. Thus every entry of $\mathbf{Q}^n$ must approach zero as n approaches infinity (i.e, $\mathbf{Q}^n$ *à* **0**).

In the following, if **u** and **v** are two vectors we say that **u** d" **v** if all components of **u** are less than or equal to the corresponding components of **v**. Similarly, if **A** and **B** are matrices then **A** d" **B** if each entry of **A** is less than or equal to the corresponding entry of **B**.

18.7 Ergodic Markov Chains

A second important kind of Markov chain we shall study in detail is an *ergodic* Markov chain, defined as follows.

A Markov chain is called an *ergodic* chain if it is possible to go from every state to every state (not necessarily in one move). In many books, ergodic Markov chains are called *irreducible*.

A Markov chain is called a *regular* chain if some power of the transition matrix has only positive elements. In other words, for some n, it is possible to go from any state to any state in exactly n steps. It is clear from this definition that every regular chain is ergodic. On the other hand, an ergodic chain is not necessarily regular.

18.8 Summary

Modern probability theory studies chance processes for which the knowledge of previous outcomes influences predictions for future experiments. In principle, when we observe a sequence of chance experiments, all of the past outcomes could influence our predictions for the next experiment.

The subject of Markov chains is best studied by considering special types of markov chains. The first type that we shall study is called an *absorbing Markov chain*. In an absorbing Markov chain, a state which is not absorbing is called *transient*.

Consider an arbitrary absorbing Markov chain. Renumber the states so that the transient states come first. If there are r absorbing states and t transient states, the transition matrix will have the following *canonical form*.

A Markov chain is called an *ergodic* chain if it is possible to go from every state to every state (not necessarily in one move). In many books, ergodic Markov chains are called *irreducible*.

18.9 Self Assessment Question

1. Explain markov chain?
2. What is transition matrix?
3. Explain ergodic Markov chains?

18.10 Further Readings

1. N D Vohra : Quantitative Techniques in Management(Tata McGraw Hill)
2. V K Kapoor : Operations Research
3. Levine, Berenson, Krehbiel, Render, Stair, & Hanna : Quantitative Techniques for Management (Pearson)
4. Sharma , J.K : Operations Research : Theory and Applications (Macmillan India Ltd)

UNIT 19 POINT ESTIMATION, INTERVAL ESTIMATION/ TIME SERIES

UNIT STRUCTURE

19.0 Objectives

In this unit we will consider how to estimate certain parameters of population distribution. Here we will study how to use estimators and the estimates they give rise to.

19.1 Introduction

If we want to estimate the mean life of a mobile phone, then we could employ the sample mean estimates of population mean. If the value of sample mean were 1022 hours, then the estimate of the population mean would be 1022 hours.

19.2 Definitions

In the words of Sheldon M. Ross the famous statistician "An estimator is a statistics whose value depends on the particular sample drawn. The value of estimator, called the estimates, is used to predict the value of estimator, called the estimators, is used to predict the value of a population parameters.

Murry R Speiegle & Loary J stephens define point estimate as "An estimate of a population parameter given by a single number is called a point estimate of the parameter". They also defined interval estimation as "An estimate of a population parameters given by two numbers between which the parameter may be considered to lie is called an interval estimate of the parameters. $\bar{X}$

19.3 Point estimator of a population mean

If $X_1 \ldots\ldots\ldots\ldots X_n$ denote a sample from a population whose mean u is not known, Then the sample mean $\bar{X}$ can be used as an estimator of therefore

$$E[\bar{X}] = u$$

Here the estimators are called un-biased.

An estimator whose expected value is equal to the parameters it is estimating is said to be an unbiased estimator of that parameter.

Lets understand this by help of an example.

To estimate average amount of damage claimed in fire at Mohan apartment complex a consumer organization sampled the files of a large insurance company to come up with the following amounts.

(In lakh of rupees) for 8 claims.

120,50,60,10,6,140,40,50

The estimates of the mean amount of damages claimed in all fires of the type being considered is thus

$$\bar{X} = \frac{120+50+60+10+06+140+40+50}{8}$$

$$= \frac{476}{8} = 59.5$$

Thus we estimate that the mean fire damage claim is Rs. 59.5 lakhs here the sample mean has expected value u, since a random variable is not likely to be too many standard deviations away from its expected value, it is important to determine the standard deviation of $\bar{X}$

$$SD(\bar{X}) = \frac{6}{\sqrt{n}}$$

6= population standard deviation

SD ($\bar{x}$) = Standard error of $\bar{X}$

Science the random variable is unlikely to be more than 2 standard deviations away from the mean.(when the random variable is approximately normal , as , will be when the sample size n is large) we say that the estimate of the population mean will be correct to within $\pm$ 2 sc..

Note that the standard error decreases by the square root of sample size as a result, to cut the standard error in half, we increase the sample size by a factor of a 4.

Let's take an example to complete understand.

In an IQ test 4 students have got 36,39,34,35 marks with a mean u and understand deviation of o, 3.

The estimates of mean IQ marks

$$\frac{3.6+3.9+3.4+3.5}{4} = 3.6$$

With the standard error of the estimate being equal to

$$SD(\bar{X}) = \frac{6}{\sqrt{n}} = \frac{3}{2} = 0.15$$

Therefore we can be quite confident that the actual mean will differ from 3.6 by more than 0.30. suppose we wanted the estimator to have standard error of 0.05. then, since, this would be reduction in standard error by a factor of 3, it follows that we would have to choose a sample of I times as large. That is, we would have to take 36 students readings

19.4 Point estimator of a population proportion

If we had to estimate for the large population proportion in a favour of a given proposition. Let unknown proportion be P, A random sample is done to choose p and then p should be estimate by the proportion of sample that is in favour. P is expressed as

$$\hat{p} = \frac{x}{n}$$

x= number of members in sample favor in proposition

n = size of sample

We know that

$E(\hat{p}) = p$

i.e , the proportion of the sample in favour of proposition, is an unbasied estimator of p, the entire population in favour. The spread of the estimator p about its mean p is measured by its standard deviation which is equal to

$$SD(\hat{p}) = \frac{\sqrt{p(1-p)}}{n}$$

the standard of is also called standard error as an estimators of the population proportion p. By the foregoing formula this standard error will be whenever the sample size n is large. In fact since it can be shown that for every value of p

$$p(1-p) \leq 1/4$$

It follow that

$$SD(\hat{p})\sqrt{1/4}n=1/2\sqrt{n}$$

Take an example a random sample of 400 is chosen. Then no matter how much population, it follows that the standard error of this proportion is less than or equal to 1(2/400) = 1/40

The preceding formula and bound on the standard error assume that we are drawing a random sample of size n form an infinitely large sample, when the population size is smaller, then so is the standard error, thus making the estimator even more precise than just indicated.

19.5 Interval estimators

"When we estimate a parameter by a point estimator, we do not expect the resulting estimator to exactly equal the parameter, but we expect that it will be "close" to it. To be more specific, we sometimes try to find an interval about point estimator in which we can be highly confident that the parameter lies. Such an interval is called interval estimator"

In statistics, interval estimation is the use of sample data to calculate an interval of possible or probable values of an unknown population parameter, in contrast to point estimation, which is a single number. Neyman (1937) identified interval estimation as distinct from point estimation. In doing so, he recognized that then-recent work quoting results in the form of an estimate plus-or-minus a standard deviation indicated that interval estimation was actually the problem statisticians really had in mind.

19.6 Disadvantages and problems associated with interval estimation

The scientific problems associated with interval estimation may be summarized as follows:

- When interval estimates are reported, they should have a commonly-held interpretation in the scientific community and more widely. In this regard, credible intervals are held to be most readily understood by the general public. Interval estimates derived from fuzzy logic have much more application-specific meanings.

- For commonly occurring situations there should be sets of standard procedures that can be used, subject to the checking and validity of any required assumptions. This applies for both confidence intervals and credible intervals.

- For more novel situations there should be guidance on how interval estimates can be formulated. In this regard confidence intervals and credible intervals have a similar standing but there are differences:

- credible intervals can readily deal with prior information, while confidence intervals cannot.

- confidence intervals are more flexible and can be used practically in more situations than credible intervals: one area where credible intervals suffer in comparison is in dealing with non-parametric models .

- There should be ways of testing the performance of interval estimation procedures. This arises because many such procedures involve approximations of various kinds and there is a need to check that the actual performance of a procedure is close to what is claimed. The use of stochastic simulations makes this is straightforward in the case of confidence intervals, but it is somewhat more problematic for credible intervals where prior information needs to taken properly into account. Checking of credible intervals can be done for situations representing no-prior-information but the check involves checking the long-run frequency properties of the procedures.

19.7 TIME SERIES

Time series analysis refers to problems in which observations are collected at regular time intervals and there are correlations among successive observations. Applications cover almost all areas of Statistics but some of the most significant include economic and financial time series, and many areas of environmental or ecological data.

A time series is a series of data points, measured normally at consecutive points in time spaced at uniform time intervals. Examples of time series are the daily closing value of the Bombay Stock Exchange (BSE) and the annual flow volume of the river Ganga at Kanpur. Time series are very often plotted via line charts. Time series are used in statistics, signal processing, pattern recognition, econometrics, mathematical finance, weather forecasting, earthquake prediction, electroencephalography, control engineering and communications engineering.

Time series analysis comprises methods for analyzing time series data in order to take out important statistics and other characteristics of the data. Time series forecasting is the use of a model to forecast future values based on earlier observed values. While regression analysis is often employed in such a way as to test theories that the current values of one or more independent time series affect the present value of another time series, this type of study of time series is not called "time series analysis".

Time series data have a normal sequential ordering. This makes time series analysis distinct from other common data analysis problems, in which there is no natural ordering of the observations. Time series analysis is also distinct from spatial data analysis where the remarks characteristically relate to geographical locations. A stochastic model for a time series will generally reflect the fact that observations close together in time will be more closely related than observations further apart. In accumulation, time series models will often make use of the natural one-way ordering of time so that values for a given period will be expressed as deriving in some way from past values, rather than from future values

19.8 Definitions

As defined by Kenny and Keeping time series is "A set of data depending on the time is called a time series."

In the words of Werner Z. Hirsch time series is "A time series is a sequence of values of the same variate corresponding to successive points in time."

Ya-lun-chou defined time series as "A time series may be defined as a collection of reading belonging to different time periods, of some economic variable or composite of variables."

Spiegel defined time series as "mathematically, a time series is defined by the values Y_1, Y_2......... Of a variable Y at time t_1, t_2......... Thus, Y is a function of t Symbolized $Y = f(t)$."

19.9 Importance Or Utility Of Time Series

1) Time series is used to analyze the past behavior. By this we can analyze the past behavior by using the past data.
2) Time series is used for comparative study. The past data are analyzed and on that basis comparison of data are made.
3) It is used to forecast the future on the basis of past.
4) It is used to assess the past performance and give the output.
5) Trade cycles can be easily analyzed by the help of time series.
6) Trends are of three types upward trend, downward trend and stable trend

19.10 Time series data

A time series is a set of statistics, usually collected at regular intervals. Time series data occur as expected in many application areas.

- Economics - e.g., monthly data for unemployment.

- Environmental - e.g., daily rainfall, air quality readings.
- Finance - e.g., daily exchange rate, share price, etc.
- Medicine - e.g., ECG brain wave activity every 2 secs.

The aims of time series analysis is to describe and review time series data, fit low-dimensional models, and make forecasts.

We write our real-valued series of observations as

.X"2,

X"1,

X0,

X1,

X2, . . .,

adoubly infinite sequence of real-valued random variables indexed by Z.

19.11 Trend, Seasonality, Cycles And Irregular Or Random Fluctuations

One simple method of describing a series is that of classical decomposition. The notion is that he series can be decomposed into four elements:

Trend (t) -- trends are long term movements in the mean; there are three types of trends upward trends, downward trend and stable trend

Seasonal effects (s) — cyclical fluctuations related to the calendar; seasonal variations are repetitive and regular.

Cycles (c) — other cyclical fluctuations (such as a business cycles); there are five phase of cycles fluctuations, prosperity, recession, depression, recovery and prosperity.

Irregular or Random fluctuations(r) — other random or systematic fluctuations like epidemic, drought, flood, earthquake etc..

19.12 Models of Time Series

There are two models of time series. The idea here is to create separate models for above mention four elements and then combine them, either additively or multiplicatively

19.12.1 Additive Model of Time Series

In additive model all four factors i.e. trends, seasonal, cycles and residuals are added to get the value. Formula for additive model is

$Xt= t+ s+ c+ r$

19.12.2 Multiplicative Model Of Time Series

In multiplicative model all four factors i.e. trends, seasonal, cycles and residuals are multiplied to get the value. Formula for multiplicative model is model is

$$Xt= t \cdot s \cdot c \cdot r$$

19.13 How to Calculate Time Series

There are two ways of finding the time series, the methods differs for odd and even numbers. The two method are described below.

19.13.1 For odd number of years

Moving averages of odd number of data is arithmetic average of a series of successive averages obtained from a series of values by averaging groups of successive values of time series.

Example : Give 3yearly moving average for the following series.

Year	Production (in lakh tons)
1996	16.1
1997	16.5
1998	17.2
1999	17.6

Solution

Year	Production	3 Year Total	3 Yearly moving Average
1996	16.1		
1997	16.5	49.8	16.6
1998	17.2	51.3	17.1
1999	17.6		

Here for calculating 3 year total we add up the data of the year1996, 1997, 1998 in one and 1997,1998,1999 in other trend. Then finally we average them by dividing it by no. of years which 3 in this case.

19.13.2 For Even number of years

Moving average of even no. of data is arithmetic average of a series of successive averages obtained from a series of values by averaging groups of successive values of time series considering it two time one with given no. of years and other by taking two at a time and finally while averaging we average over twice the no. of years.

Example

Find the 4 yearly moving average for the following data

Year	1990	1991	1992	1993	1994	1995
Production(in 1000 Tons)	23	10	21	12	16	16

Solution:

Year	Production	4 year total	2 year total of 4 year total	4 yearly moving average trend
1990	23			
1991	10	66		
1992	21	59	125	15.625
1993	12	65	124	15.500
1994	16			
1995	16			

Here for even no. of years we always make a column of average 2 years. Hence forth we have to average the data over twice the no. of years.

19.15 Terminal Questions

1. Find the bend by four year moving averages from the following data.

Year:-	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Value:-	105	115	100	90	80	95	85	75	60	65

Solve here:

Year	Value	4 year total	2 year total of 4 year Total	4 year total moving average trend

2. Find out trend by three yearly moving average form the data given

Year:-	2002	2003	2005	2006	2007	2008	2009	2010
Value:-	38	40	65	72	69	60	87	95

Solve :

Year	Income	3 year total	3 year moving Average

3. Explain point estimation?
4. How a point estimator a population mean calculated?
5. What isinterval estimator?
6. Explain time series and its importance?

7. Elucidate with example the model of time series?

19.14 Further Reading

1. ND Vohra : Quantitative Techniques in Management (Tata McGraw Hill)
2. V.K.Kapoor : Operations Research
3. Levine, Berenson, Krehbiel, Render, Stair, & Hanna : Quantitative Tech niques for Management (Pearson)
4. Sharma, J.K. : Operations Research : Theory and Applications (Macmillan India Ltd.)

Unit 20 CPM, PERT ANALYSIS, QUEEING THEORY

Unit structure

20.0 OBJECTIVES

20.1 CRITICAL PATH METHOD (CPM)

The critical path method (CPM) is a project modelling system developed in the late 1950s by Morgan R. Walker of DuPont and James E. Kelley, Jr. of Remington Rand. Kelley and Walker related their memories of the development of CPM in 1989. Kelley attributed the term "critical path" to the developers of the Program Evaluation and Review Technique which was developed at about the same time by Booz Allen Hamilton and the U.S. Navy. The precursors of what came to be known as Critical Path were developed and put into practice by DuPont between 1940 and 1943 and contributed to the success of the Manhattan Project.

CPM is normally used with all forms of projects, including construction, aerospace and defence, software development, research projects, product development, engineering, and plant maintenance, among others. Any project with mutually dependent activities can apply this process of mathematical analysis. Although the original CPM program and approach is no longer used, the term is generally applied to any approach used to analyze a project network logic diagram.

20.2 BASIC TECHNIQUE FOR CPM

The essential method for using CPM is to build a model of the project that includes the following:

1. A list of all activities required to complete the project
2. The time duration that each activity will take to completion, and
3. The dependencies between the activities.

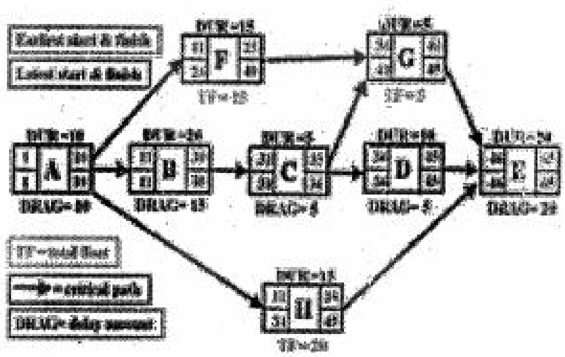
Using these values, CPM calculates the longest path of planned activities to the finish of the project, and the original and most recent that each activity can start and finish without making the project longer. This process determines which activities are "critical" i.e., on the longest path and which have "total float" i.e., can be delayed without making the project longer. In project management, a critical path is the series of project network actions which add up to the longest overall duration. This determines the shortest time possible to complete the project. Any delay of an activity on the critical path directly impacts the planned project completion date i.e. there is no float on the critical path. A project can have several, parallel, near critical paths. An additional parallel path through the network with the total durations shorter than the critical path is called a sub-critical or non-critical path.

Even though the activity-on-arrow diagram "PERT Chart" is still used in a few places, it has generally been superseded by the activity-on-node diagram, where each activity is shown as a box or node and the arrows signify the logical relationships going from predecessor to successor as shown here in the "Activity-on-node diagram".

Activity-on-node diagram showing critical path schedule, along with total float and critical path drag computations

1. If a critical path activity has nothing in parallel, its drag is equal to its duration. Thus A and E have drags of 10 days and 20 days respectively.

2. If a critical path activity has another activity in parallel, its drag is equal to whichever is less: its duration or the total float of the parallel activity with the least total float. Thus since B and C are both parallel to F (float of 15) and H (float of 20), B has a duration of 20 and drag of 15 (equal to F's float), while C has a duration of only 5 days and thus drag of only 5. Activity D, with a duration of 10 days, is parallel to G (float of 5) and H (float of 20) and therefore its drag is equal to 5, the float of G.



These results, including the drag computations, allow managers to prioritize activities for the effective management of project completion, and to shorten the planned critical path of a project by pruning critical path activities, by **"fast tracking"** i.e., performing more activities in parallel, and/or by **"crashing the critical path"** i.e., shortening the durations of critical path activities by adding resources.

20.3 Crash duration

"Crash duration" is a term referring to the shortest possible time for which an activity can be scheduled. It is achieved by shifting more resources towards the completion of that activity, resulting in decreased time spent and often a reduced quality of work, as the premium is set on speed. Crash duration is typically modelled as a linear relationship between cost and activity duration, however in many cases a convex function or a step function is more applicable.

20.4 Expansion

Initially, the critical path method considered only logical dependencies between terminal elements. Since then, it has been extended to allow for the addition of resources related to each activity, through processes called activity-based resource assignments and resource levelling. A resource-levelled schedule may include delays due to resource bottlenecks (i.e., unavailability of a resource at the required time), and may cause a previously shorter path to become the longest or most "resource critical" path. A related concept is called the critical chain, which attempts to protect activity and project durations from unforeseen delays due to resource constraints.

Since project schedules change on a regular basis, CPM allows constant monitoring of the schedule, which allows the project manager to track the critical activities, and alerts the project manager to the option that non-critical activities may be delayed beyond their total float, thus creating a new critical path and delaying project completion. In addition, the method can easily incorporate the concepts of stochastic predictions, using the Program Evaluation and Review Technique PERT and event chain methodology.

Presently, there are several software solutions available in industry that use the CPM method of scheduling.

20.5 Flexibility

A schedule generated using critical path techniques often is not realised specifically, as estimations are used to compute times: if one mistake is made, the results of the study may change. This could be reason to trouble in the implementation of a project if the estimates are blindly believed, and if changes are not addressed promptly. However, the structure of critical path analysis is such that the variance from the original schedule caused by any change can be calculated, and its crash either ameliorated or adjusted for. Indeed, an important element of project enquiry analysis is the As Built Critical Path (ABCP), which analyzes the specific causes and impacts of changes between the planned schedule and eventual schedule as actually implemented.

20.6 PROGRAMME EVALUATION REVIEW TECHNIQUE

PERT is a method to analyze the concerned tasks in carrying out a given project, particularly the time needed to complete each task, and to identify the least amount time necessary to complete the total project.

PERT was developed primarily to shorten the planning and scheduling of large and complex projects. It was developed for the U.S. Navy Special Projects Office in 1957 to support the U.S. Navy's Polaris nuclear submarine project. It was able to incorporate uncertainty by making it possible to plan a project while not knowing exactly the facts and durations of all the activities. It is more of an event-oriented method rather than start- and completion-oriented, and is used more in projects where time is the main factor rather than cost. It is applied to very one-time, large-scale, complex, non-routine infrastructure and Research and Development projects. An example of this was for the 1968 Winter Olympics in Grenoble which applied PERT from 1965 until the opening of the 1968 Games.

This project model was the first of its kind, a revival for scientific management, founded by Frederick Taylor (Taylorism) and later refined by Henry Ford (Fordism). DuPont's critical path method was invented at roughly the same time as PERT.

20.7 Some Important Points

- Two consecutive events in a PERT chart are linked by activities, which are conventionally represented as arrows.
- A PERT chart is a tool that facilitates decision making. The first sketch of a PERT chart will number its events successively in 10s (10, 20, 30, etc.) to allow the afterwards inclusion of additional events.
- The events are offered in a logical sequence and no action can begin until its immediately prior event is completed.
- A PERT chart may have multiple pages with many sub-tasks.
- The planner decides which milestones should be PERT procedures and also decides their "proper" progression.
- PERT is valuable to administer where multiple tasks are taking placeat the same time to reduce redundancy

20.8 TERMINOLOGY

- **PERT Event**: a point that marks the start or completion of one or more activities. It consumes no time and uses no resources. When it marks the completion of one or more tasks, it is not "reached" (does not occur) until *all* of the activities leading to that event have been completed.
- **Predecessor Event:** an event that instantly precedes some other event without any other events intervening. An event can have multiple predecessor events and can be the predecessor of multiple events.
- **Successor Event:** an event that immediately follows some other event without any other intervening events. An event can have multiple successor events and can be the successor of multiple events.
- **PERT Activity:** the actual performance of assignment which consumes time and requires resources (such as labour, materials, space, machinery). It can be understood as meaningful the time, effort, and resources required moving from one event to another. A PERT activity cannot be performed until the predecessor event has occurred.
- **Optimistic Time (O):**the minimum possible time required to accomplish a task, assuming everything proceeds better than is normally expected
- **Pessimistic Time (P):** the maximum possible time required to accomplish a task, assuming everything goes wrong (but excluding major catastrophes).
- **Most Likely Time (M):** the best estimate of the time required to accomplish a task, assuming everything proceeds as normal.

- **Expected Time (T_E)**: the best estimate of the time required to accomplish a task, accounting for the fact that things don't always proceed as normal (the implication being that the expected time is the average time the task would require if the task were repeated on a number of occasions over an extended period of time).

  $$T_E = (O + 4M + P) \div 6$$

- Float Or Slack is a measure of the surplus time and resources on hand to complete a task. It is the quantity of time that a project task can be deferred without causing a delay in any subsequent tasks (free float) or the whole project (total float). Positive slack would indicate ahead of schedule; negative slack would indicate behind schedule; and zero slack would indicate on schedule.

- *Critical Path*: the longest possible continuous pathway taken from the original event to the terminal event. It determines the total calendar time required for the project; and, therefore, any time delays along the critical path will delay the reaching of the terminal event by at least the same amount.

- *Critical Activity*: An activity that has total float equal to zero. An activity with zero float is not necessarily on the critical path since its path may not be the longest.

- Lead Time: the time by which a *predecessor event* must be completed in order to allow enough time for the activities that must elapse before a specific PERT event reaches finishing point.

- *Lag Time*: the original time by which a *successor event* can follow a specific PERT event.

- *Fast Tracking*: performing more critical activities in parallel

- *Crashing Critical Path*:reducing the duration of critical activities

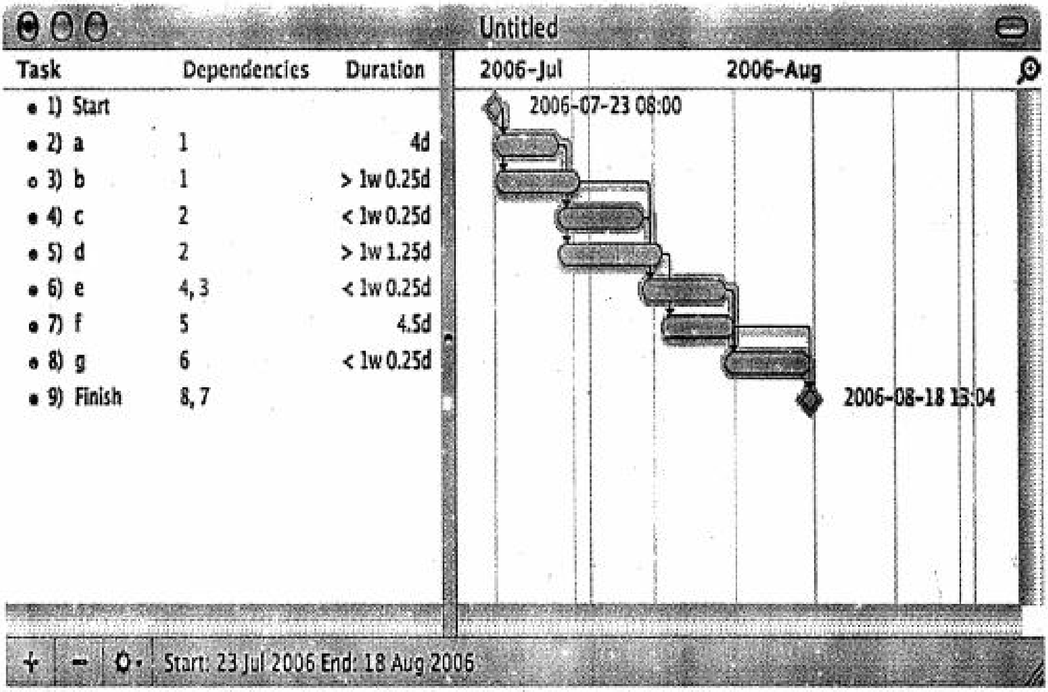
20.9 IMPLEMENTATION

The first step to scheduling the project is to decide the tasks that the project requires and the order in which they must be completed. The order may be easy to record for some tasks while difficult for others. As well, the time estimates generally reflect the normal, non-rushed time. Many times, the time required to execute the task can be reduced for an additional cost or a reduction in the quality.

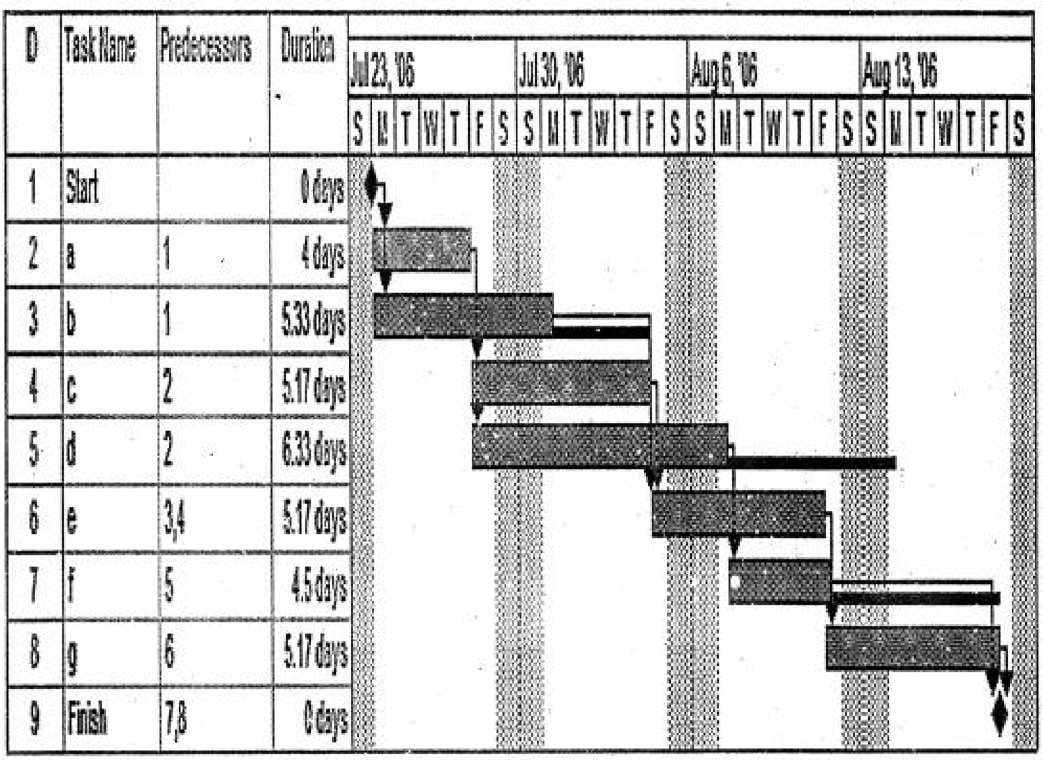
In the following example there are seven tasks, labeled*A* through *G*. Some tasks can be done concurrently (*A* and *B*) while others cannot be done until their predecessor task is complete (*C* cannot begin until *A* is complete). Additionally, each task has three time estimates: the optimistic time estimate (*O*), the most likely or normal time estimate (*M*), and the pessimistic time estimate (*P*). The expected time (T_E) is computed using the formula $(O + 4M + P) \div 6$.

Activity	Predecessor	Time estimates			Expected time
		Opt. (*O*)	Normal (*M*)	Pess. (*P*)	
A	—	2	4	6	4.00
B	—	3	5	9	5.33
C	*A*	4	5	7	5.17
D	*A*	4	6	10	6.33
E	*B*, *C*	4	5	7	5.17
F	*D*	3	4	8	4.50
G	*E*	3	5	8	5.17

Once this step is complete, one can draw a Gantt chart or a network diagram.



A Gantt chart created using Microsoft Project (MSP). Note (1) the critical path is in red, (2) the slack is the black lines connected to non-critical activities, (3) since Saturday and Sunday are not work days and are thus excluded from the schedule, some bars on the Gantt chart are longer if they cut through a weekend.



A Gantt chart created using OmniPlan. Note (1) the critical path is highlighted, (2) the slack is not specifically indicated on task 5 (d), though it can be observed on tasks 3 and 7 (b and f), (3) since weekends are indicated by a thin vertical line, and take up no additional space on the work calendar, bars on the Gantt chart are not longer or shorter when they do or don't carry over a weekend.

A network diagram can be created by hand or by using diagram software. There are two types of network diagrams,

1. Activity on Arrow (AOA)
2. Activity on Node (AON).

Activity on node diagrams are generally easier to create and interpret. To create an AON diagram, it is recommended to start with a node named *start*. This "activity" has a duration of zero (0). Then you draw each activity that does not have a predecessor activity and connect them with an arrow from start to each node.

20.10 ADVANTAGES OF PERT

- PERT chart clearly defines and makes visible dependencies between the work breakdown elements.
- PERT facilitates classification of the critical path and makes this visible.
- PERT facilitates identification of early start, late start, and slack for each activity,
- PERT provides for potentially reduced project duration due to better understanding of dependencies leading to improved overlapping of actions and tasks where feasible.

- The large amount of project data can be organized & presented in diagram for use in decision making.

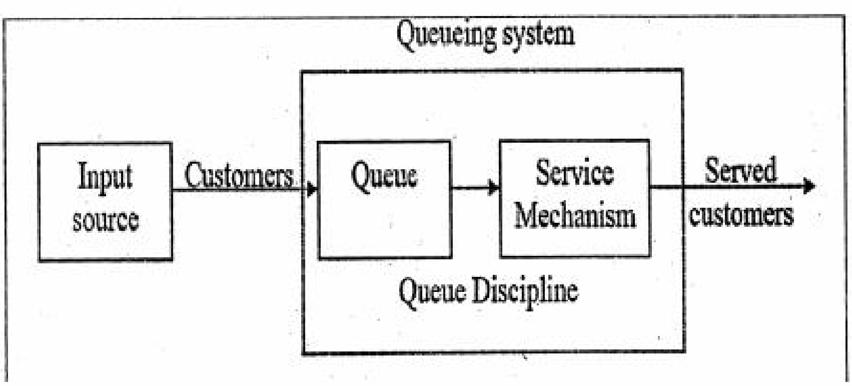
20.11 DISADVANTAGES OF PERT

- There can be potentially hundreds or thousands of activities and individual dependency relationships
- PERT is not easily scalable for smaller projects
- The network charts tend to be large and unwieldy requiring several pages to print and requiring special size paper
- The lack of a timeframe on most PERT/CPM charts makes it harder to show status although colours can help
- When the PERT/CPM charts become unwieldy, they are no longer used to manage the project.

20.12 QUEUEING THEORY

Queuing theory is the mathematical study of waiting lines, or queues. In queuing theory a model is created so that queue spans and waiting times can be predicted. The ultimate goal of the queuing theory is to accomplish an economic balance between the rate of service and the rate related with the waiting for that service. Queuing theory is the study of waiting in all these various facades.

20.13 STRUTURE OF QUEING MODEL



20.14 Queuing networks

Networks of queues are systems a number of queues are connected by customer routing. When a customer is serviced at one node it can join another node and queue for service, or leave the network. For a network of m the state of the system can be described by an m–dimensional vector $(x_1, x_2, \ldots, x_m)$ where x_i represents the number of customers at each node. The first significant results in this area were Jackson networks, for which an efficient product-form stationary distribution exists and the mean value analysis, which allows average metrics such as throughput and sojourn times to be computed.

If the total number of customers in the network remains constant the network is called a closed network and has also been shown to have a product–form stationary distribution in the Gordon–Newell theorem. This result was extended to the BCMP network. Here a network with very general service time, regimes and customer routing is shown to also exhibit a product-form stationary distribution.

Networks of customers have also been investigated; Kelly networks where customers of different classes experience different priority levels at different service nodes.

Another type of network isG-networks first proposed by Erol Gelenbe in 1993: these networks do not assume exponential time distributions like the classic Jackson Network.

20.15 Mean field limits

Mean field models consider the limiting behavior of the empirical measure (proportion of queues in different states) as the number of queues (m above) goes to infinity. The impact of other queues on any given queue in the network is approximated by a differential equation.

The deterministic model converges to the same stationary distribution as the original model.

20.16 Fluid limits

Fluid models are continuous deterministic analogs of queuing networks obtained by taking the limit when the process is scaled in time and space, allowing heterogeneous objects. This scaled trajectory converges to a deterministic equation which allows us stability of the system to be proven. It is known that a queuing network can be stable, but have an unstable fluid limit.

20.17 Self Assessment Question

1. What is CPM?
2. Explain the technique for CPM?
3. What is PERT?
4. Explain advantages & disadvantages of PERT.
5. Explain queuing theory
6. Describe queuing networks.

20.18 Further Readings

1. N D Vohra : Quantitative Techniques in Management(Tata McGraw Hill)
2. V K Kapoor : Operations Research
3. Levine, Berenson, Krehbiel, Render, Stair, & Hanna : Quantitative Techniques for Management (Pearson)
4. Sharma , J.K : Operations Research : Theory and Applications (Macmillan India Ltd)